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Power Function Solutions Of Iterative Functional Differential Equations *

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Abstract

In three recent papers [1-3], solutions of the form $x(z) = \lambda z^{\mu}$ are found for iterative functional differential equations. We find similar solutions for a more general equation

$$\prod_{i=1}^{a}\left(x^{(n_i)}(p_i z)\right)^{N_i} = Az^j \prod_{i=1}^{b}\left(x^{[m_i]}(q_i z)\right)^{M_1}.$$

Given a function f, its m-th iterate is defined as $f^{[0]}(z) = z$, $f^{[1]}(z) = f(z)$, and $f^{[m+1]}(z) = f\left(f^{[m]}(z)\right)$. In [1], it is noted that power solutions of the form $x(z) = \beta z^{\gamma}$ can be found for the iterative functional differential equation

$$x'(z) = x^{[m]}(z)$$

provided that $\gamma^m = \gamma - 1$ and $\beta^{\gamma^{m-1}+\cdots+\gamma} = \gamma$. In 2001, Li et al. [2] found solutions of the form $x(z) = \lambda z^{\mu}$ for the iterative functional differential equation of the form

$$x^{(n)}(z) = az^j\left(x^{[m]}(z)\right)^k \tag{1}$$

where k, m, n are positive integers, j is a nonnegative integer, a is a complex number and $x^{(n)}(z)$ is the n-th derivative of $x(z)$. As in [1], the proof is based on directly substituting $x(z) = \lambda z^{\mu}$ into the above equation and deduce necessary conditions on the exponent μ and the coefficient λ from the functional differential equation. Using essentially the same proof, Li et al. [3] in 2002, found power solutions for iterative equations of the form

$$x^{(n)}(z) = a\prod_{i=1}^{l}(x^{[m_i]}(q_i z))^{k_i}, \tag{2}$$

where $n, l, k_1, k_2, ..., k_l$ are positive integers, $m_1, m_2, ..., m_l$ are nonnegative integers such that $m_l \geq 2$ and $0 < m_1 < m_2 < \cdots < m_l$, and $a, q_1, q_2, ..., q_l$ are nonzero complex numbers.

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In this note, we prove the existence of power solutions for the more general equation

$$\left(x^{(n_1)}\left(p_1 z\right)\right)^{N_1}\cdots\left(x^{(n_a)}\left(p_a z\right)\right)^{N_a}=Az^j\left(x^{[m_1]}\left(q_1 z\right)\right)^{M_1}\cdots\left(x^{[m_b]}\left(q_b z\right)\right)^{M_b} \tag{3}$$

where $a, b, N_1, \ldots, N_a, M_1, \ldots, M_b$ and $n_1, ..., n_a, m_1, ..., m_b$ are positive integers such that $n_1 > n_2 > \cdots > n_a$ and $m_1 > m_2 > \cdots > m_b$. The number j is a nonnegative integer and $A, p_1, \ldots, p_a, q_1, \ldots, q_b$ are nonzero complex numbers.

By taking $a = b = 1$, $N_1 = 1$, $M_1 = k$ and $p_1 = q_1 = 1$ in (3), we obtain (1). By taking $a = 1$, $b = l$, $N_1 = 1$, $p_1 = 1$, $M_i = k_i$ for $i = 1, 2, \ldots, l$ in (3), we obtain (2).

For the sake of convenience, we employ the notation

$$(\mu)_n = \mu(\mu-1)\cdots(\mu-n+1).$$

THEOREM 1. Let Ω be a domain in the complex plane $\mathbf{C}$ which does not include the negative real axis nor the origin. Put $s(N,a) = N_1 + \cdots + N_a$, $s(M,b) = M_1 + \cdots + M_b$ and $s(Nn,a) = N_1 n_1 + \cdots + N_a n_a$. Let $\mu_1, \ldots, \mu_m$, where $1 \leq m \leq m_1$, be distinct roots of the polynomial

$$f(z) = M_1 z^{m_1} + \cdots + M_b z^{m_b} - s(N,a)\, z + s(Nn,a) + j. \tag{4}$$

If $s(N,a) \leq s(M,b)$, then (3) has m distinct, single-valued, nonzero, analytic power solutions of the form

$$x_i(z) = \lambda_i z^{\mu_i},\ i = 1, 2, \ldots, m; z \in \Omega,$$

where

$$\lambda_i = \left[\frac{\prod_{l=1}^{a} p_l^{N_l \mu_i}}{A\prod_{l=1}^{b} q_l^{M_l \mu_i^{m_l}}}(\mu_i)_{n_a}^{s(N,a)}(\mu_i - n_a)_{n_{a-1}-n_a}^{s(N,a-1)}\cdots(\mu_i - n_2)_{n_1-n_2}^{s(N,1)}\right]^{B_i}, \tag{5}$$

and

$$B_i = \frac{1-\mu_i}{s(M,b) + s(Nn,a) - s(N,a) + j}.$$

PROOF. Substituting $x(z) = \lambda z^{\mu}$ into (3), we obtain

$$P_\mu \lambda^{s(N,a)}(\mu)_{n_a}^{s(N,a)}(\mu - n_a)_{n_{a-1}-n_a}^{s(N,a-1)}\cdots(\mu - n_2)_{n_1-n_2}^{s(N,1)}\, z^{s(N,a)\mu - s(Nn,a)} = Q_\mu A \lambda^c z^r,$$

where

$$c = \sum_{l=1}^{b} M_l (1 + \mu + \cdots + \mu^{m_l - 1}),$$

$$r = \sum_{l=1}^{b} M_l \mu^{m_l} + j,$$

$$P_\mu = \prod_{l=1}^{a} p_l^{N_l \mu},$$

and

$$Q_\mu = \prod_{l=1}^{b} q_l^{M_l \mu^{m_l}}.$$

This leads to two requirements

$$P_\mu \lambda^{s(N,a)} (\mu)_{n_a}^{s(N,a)} (\mu - n_a)_{n_{a-1}-n_a}^{s(N,a-1)} \cdots (\mu - n_2)_{n_1-n_2}^{s(N,1)} = Q_\mu A \lambda^c \tag{6}$$

and

$$s(N,a)\,\mu - s(Nn,a) = \sum_{l=1}^{b} M_l \mu^{m_l} + j, \tag{7}$$

or

$$f(\mu) = 0.$$

Note that the polynomial $f(z)$ does not have any nonnegative real roots if $s(N,a) \leq s(M,b)$. Indeed, $f(0) = s(Nn,a) + j > 0$. For real $z \geq 1$, from $s(N,a) \leq s(M,b)$, we get $s(N,a)\,z \leq s(M,b)\,z \leq M_1 z^{m_1} + \cdots + M_b z^{m_b}$ and so $f(z) \geq s(Nn,a) + j > 0$. For real $z \in (0,1)$, we have $f(z) > 0 - s(N,a) + s(Nn,a) + j \geq 0$. Thus none of $\mu_1, \ldots, \mu_m$ is a nonnegative real number. Substitute $\mu = \mu_i$ into (6), we may then solve for $\lambda = \lambda_i \neq 0$ and conclude that $\lambda_i z^{\mu_i}$ is a desired solution. The proof is complete.

We remark that if the condition $s(N,a) \leq s(M,b)$ fails to hold, the theorem is not true as can be seen from the following example.

EXAMPLE 1. Consider the equation

$$\left(x^{(3)}(z)\right)\left(x^{(1)}(z)\right)^3 = x^{[1]}(z).$$

Here $s(N,2) = 4 > s(M,1) = 1$, $f(z) = z - 4z + 6$ has a unique root $\mu = 2$ with $\lambda = 0$, yielding only the trivial power function solution.

In certain cases, the number of solutions can be strengthened to m_1 as follows:

COROLLARY 1. In addition to the hypotheses in Theorem 1, suppose $m_1, \ldots, m_b$ are all even, or, m_1 is odd but $m_{2,}, \ldots, m_b$ are even. Then there exist m_1 distinct, single-valued, nonzero, analytic power function solutions.

Indeed, in the proof above, we already have $f(z) > 0$ for each $z \geq 0$. If $m_1, \ldots, m_b$ are even, then Descartes' rule of sign (see e.g. Barbeau [2, p.171]), tells us that $f(z)$ has no negative real root, while if m_1 is odd but $m_2, \ldots, m_b$ are even, then $f(z)$ has at most one negative real root. In either case, $f(z)$ cannot have repeated roots, other roots being complex conjugates. Hence, all m_1 roots of $f(z)$ are distinct.

We remark that since $s(N,a) = 1 \leq s(M,b) = k$ for equation (1), and $s(N,a) = 1 \leq s(M,b) = k_1 + \cdots + k_b$ for equation (2), we see that our result is an extension of the main results in [2] and [3].

Observe that each solution $x_i(z) = \lambda_i z^{\mu_i}$ has a nontrivial fixed point α_i of the form

$$\alpha_i = \lambda_i^{\frac{1}{1-\mu_i}} = \lambda_i^{\frac{1}{s(M,b)+s(Nn,a)-s(N,a)+j}} \neq 0,$$

thus we may write each solution $x_i(z)$ as

$$x_i(z) = \alpha_i^{1-\mu_i} z^{\mu_i}.$$

Expanding such solution about its fixed point, we immediately get the following consequence.

COROLLARY 2. Let $\mu_1, \ldots, \mu_m$ be the distinct roots of (7), and

$$\alpha_i = \lambda_i^{\frac{1}{1-\mu_i}},\ i = 1, 2, ..., m,$$

where λ_i is defined by (5). Then in a neighborhood of each point α_i, the iterative functional differential equation (3) has an analytic solution of the form

$$x_i(z) = \alpha_i + \frac{(\mu_i)_1}{1!}(z - \alpha_i) + \frac{(\mu_i)_2}{2!\alpha_i}(z - \alpha_i)^2 + \cdots + \frac{(\mu_i)_n}{n!\alpha_i^{n-1}}(z - \alpha_i)^n + \cdots.$$

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