Applied Mathematics E-Notes, 4(2004), 92-98 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

A Note On The Integral Criterion For Spectral Dichotomy Of Regular Pencils*

Miloud Sadkane †

Received 3 September 2003

Abstract

Perturbations of spectral projectors generated by linear matrix pencils are investigated. Estimates for norms of perturbed projectors are derived.

1 Introduction

Let A and B be $n \times n$ complex matrices such that the pencil $\lambda B - A$ is regular having no eigenvalues on the positively oriented closed contour γ. The spectral dichotomy methods compute the spectral projector

$$P_\gamma(A,B) = \frac{1}{2i\pi}\int_\gamma (\lambda B - A)^{-1} B \, d\lambda \tag{1}$$

onto the deflating subspace of $\lambda B - A$ corresponding to the eigenvalues inside γ. Along with $P_\gamma(A,B)$, these methods compute the so-called integral criterion for spectral dichotomy, a quantity that gives an idea about the confidence to be placed in the numerical quality of the computed spectral projector $P_\gamma(A,B)$. This quantity is the spectral norm $\|H_\gamma(A,B)\|_2$ of the matrix integral

$$H_\gamma(A,B) = \frac{1}{L_\gamma}\int_\gamma (\lambda B - A)^{-*}(\lambda B - A)^{-1} \, |d\lambda| \tag{2}$$

where $L_\gamma = \int_\gamma |d\lambda|$ is the length of γ. Here and throughout this note, an expression like $(\lambda B - A)^{-*}$ means the conjugate transpose of the inverse of $\lambda B - A$. As will be shown later, the smaller $\|H_\gamma(A,B)\|_2$, the better the stability of the projector $P_\gamma(A,B)$ with respect to perturbations in A in B. In case where the curve γ is a circle, there are now efficient algorithms that compute $P_\gamma(A,B)$ and $\|H_\gamma(A,B)\|_2$ [5] or $P_\gamma(A,B)$ and $H_\gamma(A,B)$ [3]. Moreover, in this case, $P_\gamma(A,B)$ and $H_\gamma(A,B)$ are related by a generalized Lyapunov equation (see first line of (14)).

The aim of this note is to show that for general closed contour γ, perturbation estimates for $P_\gamma(A,B)$ and $H_\gamma(A,B)$ can be derived showing that the two variables

*Mathematics Subject Classifications: 34D09, 15A22.

†Université de Bretagne Occidentale. Département de Mathématiques. 6, Av. Le Gorgeu. BP 809. 29285 Brest Cedex. France

functions $(A, B) \mapsto \|P_\gamma(A, B)\|_2$ and $(A, B) \mapsto \|H_\gamma(A, B)\|_2$ are continuous. Their modulus of continuity which involve the resolvent norm of the pencil $\lambda B - A$ give permissible bounds for the stable computation of $P_\gamma(A, B)$ and $H_\gamma(A, B)$. Also some relations connecting the norms of $P_\gamma(A, B)$ and $H_\gamma(A, B)$ are derived.

2 Perturbations of $P_\gamma(A, B)$ and $H_\gamma(A, B)$ and estimate on $\|H_\gamma(A, B)\|_2$

Let E and F be two perturbations on A and B respectively such that the perturbed pencil $\lambda(B + F) - (A + E)$ remains regular having no eigenvalues on γ. Assume that neither $\lambda B - A$ nor $\lambda(B + F) - (A + E)$ have the infinite eigenvalue $\lambda = \infty$ and that $\sqrt{\|E\|_2^2 + \|F\|_2^2} \leq \epsilon$. Define

$$m_\gamma(A, B) = \max_{\lambda \in \gamma} \left(\|(\lambda B - A)^{-1}\|_2 \sqrt{1 + |\lambda|^2} \right). \tag{3}$$

This quantity appears in a natural way when comparing the projectors $P_\gamma(A, B)$ with $P_\gamma(A + E, B + F)$. It was analyzed in the framework of ϵ-pseudospectrum of the pencil $\lambda B - A$ defined as (see [4]):

$$\Sigma_\epsilon(A, B) = \{\lambda : \quad \|(\lambda B - A)^{-1}\|_2 \sqrt{1 + |\lambda|^2} \geq \frac{1}{\epsilon}\}. \tag{4}$$

The following proposition gives a perturbation result on the spectral projector $P_\gamma(A, B)$. It is a generalization to matrix pencils of the result given in [2, Sec. 8.3].

PROPOSITION 2.1. Let $m_\gamma \equiv m_\gamma(A, B)$ and assume that $\epsilon\, m_\gamma < 1$. Then

$$\|P_\gamma(A + E, B + F) - P_\gamma(A, B)\|_2 \leq \frac{1}{2\pi} L_\gamma \epsilon\, m_\gamma \frac{1 + m_\gamma\, \|B\|_2}{1 - \epsilon\, m_\gamma}. \tag{5}$$

PROOF. A direct computation gives

$$P_\gamma(A + E, B + F) = \frac{1}{2i\pi} \int_\gamma \left(\lambda(B + F) - (A + E)\right)^{-1} (B + F)\, d\lambda =$$

$$\frac{1}{2i\pi} \int_\gamma \left(I + (\lambda B - A)^{-1}(\lambda F - E)\right)^{-1} (\lambda B - A)^{-1}(B + F)\, d\lambda.$$

Let

$$X(\lambda) = (\lambda B - A)^{-1} (\lambda F - E).$$

Then

$$P_\gamma(A + E, B + F) - P_\gamma(A, B) =$$

$$\frac{1}{2i\pi} \int_\gamma \left(I + X(\lambda)\right)^{-1} (\lambda B - A)^{-1} \left(F - (\lambda F - E)(\lambda B - A)^{-1} B\right)\, d\lambda.$$

Taking the norm we obtain

$$\|P_\gamma(A+E,B+F)-P_\gamma(A,B)\|_2 \le \frac{1}{2\pi}\int_\gamma \|\left(I+X(\lambda)\right)^{-1}\|_2 \|(\lambda B-A)^{-1}\|_2 \times \left(\|F\|_2+\|\lambda F-E\|_2\ \|(\lambda B-A)^{-1}\|_2\ \|B\|_2\right)\ |d\lambda|.$$

But

$$\|X(\lambda)\|_2 \le \|(\lambda B-A)^{-1}\|_2\sqrt{1+|\lambda|^2}\sqrt{\|E\|_2^2+\|F\|_2^2} \le \epsilon m_\gamma < 1.$$

Therefore

$$\left\|(I+X(\lambda))^{-1}\right\|_2 \le \frac{1}{1-\epsilon m_\gamma},$$

from which the proof easily follows.

REMARKS

1. The proof of Proposition 2.1 excludes the case where $\lambda=\infty$ is an eigenvalue of the pencil $\lambda B - A$. This happens when B is singular. Then the pencil $\lambda A - B$ has the eigenvalue $\lambda = 0$ (see [6]) and it suffices to consider the projector

$$P_\infty(A,B) := P_{\gamma_0}(B,A) = \frac{1}{2i\pi}\int_{\gamma_0}(\lambda A-B)^{-1}A\ d\lambda \tag{6}$$

onto the deflating subspace of $\lambda A - B$ corresponding to the eigenvalue $\lambda = 0$ enclosed by a contour γ_0. Similarly to Proposition 2.1, it can be shown that

$$\|P_\infty(A+E,B+F)-P_\infty(A,B)\|_2 \le \frac{1}{2\pi}L_{\gamma_0}\epsilon\ m_{\gamma_0}\frac{1+m_{\gamma_0}\ \|A\|_2}{1-\epsilon\ m_{\gamma_0}}. \tag{7}$$

where L_{γ_0} is the length of γ_0, $m_{\gamma_0} = \max_{\lambda\in\gamma_0}\left(\|(\lambda A-B)^{-1}\|_2\ \sqrt{1+|\lambda|^2}\right)$, E and F are perturbations such that $\sqrt{\|E\|_2^2+\|F\|_2^2} \le \epsilon$ and $\epsilon\ m_{\gamma_0} < 1$.

2. The condition $\epsilon\ m_\gamma < 1$ in Proposition 2.1 is clearly satisfied if $\partial\Sigma_\epsilon(A,B)\cap\gamma=\emptyset$ where $\partial\Sigma_\epsilon(A,B)$ denotes the boundary of $\Sigma_\epsilon(A,B)$. The stability of the projector $P_\gamma(A,B)$, as a function of the variables A and B, is ensured provided that $\epsilon\ m_\gamma < 1$ and $L_\gamma\epsilon\ m_\gamma\,(1+m_\gamma\|B\|_2) \ll 1$. This implies that the number of eigenvalues enclosed by γ remains constant. For example, the condition $m_\gamma^2 \ll 1/\epsilon$ is sufficient for the stability of $P_\gamma(A,B)$ with respect to perturbations E and F. The quantity m_γ is actually a modification (up to the term $\sqrt{1+|\lambda|^2}$) of the stability radius of the pencil $\lambda B - A$. It is difficult to compute and our aim (see Proposition 2.4) is to show that the largest eigenvalue of the Hermitian positive definite matrix $H_\gamma(A,B)$ gives the same information as m_γ.

Using analogous perturbation techniques, the following proposition shows the continuity of the function $(A,B)\mapsto\|H_\gamma(A,B)\|_2$.

PROPOSITION 2.2. Assume that $\epsilon\, m_\gamma < 1$. Then

$$\|H_\gamma(A+E,B+F) - H_\gamma(A,B)\|_2 \le \frac{\epsilon\, m_\gamma(2+\epsilon\, m_\gamma)}{(1-\epsilon m_\gamma)^2}\, \|H_\gamma(A,B)\|_2 \tag{8}$$

PROOF.

$$H_\gamma(A+E,B+F) = \frac{1}{L_\gamma}\int_\gamma \left(\lambda(B+F)-(A+E)\right)^{-*}\left(\lambda(B+F)-(A+E)\right)^{-1}\ |d\lambda|.$$

A few calculations show that

$$\left(\lambda(B+F)-(A+E)\right)^{-*}\left(\lambda(B+F)-(A+E)\right)^{-1} = (\lambda B-A)^{-*}\left(I-S(\lambda)\right)(\lambda B-A)^{-1}$$

where

$$\begin{array}{rcl} I - S(\lambda) & = & (I+X(\lambda))^{-*}\,(I+X(\lambda))^{-1} \\ X(\lambda) & = & (\lambda B-A)^{-1}(\lambda F-E)\,. \end{array}$$

Thus

$$\begin{gathered} \|H_\gamma(A+E,B+F) - H_\gamma(A,B)\|_2 = \\ \max_{\|x\|_2=1} \frac{1}{L_\gamma}\left|\int_\gamma x^*\,(\lambda B-A)^{-*}\,S(\lambda)\,(\lambda B-A)^{-1}\,x|d\lambda|\right| \le \\ \max_{\lambda\in\gamma}\|S(\lambda)\|_2\ \max_{\|x\|_2=1}\frac{1}{L_\gamma}\int_\gamma x^*\,(\lambda B-A)^{-*}\,(\lambda B-A)^{-1}\,x\ |d\lambda| = \\ \max_{\lambda\in\gamma}\|S(\lambda)\|_2\ \|H_\gamma(A,B)\|_2. \end{gathered}$$

The proof terminates by noting that (see the proof of Proposition 2.1)

$$\|X(\lambda)\|_2 \le \epsilon m_\gamma, \quad \|\,(I+X(\lambda))^{-1}\,\|_2 \le \frac{1}{1-\epsilon m_\gamma},$$

and that $\|S(\lambda)\|_2 \equiv \|\,(I+X(\lambda))^{-*}\,(X(\lambda)+X(\lambda)^*+X(\lambda)^*X(\lambda))\,(I+X(\lambda))^{-1}\,\|_2 \le \frac{\epsilon\, m_\gamma(2+\epsilon m_\gamma)}{(1-\epsilon m_\gamma)^2}$.

The following proposition shows how the norms of $P_\gamma(A,B)$ and $H_\gamma(A,B)$ are related.

PROPOSITION 2.3. The projector $P_\gamma(A,B)$ and the matrix $H_\gamma(A,B)$ satisfy

$$\|P_\gamma(A,B)\|_2 \le \frac{L_\gamma}{2\pi}\sqrt{\|B^*H_\gamma(A,B)B\|_2}. \tag{9}$$

PROOF.

$$\begin{array}{rcl} \|P_\gamma(A,B)\|_2^2 & = & \displaystyle\max_{\|x\|_2=1}\|P_\gamma(A,B)x\|_2^2 \\ & \le & \displaystyle\max_{\|x\|_2=1}\frac{1}{4\pi^2}\left(\int_\gamma \|(\lambda B-A)^{-1}Bx\|_2|d\lambda|\right)^2 \\ & \le & \displaystyle\max_{\|x\|_2=1}\frac{L_\gamma}{4\pi^2}\int_\gamma \|(\lambda B-A)^{-1}Bx\|_2^2|d\lambda| \\ & = & \displaystyle\frac{L_\gamma^2}{4\pi^2}\|B^*H_\gamma(A,B)B\|_2. \end{array}$$

The second inequality above comes from the Cauchy-Schwarz inequality.

Next we show how m_γ is related to the norm of $H_\gamma(A,B)$. But first we need the following lemma.

LEMMA 2.1. If $\lambda_0 \in \gamma$ and $\alpha > 0$, then

$$\int_\gamma \frac{|d\lambda|}{(1+\alpha|\lambda-\lambda_0|)^2} \geq \frac{L_\gamma}{1+\alpha L_\gamma}.$$

PROOF. Consider the parametric representation of the contour γ as : $\lambda = \lambda(\theta)$ and denote by $L(\theta) = \int_{\theta_0}^{\theta} |\lambda'(\varphi)|\, d\varphi$ the arc length between $\lambda_0 \equiv \lambda(\theta_0)$ and $\lambda(\theta)$. Then

$$\int_\gamma \frac{|d\lambda|}{(1+\alpha|\lambda-\lambda_0|)^2} = \int_{\theta_0}^{\theta_0+L_\gamma} \frac{|\lambda'(\theta)|}{(1+\alpha|\lambda(\theta)-\lambda(\theta_0)|)^2} d\theta.$$

But

$$|\lambda(\theta)-\lambda(\theta_0)| = \left| \int_{\theta_0}^{\theta} \lambda'(\varphi)\, d(\varphi) \right| \leq L(\theta).$$

Hence

$$\int_\gamma \frac{|d\lambda|}{(1+\alpha|\lambda-\lambda_0|)^2} \geq \int_{\theta_0}^{\theta_0+L_\gamma} \frac{|L'(\theta)|}{(1+\alpha L(\theta))^2} d\theta = \frac{L_\gamma}{1+\alpha L_\gamma}.$$

PROPOSITION 2.4. We have

$$\frac{1}{1+|\lambda_0|^2}\, \frac{m_\gamma^2}{1+m_\gamma\, \|B\|_2\, L_\gamma} \leq \|H_\gamma(A,B)\|_2 \leq m_\gamma^2, \tag{10}$$

$$\frac{d_\gamma^2}{1+d_\gamma\, \|B\|_2\, L_\gamma} \leq \|H_\gamma(A,B)\|_2 \leq d_\gamma^2, \tag{11}$$

where $\lambda_0 \in \gamma$ and $d_\gamma = \max_{\lambda\in\gamma} \|(\lambda B - A)^{-1}\|_2$.

PROOF.

$$\begin{aligned} \|H_\gamma(A,B)\|_2 &= \max_{\|x\|_2=1} (H_\gamma(A,B)x, x) \\ &= \max_{\|x\|_2=1} \frac{1}{L_\gamma} \int_\gamma \|(\lambda B - A)^{-1} x\|_2^2\, |d\lambda| \\ &\leq \frac{1}{L_\gamma} \int_\gamma d_\gamma^2\, |d\lambda| = d_\gamma^2 \leq m_\gamma^2. \end{aligned}$$

Now let $\lambda_0 \in \gamma$ and $x_0 \in \mathbf{C}^n$ with $\|x_0\|_2 = 1$ such that

$$m_\gamma(A,B) = \|(\lambda_0 B - A)^{-1}\|_2\, \sqrt{1+|\lambda_0|^2}$$

and

$$\|(\lambda_0 B - A)^{-1}\|_2 = \|(\lambda_0 B - A)^{-1} x_0\|_2.$$

From the identity

$$(\lambda B - A)^{-1} = (\lambda_0 B - A)^{-1} + (\lambda_0 - \lambda)(\lambda_0 B - A)^{-1} B (\lambda B - A)^{-1},$$

we obtain

$$\|(\lambda B - A)^{-1} x_0\|_2 \geq \|(\lambda_0 B - A)^{-1} x_0\|_2 - |\lambda - \lambda_0| \, \|(\lambda_0 B - A)^{-1}\|_2 \, \|B\|_2 \|(\lambda B - A)^{-1} x_0\|_2.$$

Hence

$$\|(\lambda B - A)^{-1} x_0\|_2 \geq \frac{\|(\lambda_0 B - A)^{-1}\|_2}{1 + |\lambda - \lambda_0| \, \|(\lambda_0 B - A)^{-1}\|_2 \|B\|_2}.$$

Therefore

$$\begin{aligned} \|H_\gamma(A,B)\|_2 &\geq \frac{1}{L_\gamma} \int_\gamma \|(\lambda B - A)^{-1} x_0\|_2^2 \, |d\lambda| \\ &\geq \frac{1}{L_\gamma} \|(\lambda_0 B - A)^{-1}\|_2^2 \int_\gamma \frac{|d\lambda|}{\left(1 + \|(\lambda_0 B - A)^{-1}\|_2 \|B\|_2 \, |\lambda - \lambda_0|\right)^2}, \end{aligned}$$

and from Lemma 2.1 we obtain

$$\begin{aligned} \|H_\gamma(A,B)\|_2 &\geq \frac{1}{L_\gamma} \|(\lambda_0 B - A)^{-1}\|_2^2 \, \frac{L_\gamma}{1 + \|(\lambda_0 B - A)^{-1}\|_2 \|B\|_2 L_\gamma} \\ &\geq \frac{1}{1 + |\lambda_0|^2} \, \frac{m_\gamma^2}{1 + m_\gamma \|B\|_2 L_\gamma}. \end{aligned}$$

With the same reasoning, we prove the bounds (11).

REMARKS

1. Proposition 2.3 shows that when $\|P_\gamma(A,B)\|_2$ is large, then so is the quantity $\sqrt{\|B^* H_\gamma(A,B) B\|_2}$. Then Proposition 2.4 shows that $d_\gamma \ \|B\|_2$ and hence $m_\gamma \ \|B\|_2$ are also large. Conversely, a large m_γ means that the $\epsilon-$pseudospectrum of $\lambda B - A$ intersects the contour γ (see [4]) and therefore that the projector $P_\gamma(A,B)$ may not be well defined.

   Also, Proposition 2.4 shows that $\|H_\gamma(A,B)\|_2$ can be as large as d_γ^2. The lower bounds in (10) and (11) are probably not optimal, but they show that

   $$\mathcal{O}(m_\gamma) \leq \|H_\gamma(A,B)\|_2 \leq d_\gamma^2 \leq m_\gamma^2.$$

2. The case where γ is a circle is important in stability analysis of discrete-time systems (or difference equations). If for instance $\gamma = C$ is the unit circle, then the projector $P_\gamma(A,B)$ and the matrix $H_\gamma(A,B)$ become

$$P \equiv P_C(A,B) = \frac{1}{2\pi} \int_0^{2\pi} \left(B - e^{-i\theta} A\right)^{-1} B \, d\theta, \qquad (12)$$

$$H \equiv H_C(A,B) = \frac{1}{2\pi} \int_0^{2\pi} \left(B - e^{-i\theta} A\right)^{-*} \left(B - e^{-i\theta} A\right)^{-1} \, d\theta. \qquad (13)$$

Using the Kronecker decomposition [1, 6] of A and B, it can easily be shown that P and H satisfy the following properties

$$\begin{cases} B^*HB - A^*HA = P^*P - (I-P)^*(I-P), \\ P^2 = P,\ (\tilde{H}P)^* = \tilde{H}P \ \text{ with } \tilde{\mathrm{H}} = (\mathrm{A} \pm \mathrm{B})^*\mathrm{H}(\mathrm{A} \pm \mathrm{B}). \end{cases} \tag{14}$$

For that special case, an algorithm has recently been proposed in [3]. It computes in a stable way the projector P and the scaled matrix H taken in the following form:

$$H = \frac{1}{2\pi}\int_0^{2\pi} \left(B - e^{-i\theta}A\right)^{-*} H^{(0)} \left(B - e^{-i\theta}A\right)^{-1}\, d\theta$$

where $H^{(0)}$ is an arbitrary hermitian positive definite matrix used for scaling purposes.

3. It would be interesting to derive systems analogous to (14) for the contour γ.

References

[1] F. R. Gantmacher, Théorie des matrices, Vol 2., Dunod, Paris, 1966.

[2] S. K. Godunov, Modern Aspects of Linear Algebra, The Scientific Book, Novosibirsk, 1997, (In Russian). English transl. Amer. Math. Soc., Vol. 175, 1998.

[3] S. K. Godunov and M. Sadkane, Some new algorithms for the spectral dichotomy methods, Lin. Alg. Applic, 358(2003), 173–194.

[4] P.-F. Lavallée and M. Sadkane, Pseudospectra of linear matrix pencils by block diagonalization, Computing, 60(1998), 133–156.

[5] A. N. Malyshev, Parallel algorithm for solving some spectral problems of linear algebra, Lin. Alg. Applic, 188-189(1993), 489–520.

[6] G. Stewart and J.-G. Sun, Matrix Perturbation Theory, Academic Press, New York, 1990.