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Optimal Singular Stochastic Problem On Harvesting System*

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Abstract

This paper presents an optimal singular stochastic problem on harvesting system. For the first time we introduce the running function into the total expected discounted harvested value. By relying on both stochastic calculus and the classical theory of singular control, we give a set of sufficient conditions for its solution in terms of optimal return function. Moreover, we also derive its optimal harvesting strategies and explicit form of optimal return function.

1 Introduction

Most models on stochastic harvesting model can be divided into two types. One is impulse rotation control problem (see e.g. [1]), and the other is singular control problem (see e.g. [2, 3, 4]). The above cited papers consider the determination of a harvesting planning maximizing the expected cumulative present value of future yields. However, they overlook an important factor affecting realistic models of stochastic control. Namely, they neglect the running costs in harvesting systems. We consider a class of singular harvesting problem that includes the running function in the total expected discounted harvested value. Other similar singular and impulse control problems can be found in [5, 6].

Let $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ be a complete probability space with filtration $\{\mathcal{F}_t\}$, which is assumed to be right-continuous, and $\mathcal{F}_0$ contains all the P-null sets in $\mathcal{F}$. We assume that a one-dimensional Brownian motion $W = \{W(t) : t \geq 0\}$ with respect to $\{\mathcal{F}_t\}$ is given on this probability space.

Consider a large population having a size $\hat{X} = \{\hat{X}(t); t \geq 0\}$ which in the absence of harvesting evolves according to the Itô stochastic differential equation,

$$d\hat{X}(t) = \mu(\hat{X}(t))dt + \sigma(\hat{X}(t))dW(t), \quad \hat{X}(0) = x, \tag{1}$$

where $x > 0$, $\mu : \mathrm{R}_+ \to \mathrm{R}$ and $\sigma : \mathrm{R}_+ \to \mathrm{R}$ are known to be sufficiently smooth (at least continuous) mapping guaranteeing the existence of a solution to the differential equation (1) (see [7]). In line with standard models for the size dynamics of a population

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stand, we assume that the upper boundary ∞ of the state space is nature and the lower boundary 0 is unattainable for the controlled diffusion (1).

If the population is subjected to harvesting, and ξ_t is the total number of individuals harvested up to time t, then the *size* of the harvested population, $X = \{X(t) : t \geq 0\}$, satisfies the stochastic differential equation

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dW(t) - d\xi_t, \quad X(0) = x,$$

where $x > 0$, the harvesting admissible strategy ξ_t is assumed to be non-negative, non-decreasing, left-continuous, $\{\mathcal{F}_t\}$-adapted process with $\xi_0 = 0$, and such that $X(t) > 0$.

We denote all the *admissible strategies* by Π. For every harvesting strategy $\xi = \{\xi_t\} \in \Pi, t \geq 0$, define the *expected total time-discounted value* of the harvested individuals starting with a population of size, x, i.e.

$$V_\xi(x) = \mathrm{E}_x \int_0^\infty e^{-\alpha t}\{\lambda d\xi_t - f(X(t))dt\}.$$

where $\lambda > 0$ represents the value of per unit harvested population, f is non-negative running function which represents the running cost.

As usual, we denote by $\mathcal{L}^1(\mathrm{R}_+)$ the class of $\pi(x)$ with finite expected cumulative present cost and denote this present cost by

$$(R_\alpha \pi)(x) = \mathrm{E}_x \int_0^\infty e^{-\alpha s}\pi(X(s))ds. \tag{2}$$

In order to guarantee the finiteness of the objective function (2), we shall assume further that the expected cumulative running cost is finite. i.e. $f \in \mathcal{L}^1(R_+)$.

Our objective is to choose an optimal harvesting strategy $\xi^* = \{\xi_t^*, t \geq 0\} \in \Pi$ such that

$$V(x) = V_{\xi^*}(x) = \sup_{\xi \in \Pi} V_\xi(x). \tag{3}$$

In the terminological spirit of Operations Research, we call the function V given by (3) the *optimal return function*.

2 A Verification Theorem for Harvesting System

In this section, by applying the theory of stochastic calculus and Doléans-Dade-Meyer formula, we obtain the verification theorem that the optimal return function and strategy satisfy.

LEMMA 1. There exists a unique control strategy $\xi^* \in \Pi$ such that (i) $0 < X^*(t) = x + \int_0^t \mu(X^*(s))ds + \int_0^t \sigma(X^*(s))dW(s) - {\xi_t}^* \leq b$ *for* $t > 0$; (ii) ξ^* is flat off for $\{t > 0 : X^*(t) > b\}$; and (iii) $\xi_{0+}^* = (x-b)\mathrm{I}_{\{x>b\}}$.

PROOF. Let

$${\xi_t}^* = \max\left[0, \max_{u\in(0,t]}\{x + \int_0^u \mu(X^*(s))ds + \int_0^u \sigma(X^*(s))dW(s) - b\}\right].$$

Then Lemma 1 is a straight consequence of the properties of the Skorokhod equation (see [8]).

REMARK 1. Except for a possible initial jump at $t = 0$ for $x > b$, which brings X^*_{0+} on $(0, b]$, the functions ξ^* are actually continuous. Intuitively, if the initial size $x \leq b$, ξ^* increases only when X^* is at the point b so as to ensure $X^* \leq b$. On the other hand, if the initial size $x > b$, the conclusion (iii) implies that ${\xi_{0+}}^* = x - b$, that is, $X^*(0)$ jumps to point b immediately and such that $X^*(0+) = b$, then evolves on as the case of X^* with the initial point b.

As usual, we denote the *infinitesimal generator* $\mathcal{A}$ associated with the controlled process $X(t)$ by $\mathcal{A} = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx}$.

THEOREM 1.

(I) Suppose the mapping $F : \mathrm{R}_+ \to \mathrm{R}$ is continuous, has two continuous derivatives, and satisfies the conditions (i) $F'(x) \geq \lambda$ for all $x \in \mathrm{R}_+$, (ii) $((\mathcal{A} - \alpha)F)(x) \leq f(x)$ for all $x \in \mathrm{R}_+$, and (iii) $\liminf_{t\to\infty} \mathrm{E}e^{-\alpha t}F(X(t)) = 0$. Then, $V(x) \leq F(x)$ for all $x \in \mathrm{R}_+$.

(II) Suppose the mapping F, in addition to the conditions of (I), also satisfies the following conditions: (iv) $(\mathcal{A} - \alpha)F(x) = f(x)$ for all $x \in (0, c]$, and (v) $F(x) = \lambda(x - c) + k$ for all $x \geq c$, where k, c are constants and $c > 0$. Then, there exists an optimal strategy $\xi^* \in \Pi$ such that

$$F(x) = V_{\xi^*}(x), \tag{4}$$

that is, $F(x)$ is the optimal return function and ξ^* is the corresponding optimal strategy.

PROOF. Let $\xi \in \Pi$ be an arbitrary admissible strategy. Then, according to the Doléans-Dade-Meyer formula (see [8]) and conditions (i) and (ii), we have

$$\begin{aligned} F(x) \quad \geq \quad & e^{-\alpha T^*}F(X(T^*)) - \int_0^{T^*} e^{-\alpha t}f(X(t))dt - \int_0^{T^*} e^{-\alpha t}F'(X(t))\sigma(X(t))dW(t) \\ & + \int_0^{T^*} e^{-\alpha t}\lambda d{\xi_t}^c - \sum_{0\leq t\leq T^*} e^{-\alpha t}[F(X(t+)) - F(X(t))], \end{aligned} \tag{5}$$

where ${\xi_t}^c$ denotes the continuous part of the admissible strategy ξ_t, $T^* = r \wedge \tau(r)$, $\tau(r) = \inf\{t \geq 0 : X(t) \geq r\}$, and $r > 0$.

The definition of T^* implies $\mathrm{E}_x \int_0^{T^*} e^{-\alpha t}F'(X(t))\sigma(X(t))dW(t) = 0$. Therefore, by taking expectations to (5) and rearranging terms, we obtain

$$\begin{aligned} F(x) \quad \geq \quad & \mathrm{E}_x e^{-\alpha T^*}F(X(T^*)) - \mathrm{E}_x \int_0^{T^*} e^{-\alpha t}f(X(t))dt + \mathrm{E}_x \int_0^{T^*} e^{-\alpha t}\lambda d{\xi_t}^c \\ & -\mathrm{E}_x \left\{ \sum_{0\leq t\leq T^*} e^{-\alpha t}[F(X(t+)) - F(X(t))] \right\}. \end{aligned}$$

In addition, we have that

$$F(X(t+)) - F(X(t)) = -F'(\theta)\Delta\xi_t \leq -\lambda\Delta\xi_t,$$

here $\theta \in (X(t+), X(t))$, $\triangle\xi_t = X(t) - X(t_+)$. Therefore, for any admissible strategy $\xi \in \Pi$ and all $x \in \mathrm{R}_+$, we have

$$F(x) \geq \mathrm{E}_x e^{-\alpha T^*} F(X(T^*)) + \mathrm{E}_x \int_0^{T^*} e^{-\alpha t} \lambda d\xi_t - \mathrm{E}_x \int_0^{T^*} e^{-\alpha t} f(X(t))dt.$$

Now invoking the condition (iii), by letting $r \to \infty$ and applying monotone convergence theorem (see [8]), we see that

$$F(x) \geq \mathrm{E}_x \int_0^{\infty} e^{-\alpha t} \{\lambda d\xi_t - f(X(t))dt\},$$

i.e., $F(x) \geq \sup_{\xi \in \Pi} V_\xi(x) = V(x)$, which completes the proof of (**I**).

Next, we will find a strategy $\xi^* \in \Pi$ such that $F(x) = V_{\xi^*}(x)$. First, let

$${\xi_t}^* = \max\left\{0, \max_{u\in(0,t]} \{x + \int_0^t \mu(X^*(s))ds + \int_0^t \sigma(X^*(s))dW(s) - c\}\right\},$$

then we will finish our job in two steps.

Step 1. Suppose $x \in (0, c]$. From Lemma 1, we know that ξ^* is continuous for all $x \in (0, c]$ and increases only when $X^* = c$. Moreover, $F''(x)$ implies that $F'(c) = \lambda$. Then by applying Doléans-Dade-Meyer formula once more and condition (iv), we see that

$$\begin{aligned} F(x) &= e^{-\alpha T^*} F(X^*(T^*)) \\ &\quad - \int_0^{T^*} e^{-\alpha t} \left\{ f(X^*(t)dt - F'(X^*(t))\sigma(X^*(t))dW(t) - \lambda d\xi_t^* \right\}. \end{aligned}$$

For all $x \in (0, c]$, by taking expectation and letting $r \to \infty$, we see that $F(x) = V_{\xi^*}(x)$.

Step 2. Suppose $x > c$. Applying Lemma 1 we know that $V_{\xi^*}(x) = \lambda(x-c) + V_{\xi^*}(c)$. Step 1 has shown that $V_{\xi^*}(c) = F(c)$. In light of the continuous property of $F(x)$, we know that $F(c) = k$. Thus, for all $x > c$, $V_{\xi^*}(x) = \lambda(x - c) + F_{\xi^*}(c) = F(x)$.

The proof of (**II**) is complete.

3 Determination of the Optimal Return Function and Corresponding Controls

If a function satisfies all the conditions of Theorem 1, then it is the optimal function to problem (3). Therefore, we will find a value function that have all the properties of $F(x)$ of Theorem 1 in this section.

LEMMA 2. The mapping $R_\alpha f : \mathrm{R}_+ \to \mathrm{R}_+$ satisfies the following equation

$$((\mathcal{A} - \alpha)(R_\alpha f))(x) + f(x) = 0.$$

The proof can be found in [9].

Let $\varphi(x)$ and $\phi(x)$ be two linearly independent *fundamental solutions*, with $\varphi(x)$ monotonically increasing and $\phi(x)$ monotonically decreasing, which span the set of solutions of the ordinary differential equation $(\mathcal{A}u)(x) = \alpha u(x)$.

LEMMA 3. Suppose $\Lambda(x) = \lambda(\mu(x) - \alpha x) - f(x)$ satisfies $\Lambda'(x) \geq 0$ for $x \in (0, M_1]$, $\Lambda'(x) \leq 0$ for $x \in [M_1, \infty)$ and $\lim_{x\to\infty} \Lambda(x) < 0$, where M_1 is a positive constant. Then there exists some point $m \geq M_1$ such that

$$\begin{aligned} m &= \arg \max_{x\in(0,\infty)} \left\{ \frac{\lambda + (R_\alpha f)'(x)}{\varphi'(x)} \right\} \\ &= \left\{ x^* : \frac{\lambda + (R_\alpha f)'(x^*)}{\varphi'(x^*)} \geq \frac{\lambda + (R_\alpha f)'(x)}{\varphi'(x)},\ x \in (0, \infty) \right\} \end{aligned} \tag{6}$$

and

$$\left(\frac{\lambda + (R_\alpha f)'(x)}{\varphi'(x)} \right)' \Bigg|_{x=m} = 0. \tag{7}$$

PROOF. Our assumption $f \in \mathcal{L}^1(R_+)$ implies that the expected cumulative running costs value $(R_\alpha f)(x)$ can be expressed in terms of the Green function [7] as

$$(R_\alpha f)(x) = B^{-1}\phi(x) \int_0^x \varphi(y) f(y) m'(y) dy + B^{-1}\varphi(x) \int_x^\infty \phi(y) f(y) m'(y) dy \tag{8}$$

where $m'(x) = 2/(\sigma^2(x)S'(x))$; $B = (\varphi'(x)\phi(x) - \phi'(x)\varphi(x))/S'(x) > 0$ denotes the *constant Wronskian* of the fundamental solutions and $S'(x) = \exp\left(-\int^x \frac{2\mu(s)}{\sigma^2(s)} ds\right)$. Since B is a constant independent of x, standard differentiation of (8) and rearrangements of terms yield

$$\frac{\varphi'(x)}{S'(x)}(R_\alpha f)(x) - \frac{(R_\alpha f)'(x)}{S'(x)}\varphi(x) = \int_0^x \varphi(y) f(y) m'(y) dy. \tag{9}$$

Observe that if $h : \mathrm{R}_+ \to \mathrm{R}$ is a twice continuously differentiable mapping on R_+, then applying Dynkin's theorem (see [9]) to $h(x)$, we have

$$\mathrm{E}_x[e^{-\alpha\tau(a,b)} h(X(\tau(a,b)))] = h(x) + \mathrm{E}_x \int_0^{\tau(a,b)} e^{-\alpha t} L(X(t)) dt, \tag{10}$$

where $L(x) = (\mathcal{A}h)(x) - \alpha h(x)$ and $\tau(a,b) = \inf\{t \geq 0 : X(t) \notin (a,b)\}$ denotes the first exit time of the underlying process $X(t)$ from the open set (a,b) for which $0 < a < b < \infty$. Since the left-hand side of equation (10) satisfies the ordinary differential equation $(\mathcal{A}u)(x) - \alpha u(x) = 0$ subject to the boundary conditions $u(a) = h(a)$ and $u(b) = h(b)$ we observe that

$$\mathrm{E}_x[e^{-\alpha\tau(a,b)} h(X(\tau(a,b)))] = h(a)\frac{\hat{\phi}(x)}{\hat{\phi}(a)} + h(b)\frac{\hat{\varphi}(x)}{\hat{\varphi}(b)}, \tag{11}$$

where $\hat{\varphi}(x) = \varphi(x) - \varphi(a)\frac{\phi(x)}{\phi(a)}$ and $\hat{\phi}(x) = \phi(x) - \phi(b)\frac{\varphi(x)}{\varphi(b)}$. On the other hand, since the expected cumulative value in the right hand side of (10) satisfies the ordinary differential equation $(\mathcal{A}v)(x) - \alpha v(x) + L(x) = 0$ subject to the boundary conditions $v(a) = v(b) = 0$, we see that

$$\begin{aligned} \mathrm{E}_x \int_0^{\tau(a,b)} e^{-\alpha s} L(X(t)) ds &= \hat{B}^{-1} \hat{\phi}(x) \int_a^x \hat{\varphi}(y) L(y) m'(y) dy \\ &\quad + \hat{B}^{-1} \hat{\varphi}(x) \int_x^b \hat{\phi}(y) L(y) m'(y) dy. \end{aligned}$$

where $\hat{B} = B\hat{\phi}(a)/\phi(a) = B\hat{\varphi}(b)/\varphi(b)$ denotes the constant Wronskian of $\hat{\varphi}(x)$ and $\hat{\phi}(x)$. Combining these results, (10) can now be expressed as

$$\begin{aligned} h(x) &= h(a)\frac{\hat{\phi}(x)}{\hat{\phi}(a)} + h(b)\frac{\hat{\varphi}(x)}{\hat{\varphi}(b)} - \hat{B}^{-1}\hat{\phi}(x) \int_a^x \hat{\varphi}(y) L(y) m'(y) dy \\ &\quad - \hat{B}^{-1} \hat{\varphi}(x) \int_x^b \hat{\phi}(y) L(y) m'(y) dy, \end{aligned}$$

which, in turn, implies that

$$\begin{aligned} \frac{h(x)}{\hat{\varphi}(x)} &= \frac{h(a)\hat{\phi}(x)}{\hat{\phi}(a)\hat{\varphi}(x)} + \frac{h(b)}{\hat{\varphi}(b)} - \hat{B}^{-1}\frac{\hat{\phi}(x)}{\hat{\varphi}(x)} \int_a^x \hat{\varphi}(y) L(y) m'(y) dy \\ &\quad - \hat{B}^{-1} \int_x^b \hat{\phi}(y) L(y) m(y) dy. \end{aligned} \tag{12}$$

By differentiation of (12) and rearranging terms, we see that

$$\frac{h'(x)}{S'(x)}\hat{\varphi}(x) - \frac{\hat{\varphi}'(x)}{S'(x)} h(x) = \int_a^x \hat{\varphi}(y) L(y) m'(y) dy - \frac{Bh(a)}{\phi(a)}.$$

Since the lower boundary 0 is unattainable for controlled process, then $\phi(0+) = \infty$. And if $h(x)$ is bounded at the origin and $L \in \mathcal{L}^1(\mathrm{R}_+)$, then letting $a \downarrow 0$ yields

$$\frac{h'(x)}{S'(x)}\varphi(x) - \frac{\varphi'(x)}{S'(x)} h(x) = \int_0^x \varphi(y) L(y) m'(y) dy. \tag{13}$$

Applying now (13) to the mapping λx gives rise to

$$\frac{\lambda\varphi(x)}{S'(x)} - \frac{\varphi'(x)}{S'(x)}\lambda x = \int_0^x \varphi(y)(\lambda(\mu(y) - \alpha y)) m'(y) dy. \tag{14}$$

Combining (14) and (9), we see that

$$\frac{\lambda + (R_\alpha f)'(x)}{S'(x)}\varphi(x) - \frac{\varphi'(x)}{S'(x)}(\lambda x + (R_\alpha f)(x)) = \int_0^x \varphi(y)\Lambda(y) m'(y) dy. \tag{15}$$

Equation (8) still holds for $f(x) \equiv 1$, that is $\frac{\varphi'(x)}{S'(x)} = \alpha \int_0^x \varphi(y) m'(y) dy$. Thus

$$\Lambda(x)\frac{\varphi'(x)}{S'(x)} - \alpha \int_0^x \varphi(y)\Lambda(y) m'(y) dy = g(x)$$

where

$$g(x) = \alpha \left(\Lambda(x) \int_0^x \varphi(y)m'(y)dy - \int_0^x \varphi(y)\Lambda(y)m'(y)dy \right).$$

Since $\varphi(x)$ is non-negative, and the assumptions on $\Lambda(x)$ imply $\Lambda(M_2) < 0$ for some $M_2 > M_1$, thus, for $x > M_2$,

$$\int_0^x \varphi(y)\Lambda(y)m'(y)dy \geq \Lambda(x) \int_0^x \varphi(y)m'(y)dy. \tag{16}$$

From (13), we know that $g'(x) = \alpha\Lambda'(x) \int_0^x \varphi(y)m'(y)dy$. By the definition of $g(x)$ and (16), we know that $g(0) = 0$ and $\lim_{x\to\infty} g(x) \leq 0$. Then, there exists some $m > M_1$ such that $g(m) = 0$, $g(x) \geq 0$ for all $x \in (0, m)$, and $g(x) \leq 0$ for $x > m$.

On the other hand,

$$\frac{d}{dx}\left\{\frac{\lambda + (R_\alpha f)'(x)}{\varphi'(x)}\right\} = \frac{(R_\alpha f)''(x)\varphi'(x) - (\lambda + (R_\alpha f)'(x))\varphi''(x)}{(\varphi'(x))^2}. \tag{17}$$

Since $((\mathcal{A} - \alpha)(R_\alpha f))(x) + f(x) = 0$ and $((\mathcal{A} - \alpha)\varphi)(x) = 0$, together with (15), we can rewrite the right side of (17) as follows:

$$\frac{2S'(x)}{\sigma^2(x)\varphi'^2(x)} \left\{\Lambda(x)\frac{\varphi'(x)}{S'(x)} - \alpha \int_0^x \varphi(y)\Lambda(y)m'(y)dy\right\}.$$

Consequently, (17) can be written as

$$\frac{d}{dx}\left\{\frac{\lambda + (R_\alpha f)'(x)}{\varphi'(x)}\right\} = \frac{2S'(x)}{\sigma^2(x)\varphi'^2(x)} g(x). \tag{18}$$

From the properties of $g(x)$ just shown, we know that $\left(\frac{\lambda+(R_\alpha f)'(x)}{\varphi'(x)}\right)' \geq 0$ for all $x \in (0, m)$, $\left(\frac{\lambda+(R_\alpha f)'(x)}{\varphi'(x)}\right)' \leq 0$ for all $x \in (m, \infty)$ and $\frac{d}{dx}\left(\frac{\lambda+(R_\alpha f)'(x)}{\varphi'(x)}\right)\Big|_{x=m} = 0$, the conclusion of our Lemma follows readily from the above results, which completes the proof.

THEOREM 2. Suppose that for any control strategy $\xi \in \Pi$,

$$\liminf_{t\to\infty} \mathrm{E}_x e^{-\alpha t} X(t) = 0, \tag{19}$$

and $\Lambda(x) = \lambda(\mu(x) - \alpha x) - f(x)$ satisfies the conditions in Lemma 3. Then, there exists a function $w(x)$ satisfies all the conditions of Theorem 1 for all $x \in \mathrm{R}_+$.

PROOF. Define $w(x)$ by

$$w(x) = \begin{cases} -(R_\alpha f)(x) + \frac{\lambda+(R_\alpha f)'(m)}{\varphi'(m)}\varphi(x) & x \leq m \\ \lambda(x - m) - (R_\alpha f)(m) + \frac{\lambda+(R_\alpha f)'(m)}{\varphi'(m)}\varphi(m) & x \geq m \end{cases}. \tag{20}$$

Eq. (20) implies that $w(x) \in C(\mathrm{R}_+) \cap C^1(\mathrm{R}_+)$. And (6) as well as (7) show that

$$(R_\alpha f)''(m)\varphi'(m) - (\lambda + (R_\alpha f)'(m))\varphi''(m) = 0.$$

Therefore, $w''(x)$ is continuous on $(0,\infty)$. Since $\frac{\lambda+(R_\alpha f)'(x)}{\varphi'(x)} \le \frac{\lambda+(R_\alpha f)'(m)}{\varphi'(m)}$, then,

$$-(R_\alpha f)'(x) + \frac{\lambda+(R_\alpha f)'(m)}{\varphi'(m)}\varphi'(x) \ge \lambda.$$

This implies that condition (i) in Theorem 1 is satisfied. Put $c = m$ and $k = -(R_\alpha f)(m) + \frac{\lambda+(R_\alpha f)'(m)}{\varphi'(m)}\varphi(m)$. Then $w(x)$ satisfies condition (v) in Theorem 1. Obviously, for all $x \in (0, m]$, Lemma 2 and the definition of $\varphi(x)$ show that

$$(\mathcal{A} - \alpha)w(x) = f(x),$$

thus, condition (iv) in Theorem 1 is satisfied.

Next we will prove $(\mathcal{A} - \alpha)w(x) \le f(x)$ for all $x \ge m$. That is, $\lambda\mu(x) - \alpha(\lambda(x - m) + w(m)) \le f(x)$. Since $\lambda(\mu(x) - \alpha x) - f(x)$ is decreasing for $x > m$, we have that

$$\lambda(\mu(x) - \alpha x) - f(x) \le \lambda(\mu(m) - \alpha m) - f(m).$$

And (20) implies that $\lambda\mu(m) - \alpha w(m) - f(m) = 0$, we have that

$$\lambda\mu(x) - \lambda\alpha(x - m) - \alpha w(m) - f(x) \le \lambda\mu(m) - \alpha w(m) - f(m) = 0,$$

condition (ii) in Theorem 1 follows.

Obviously, condition (iii) in Theorem 1 holds for $X(t) \in (0, m]$. And for $X(t) > m$, by the definition of $w(x)$ and assumption (19), we know that condition (iii) in Theorem 1 is satisfied. Therefore, for all $x \in \mathrm{R}_+$, condition (iii) in Theorem 1 is satisfied. An application of Theorem 1 yields our conclusion.

From the above knowledge, we arrive at the following result.

THEOREM 3. Suppose the assumption in Lemma 3 and (19) hold. Then $w(x)$ given by (20) is the optimal return function to problem (3), and the corresponding singular control strategy ξ^* is given by

$$\xi_t{}^* = \max[0, \max_{u\in(0,t]}\{x + \int_0^t \mu(X^*(s))ds + \int_0^t \sigma(X^*(s))dW(s) - m\}].$$

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