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A COUNTEREXAMPLE TO MERIKOSKI-KUMAR CONJECTURE ON THE PRODUCT OF NORMAL MATRICES*

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Abstract

We give a counterexample to Merikoski-Kumar conjecture on the eigenvalues of two normal matrices and their product.

1 A Counterexample to Merikoski-Kumar's Conjecture

Let $A \in \mathbb{C}_{n\times n}$ and denote by $\lambda_1(A),\ldots,\lambda_n(A)$ the eigenvalues of A in the order $|\lambda_1(A)| \geq \cdots \geq |\lambda_n(A)|$. The singular values of A are the nonnegative square roots of the eigenvalues of the positive semidefinite Hermitian A^*A and are denoted by $s_1(A) \geq \cdots \geq s_n(A)$. Weyl's theorem provides a very nice relation between the eigenvalues and singular values of A.

THEOREM 1 (Weyl's inequalities [9]) Let $A \in \mathbb{C}_{n\times n}$. Then

$$\prod_{j=1}^{k} |\lambda_j(A)| \leq \prod_{j=1}^{k} s_j(A), \quad k = 1,\ldots,n-1, \tag{1}$$

$$\prod_{j=1}^{n} |\lambda_j(A)| = \prod_{j=1}^{n} s_j(A). \tag{2}$$

A. Horn [3] established the converse of Weyl's theorem, that is, if $|\lambda_1| \geq \cdots \geq |\lambda_n|$ and $s_1 \geq \cdots \geq s_n$ satisfy (1) and (2), then there exists $A \in \mathbb{C}_{n\times n}$ such that λ's are the eigenvalues of A and s's are the singular values of A. When A is normal, the moduli of the eigenvalues of A are the singular values of A, counting multiplicities.

Very recently Merikoski and Kumar [7, p.159] made the following

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Merikoski-Kumar conjecture: If A and B are normal matrices (not necessarily commute), and if $1 \le k \le i \le n$ and $1 \le \ell \le n-i+1$, then

$$|\lambda_{i+\ell-1}(A)||\lambda_{n-\ell+1}(B)| \le |\lambda_i(AB)| \le |\lambda_{i-k+1}(A)||\lambda_k(B)|. \tag{3}$$

We obtain the following

Counterexample: Let $c > 0$. By Horn's result [3] there exists a matrix $C \in \mathbb{C}_{n\times n}$ with eigenvalues $c, \dots, c$ and singular values $c^n, 1, \dots, 1$. By the polar decomposition, $C = AB$ where $A \in \mathbb{C}_{n\times n}$ is unitary and $B \in \mathbb{C}_{n\times n}$ is positive semidefinite. So the eigenvalues of AB are $c, \dots, c$ and the eigenvalues of B are $c^n, 1, \dots, 1$. Clearly A and B are normal. The eigenvalues of the unitary A are on the unit circle.

1. When $c > 1$,

$$|\lambda_i(AB)| = c > 1 = |\lambda_k(B)| = |\lambda_{i-k+1}(A)||\lambda_k(B)|, \quad 2 \le k \le i.$$

2. When $0 < c < 1$,

$$|\lambda_{i+\ell-1}(A)||\lambda_{n-\ell+1}(B)| = |\lambda_{n-\ell+1}(B)| = 1 > c = |\lambda_i(AB)|, \quad 2 \le i \le n-\ell+1.$$

2 Remarks

1. The inequalities (1) and (2) are closely related to a notion called majorization [6, 7] which has numerous applications in different areas. See [2] for a recent application in Physics. So (1) and (2) are sometimes called multiplicative majorization or log majorization. Kostant [4] extended Weyl-Horn's result in the context of semisimple Lie groups.

2. Based on the Horn's original construction, a fast recursive algorithm was recently given by Chu [1] to construct numerically a matrix with prescribed eigenvalues and singular values. The technique can be employed to create test matrices with desired spectral features for mathematical softwares . See [1] for the robustness of the algorithm.

3. Very often one encounters real matrices. So the construction of a real matrix with given singular values and eigenvalues is of interest. Clearly the nonreal eigenvalues of a real matrix must occur in complex conjugate pairs. Indeed, this is the only condition in addition to (1) and (2) for the construction of a real matrix, established by Thompson [8].

4. Very recently Li and Mathias [5] refined the proofs of Horn and Thompson [3, 8] so that they can control the order of the eigenvalues appearing in the diagonal of the resulting matrix under a numerically stable construction scheme.

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