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On Equilibrium Problems*

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Abstract

Some iterative methods for solving equilibrium problems are suggested and analyzed by using the technique of the auxiliary principle. We have shown that the convergence of the proposed methods either requires only pseudomonotonicity, which is a weaker condition than monotonicity or partially relaxed strongly monotonicity. Our results represent an improvement and refinement of previously known results. Since the equilibrium problems include variational inequalities and complementarity problems as special cases, results proved in this paper continue to hold for these problems

1 Introduction

Equilibrium problems theory provides us with a unified, natural, innovative and general framework to study a wide class of problems arising in finance, economics, network analysis, transportation, elasticity and optimization. This theory has witnessed an explosive growth in theoretical advances and applications across all disciplines of pure and applied sciences. Equilibrium problems include variational inequalities as special cases. In recent years, several numerical techniques including projection, resolvent and auxiliary principle have been developed and analyzed for solving variational inequalities, see [1-13]. It is well-known and projection and resolvent type methods cannot be extended for mixed quasi variational inequalities. To overcome this drawback, one usually uses the auxiliary principle technique. Glowinski et al. [5] used this technique to study the existence of a solution of mixed variational inequalities, whereas Noor [7,8,10] used this technique to suggest and analyze a number of predictor-corrector and proximal methods for solving various classes of variational inequalities. In this paper, we again use the auxiliary principle technique to suggest and analyze some iterative methods for equilibrium problems. We have studied the convergence criteria of these methods under some mild conditions. As a consequence of this approach, we construct the gap (merit) function for equilibrium problems, which can be used to develop descent-type methods for solving equilibrium problems. Our results can be viewed as significant extension and generalization of the previously known results for solving equilibrium problems.

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2 Preliminaries

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$ respectively. Let K be a nonempty closed convex set in H. Let $T : H \longrightarrow H$ be a nonlinear operator. For a given nonlinear function $F(.,.) : H \times H \longrightarrow R$, consider the problem of finding $u \in K$ such that

$$F(u,v) \geq 0, \quad \forall v \in K, \tag{1}$$

which is called the *equilibrium problem,* considered and investigated by Blum and Oettli [1] and Noor and Oettli [2] in 1994. For applications and numerical results, see [1-4, 7,8,10].

If $F(u,v) = \langle Tu, v-u \rangle$, where $T : H \longrightarrow H$ is a nonlinear operator, then problem (1) is equivalent to finding $u \in K$ such that

$$\langle Tu, v-u \rangle \geq 0, \quad \forall v \in K, \tag{2}$$

which is known as the classical variational inequality, introduced and studied by Stampacchia [12] in 1964. It is well-known that a wide class of obstacle, unilateral, contact, free, moving and equilibrium problems arising in mathematical, engineering, economics and finance can be studied in the unified and general framework of the variational inequalities of type (2). For the physical and mathematical formulation of problems (1) and (2), see [1-21] and the references therein.

We also need the following concepts and results.

LEMMA 2.1. For $u, v \in H$,

$$2\langle u, v \rangle = \|u+v\|^2 - \|u\|^2 - \|v\|^2. \tag{3}$$

DEFINITION 2.1. The function $F(.,.) : K \times K \rightarrow H$ is said to be *pseudomonotone* if

$$F(u,v) \geq 0 \quad \Longrightarrow -F(v,u) \geq 0, \quad \forall u, v \in K,$$

and *partially relaxed strongly monotone,* if there exists a constant $\alpha > 0$ such that

$$F(u,v) + F(v,z) \leq \alpha \|z-u\|^2, \quad \forall u, v, z \in K.$$

Note that for $z = u$, partially relaxed strongly monotonicity reduce to

$$F(u,v) + F(v,u) \leq 0, \quad \forall u, v \in K,$$

which is known as the monotonicity of $F(.,.)$. It is known [4] that monotonicity implies pseudomonotonicity, but the converse is not true.

3 Iterative Schemes

We suggest and analyze some iterative methods for equilibrium problems (1) using the auxiliary principle technique of Glowinski et al. [5] as developed by Noor [7,8,10].

For a given $u \in K$, consider the auxiliary problem of finding a unique $w \in K$ such that

$$\rho F(w,v) + \langle w - u + \gamma(u - u), v - w\rangle \geq 0, \quad \forall v \in K, \tag{4}$$

where $\rho > 0$ and $\gamma > 0$ are constants. We note that if $w = u$, then clearly w is solution of the equilibrium problem (1). This observation enables us to suggest and analyze the following iterative method for solving (1).

Algorithm 3.1. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\rho F(u_{n+1}, v) + \langle u_{n+1} - u_n + \gamma_n(u_n - u_{n-1}), v - u_{n+1}\rangle \geq 0, \quad \forall v \in K,$$

which is known as the inertial proximal method for solving equilibrium problem (1).

Such type of inertial proximal methods have been considered by Alvarez and Attouch [11] and Noor [7,8,10] for solving variational inequalities (2).

For $\gamma_n = 0$, Algorithm 3.1 collapses to:

Algorithm 3.2. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\rho F(u_{n+1}, v) + \langle u_{n+1} - u_n, v - u_{n+1}\rangle \geq 0, \quad \forall v \in K, \tag{5}$$

which is called the proximal method for solving problem (1).

This shows that the inertial proximal method includes the classical proximal method as a special case.

If $F(u,v) = \langle Tu, v - u\rangle$, then Algorithm 3.1 reduces to:

Algorithm 3.3. For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho T u_{n+1} + u_{n+1} - u_n + \gamma_n(u_n - u_{n-1}), v - u_{n+1}\rangle \geq 0, \quad \forall \quad v \in K,$$

which can be written as

$$u_{n+1} = P_K[u_n - \rho T u_{n+1} + \gamma_n(u_n - u_{n-1})], \quad n = 0, 1, 2, \ldots,$$

where P_K is the projection of H onto the convex set K.

Algorithm 3.3 is known as the inertial proximal point algorithm for solving variational inequalities and has been studied by Noor [7,10].

We now study the convergence analysis of Algorithm 3.2. The analysis is in the spirit of Noor [7,8,10].

THEOREM 3.1. Let $\bar{u} \in K$ be a solution of (1) and u_{n+1} be the approximate solution obtained from Algorithm 3.2. If $F(.,.)$ is pseudomonotone, then

$$\|u_{n+1} - \bar{u}\|^2 \leq \|u_n - \bar{u}\|^2 - \|u_{n+1} - u_n\|^2. \tag{6}$$

PROOF. Let $\bar{u} \in K$ be a solution of (1). Then

$$-F(v, \bar{u}) \geq 0, \quad \forall v \in K, \tag{7}$$

since $F(.,.)$ is pseudomonotone. Taking $v = u_{n+1}$ in (7) and $v = \bar{u}$ in (5), we have

$$-F(u_{n+1}, \bar{u}) \geq 0. \tag{8}$$

and

$$\rho F(u_{n+1}, \bar{u}) + \langle u_{n+1} - u_n, \bar{u} - u_{n+1} \rangle \geq 0. \tag{9}$$

From (8) and (9), we have

$$\langle u_{n+1} - u_n, \bar{u} - u_{n+1} \rangle \geq -\rho F(u_{n+1}, \bar{u}) \geq 0. \tag{10}$$

Setting $u = \bar{u} - u_{n+1}$ and $v = u_{n+1} - u_n$ in (3), we obtain

$$2\langle u_{n+1} - u_n, \bar{u} - u_{n+1} \rangle = \|\bar{u} - u_n\|^2 - \|\bar{u} - u_{n+1}\|^2 - \|u_n - u_{n+1}\|^2. \tag{11}$$

Combining (10) and (11), we obtain the required result (6).

THEOREM 3.2. Let H be a finite dimensional space. If u_{n+1} is the approximate solution obtained from Algorithm 3.1 and $\bar{u} \in K$ is a solution of (1), then $\lim_{n \longrightarrow \infty} u_n = \bar{u}$.

PROOF. Let $\bar{u} \in K$ be a solution of (1). From (6), it follows that the sequence $\{\|\bar{u} - u_n\|\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Also from (6), we have

$$\sum_{n=0}^{\infty} \|u_{n+1} - u_n\|^2 \leq \|u_0 - \bar{u}\|^2,$$

which implies that

$$\lim_{n \longrightarrow \infty} \|u_{n+1} - u_n\| = 0. \tag{12}$$

Let $\hat{u}$ be a cluster point of $\{u_n\}$ and the subsequence $\{u_{n_j}\}$ of the sequence $\{u_n\}$ converge to $\hat{u} \in H$. Replacing u_n by u_{n_j} in (4) and taking the limit $n_j \longrightarrow \infty$ and using (12), we have

$$F(\hat{u}, v) \geq 0, \quad \forall v \in K,$$

which implies that $\hat{u}$ solves the equilibrium problem (1) and

$$\|u_{n+1} - u_n\|^2 \leq \|u_n - \bar{u}\|^2.$$

Thus it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and $\lim_{n \longrightarrow \infty} u_n = \hat{u}$, the required result.

It is known that in order to implement the inertial proximal and proximal algorithms, one has to find the approximate solution implicitly, which is itself a difficult problem. To overcome this drawback, we suggest another iterative method for solving problem (1).

For a given $u \in K$, consider the auxiliary problem of finding a unique $w \in K$ such that

$$\rho F(u, v) + \langle w - u, v - w \rangle \geq 0, \quad \forall v \in K, \tag{13}$$

where $\rho > 0$ is a constant. We note that if $w = u$, then clearly w is solution of the equilibrium problem (1). This observation enables us to suggest and analyze the following iterative method for solving equilibrium problem (1).

Algorithm 3.4. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\rho F(u_n, v) + \langle u_{n+1} - u_n, v - u_{n+1}\rangle \geq 0, \quad \forall v \in K. \tag{14}$$

If $F(u, v) \equiv \langle Tu, v - u\rangle$, then Algorithm 3.4 collapses to:

Algorithm 3.5. For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho Tu_n + u_{n+1} - u_n, v - u_{n+1}\rangle \geq 0, \quad \forall v \in K.$$

Algorithm 3.5 has been studied extensively, see [7-10].

We now study the convergence analysis of Algorithm 3.4 using essentially the technique of Theorem 3.1.

THEOREM 3.3. Let $\bar{u} \in K$ be a solution of (1) and u_{n+1} be the approximate solution obtained from Algorithm 3.4. If $F(.,.)$ is partially strongly monotone with constant $\alpha > 0$, then

$$\|u_{n+1} - \bar{u}\|^2 \leq \|u_n - \bar{u}\|^2 - (1 - 2\alpha\rho)\|u_{n+1} - u_n\|^2. \tag{15}$$

PROOF. Let $\bar{u} \in K$ be a solution of (1). Taking $v = u_{n+1}$ in (1), we have

$$F(\bar{u}, u_{n+1}) \geq 0. \tag{16}$$

Now taking $v = \bar{u}$ in (14), we obtain

$$\rho F(u_n, \bar{u}) + \langle u_{n+1} - u_n, \bar{u} - u_{n+1}\rangle \geq 0. \tag{17}$$

From (16) and (17), we have

$$\begin{aligned} \langle u_{n+1} - u_n, \bar{u} - u_{n+1}\rangle &\geq -\rho\{F(u_n, \bar{u}) + F(\bar{u}, u_{n+1})\} \\ &\geq -\alpha\rho\|u_n - u_{n+1}\|^2, \end{aligned} \tag{18}$$

since $F(.,.)$ is partially relaxed strongly monotone with a constant $\alpha > 0$. Combining (11) and (18), we obtain the required result (15).

THEOREM 3.4. Let H be a finite dimensional space and let $0 < \rho < 1/(2\alpha)$. If u_{n+1} is the approximate solution obtained from Algorithm 3.4 and $\bar{u} \in H$ is a solution of (1), then $\lim_{n\longrightarrow\infty} u_n = \bar{u}$.

Its proof is similar to Theorem 3.2.

It is obvious that the auxiliary equilibrium problem (14) is equivalent to finding the minimum of the functional $I[w]$ over the convex set K, where

$$I[w] = (1/2)\langle w - u, w - u\rangle - \rho F(u, w), \tag{19}$$

which is known as the auxiliary energy (virtual work, potential) function associated with the problem (13). Using this functional $I[w]$, one can reformulate the equilibrium problem (1) as an equivalent optimization problem:

$$\Psi_\alpha(u) = \max_{w\in K}\{-\rho F(u, w) - (\alpha/2)\|u - w\|^2\}, \tag{20}$$

where $\alpha > 0$ is a constant. Function of the type $\Psi(u)$ defined by (20) is called the regular gap function for the equilibrium problem. Note that for $\alpha = 0$, and $F(u, v) \equiv \langle Tu, v-u\rangle$, we obtain the original gap function for the variational inequality (2), which is due to Fukushima [13]. From the above discussion and observation, it is clear that can obtain the gap (merit) function for the equilibrium problems (1) by using the auxiliary principle technique. In passing, we remark this is observation is due to Noor [10], where it has been shown that the auxiliary principle technique can be used to construct gap functions for several variational inequalities. This equivalent optimization formulation of the equilibrium problems can be used to develop some descent-type algorithms for solving equilibrium problems under suitable conditions on the function $F(.,.)$, by using the technique of Fukushima [13].

4 Extensions

We would like to point out that the techniques and ideas of section 3 can be extended for solving the uniformly regular equilibrium problems, which are defined over the uniformly prox-regular sets K in H. It is known [14,15] that the uniformly prox-regular sets are nonconvex and include the convex sets as a special case. For this purpose, we need the following concepts from nonsmooth analysis, see [14,15].

DEFINITION 4.1. The proximal normal cone of K at u is given by

$$N^P(K;u) := \{\xi \in H : u \in P_K[u + \alpha\xi]\},$$

where $\alpha > 0$ is a constant and

$$P_K[u] = \{u^* \in S : d_K(u) = \|u - u^*\|\}.$$

Here $d_K(.)$ is the usual distance function to the subset K, that is

$$d_K(u) = \inf_{v\in K} \|v - u\|.$$

The proximal normal cone $N^P(K;u)$ has the following characterization.

LEMMA 4.1. Let K be a closed subset in H. Then $\zeta \in N^P(K;u)$ if and only if there exists a constant $\alpha > 0$ such that

$$\langle \zeta, v - u\rangle \leq \alpha\|v - u\|^2, \quad \forall v \in K.$$

DEFINITION 4.2. The Clarke normal cone, denoted by $N^C(K;u)$, is defined as

$$N^C(K;u) = \overline{co}[N^P(K;u)],$$

where $\overline{co}$ means the closure of the convex hull.

Clearly $N^P(K;u) \subset N^C(K;u)$, but the converse is not true. Note that $N^P(K;u)$ is always closed and convex, whereas $N^P(K;u)$ is convex, but may not be closed, see [15].

Poliquin et al. [15] and Clarke et al. [14] have introduced and studied a new class of nonconvex sets, which are also called uniformly prox-regular sets. This class of uniformly prox-regular sets has played an important part in many nonconvex applications such as optimization, dynamic systems and differential inclusions. In particular, we have

DEFINITION 4.3. For a given $r \in (0, \infty]$, a subset K is said to be uniformly r-prox-regular if and only if every nonzero proximal normal to K can be realized by an r-ball, that is, $\forall u \in K$ and $0 \neq \xi \in N^P(K; u)$, one has

$$\langle (\xi)/\|\xi\|, v - u\rangle \leq (1/2r)\|v - u\|^2, \quad \forall v \in K.$$

It is clear that the class of uniformly prox-regular sets is sufficiently large to include the class of convex sets, p-convex sets, $C^{1,1}$submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets; see [14, 15]. It is clear that if $r = \infty$, then uniform r-prox-regularity of K is equivalent to the convexity of K. This fact plays an important part in this paper.

It is known that if K is a uniformly r-prox-regular set, then the proximal normal cone $N^P(K; u)$ is closed as a set-valued mapping. Thus, we have $N^C(K; u) = N^P(K; u)$.

We consider the problem of finding $u \in K$ such that

$$F(u, v) + (k/2r)\|v - u\|^2 \geq 0, \quad \forall v \in K, \tag{21}$$

where k is a positive constant. Problem of the type (21) is called the uniformly regular equilibrium problem. Note that if $r = \infty$, then the uniformly prox-regular set K becomes the convex set K. Consequently problem (21) is exactly the equilibrium problem (1). Using essentially the technique of section 3, one can suggest and analyze similar iterative schemes for solving uniform regular equilibrium problems (21) with minor modification and adjustments.

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