Applied Mathematics E-Notes, 4(2004), 68-73 © ISSN 1607-2510
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Stability In A Nonlinear Four-Term Recurrence Equation*

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Received 9 September 2003

Abstract

In this paper, we provide sufficient conditions for the existence of unbounded solutions and the global attractivity of solutions of a four-term recurrence equation.

1 Introduction

Difference equations appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations having applications in biology, ecology, physics, etc. Recently there has been a lot of work concerning the boundedness, and the global asymptotic stability of the solutions of nonlinear difference equations (see [1-6] and the references cited therein). In this paper, we study the difference equation

$$x_{n+2} = f(x_{n+1}, x_n, x_{n-1}),\ n = 0, 1, ..., \tag{1}$$

under the initial conditions $x_{-1}, x_0, x_1 \geq 0$ and $x_{-1}^2 + x_0^2 + x_1^2 > 0$, where the function f satisfies some of the following conditions:

(H_1) $f \in C[[0,\infty)^3 \backslash \{(0,0,0)\}, (0,\infty)]$;

(H_2) $f(u, v, w)$ is decreasing in u, v and w;

(H_3) the equation $x = f(x, x, x)$ has a unique positive equilibrium $x = \overline{x} > 0$, that is, $\bar{x}$ is a positive fixed point of f;

(H_4) there exist $M_1, M_2, M_3 \geq \bar{x}$ such that

$$f(M_1, 0, 0) \leq \bar{x},\ f(0, M_2, 0) \leq \bar{x},\ f(0, 0, M_3) \leq \bar{x};$$

(H_5) $H^2(x) > x$ for $0 < x < \bar{x}$, where $H(x) = f(x, x, x)$;

(H_6) there exists a $K \geq \bar{x}$ such that for all $u > K$,

$$G(u) = f(f(0, 0, u), f(0, u, 0), f(u, 0, 0)) > u.$$

*Mathematics Subject Classifications: 39A10

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Our aim in this paper is to investigate the existence of unbounded solutions and the attractivity of solutions of (1).

The initial conditions $x_{-1}, x_0, x_1 \geq 0$ and $x_{-1}^2 + x_0^2 + x_1^2 > 0$ determine a corresponding unique solution $\{x_n\}_{n=-1}^{\infty}$ of (1). The set of all such solutions will be denoted by Ω. The equilibrium $\bar{x}$ of (1) is called a global attractor if every solution $\{x_n\}$ in Ω satisfies $\lim_{n\to\infty} x_n = \overline{x}$. A real interval I is called an invariant interval for (1) if the additional conditions $x_{-1}, x_0, x_1 \in I$ imply the corresponding solution $\{x_n\}_{n=-1}^{\infty} \subset I$. $\bar{x}$ is a global attractor for solutions of (1) originated from I if every solution in Ω under the additional condition that $x_{-1}, x_0, x_1 \in I$ satisfies $\lim_{n\to\infty} x_n = \overline{x}$.

2 Existence of Unbounded Solutions

We first establish the existence of an unbounded solution of (1).

THEOREM 1. Assume that the hypotheses (H_1)-(H_4) and (H_6) are satisfied. Then there exist unbounded solutions in Ω.

PROOF. Consider any solution $\{x_n\}_{n=-1}^{\infty}$ in Ω that satisfies $x_1 > K > 0$. Then $x_2 = f(x_1, x_0, x_{-1}) < f(x_1, 0, 0)$, $x_3 = f(x_2, x_1, x_0) < f(0, x_1, 0)$, $x_4 = f(x_3, x_2, x_1) < f(0, 0, x_1)$ and

$$x_5 = f(x_4, x_3, x_2) > f(f(0, x_1, 0), f(0, x_1, 0), f(x_1, 0, 0)) > x_1.$$

By induction, we obtain

$$x_{4k+5} > f(f(0, 0, x_{4k+1}), f(0, x_{4k+1}, 0), f(x_{4k+1}, 0, 0)) > x_{4k+1} \tag{2}$$

for $k = 0, 1, 2, \ldots$. Assume to the contrary that $\{x_{4k+1}\}$ is bounded above. Since $\{x_{4k+1}\}$ is increasing, it must converge. Let

$$\lambda = \lim_{k\to\infty} x_{4k+1}.$$

Since $\lambda > K$, from (H_5), it follows that

$$f(f(0, 0, \lambda), f(0, \lambda, 0), f(\lambda, 0, 0)) > \lambda.$$

On the other hand, by letting $k \to \infty$ in (2), we find

$$\lambda \geq f(f(0, 0, \lambda), f(0, \lambda, 0), f(\lambda, 0, 0)),$$

which is a contradiction. The proof is complete.

EXAMPLE 1. Consider the Equation

$$x_{n+2} = \frac{1}{x_{n+1}^2 + x_n^2 + x_{n-1}^2}, \ n = 0, 1, \ldots . \tag{3}$$

Let $M_1 = M_2 = M_3 = \sqrt[6]{3}$, $K = 3$ and $\bar{x} = \frac{1}{\sqrt[3]{3}}$. Then it is easy to show that $f(u, v, w) = \frac{1}{u^2+v^2+w^2}$ satisfies the hypotheses (H_1)-(H_4) and (H_6). Hence by Theorem 1, there exists a solution of (3) that is unbounded.

3 Attractivity

In this section, we study the attractivity of the positive equilibrium $\bar{x}$ of (1). Let $I \subset (0, \infty)$ denote the maximal interval containing $\bar{x}$ such that the function h, defined by

$$h(x) = f(f(x,x,x), f(x,x,x), f(x,x,x)) \tag{4}$$

satisfies the weak negative feedback condition

$$(h(x) - x)(x - \bar{x}) \leq 0, \; x \in I.$$

Also, let $a = \inf I$ and $b = \sup I$.

LEMMA 1. Assume that the hypotheses (H_1)-(H_4) are satisfied. Then the following statements are true:

(a) $0 \leq a \leq \bar{x} \leq b \leq \infty$.
(b) If $a > 0$, then $a \in I$ and $h(a) = a$.
(c) If $b < \infty$, then $b \in I$ and $h(b) = b$.
(d) If either $a = \bar{x}$ or $b = \bar{x}$, then $I = \{\bar{x}\}$.
(e) $a > 0$ if, and only if, $b < \infty$. If $a > 0$, then $a = f(b,b,b)$ and $b = f(a,a,a)$.
(f) $a = 0$ if, and only if, $b = \infty$.

PROOF.

(a) This is trivial.

(b) Clearly, $h(x) \geq x$ for $a < x \leq \bar{x}$. Assume to the contrary that $a \notin I$. Then $h(a) < a$. Since h is continuous, there exists an $\epsilon > 0$ such that $h(x) < x$ for $x \in (a-\epsilon, a+\epsilon)$, which is a contradiction. Therefore, $h(a) \geq a$. If $h(a) > a$, there exists an $\epsilon > 0$ such that $h(x) > x$ for $x \in (a-\epsilon, a+\epsilon)$. So $a \neq \inf I$, which is a contradiction. Consequently, we obtain $h(a) = a$.

(c) Similar to (b).

(d) Let $a = \bar{x}$. Assume to the contrary that $b > \bar{x}$. Then for all $x \in [\bar{x}, c]$, where $\bar{x} < c < b$, we have

$$f(x,x,x) \leq f(\bar{x},\bar{x},\bar{x}) = \bar{x}, f(x,x,x) \geq f(c,c,c), h(x) \leq x.$$

Furthermore, $f(x,x,x) \in [f(c,c,c), \bar{x}]$ and

$$h(f(x,x,x)) = f(h(x), h(x), h(x)) \geq f(x,x,x),$$

so that $[f(c,c,c), \bar{x}] \subset I$, which is a contradiction. The case where $b = \bar{x}$ is similarly proved.

(e) Let $0 < a < \bar{x}$. Since $f(x,x,x)$ is continuous and decreasing, we find

$$f([a,\bar{x}], [a,\bar{x}], [a,\bar{x}]) = [\bar{x}, f(a,a,a)].$$

For every $x \in [\bar{x}, f(a,a,a)]$, there exists a unique $x' \in [a, \bar{x}]$ such that $f(x',x',x') = x$. As a result, $h(x') \geq x'$ and

$$h(x) = h(f(x',x',x')) = f(h(x'), h(x'), h(x')) \leq f(x',x',x') = x,$$

which implies $[\bar{x}, f(a,a,a)] \subset I$ and $f(a,a,a) \leq b$. Assume to the contrary that $f(a,a,a) < b$. Let $c \in (f(a,a,a), b)$. Using similar arguments as above, we find

$$f([\bar{x}, c], [\bar{x}, c], [\bar{x}, c]) = [f(c,c,c), \bar{x}] \subset I$$

and

$$a \leq f(c,c,c).$$

Since $c > f(a,a,a)$ and $h(a) = a$, we find

$$f(c,c,c) < f(f(a,a,a), f(a,a,a), f(a,a,a)) = h(a) = a,$$

which is a contradiction. Therefore, $b = f(a,a,a) < \infty$. The case when $b < \infty$ is similarly proved.

(f) This follows from (e).

COROLLARY 1. Assume that the hypotheses (H_1)-(H_4) are satisfied. Then I can be $\{\bar{x}\}, [a,b]$, or $(0, \infty)$, where $0 < a < \bar{x} < b < \infty$, $a = f(b,b,b)$ and $b = f(a,a,a)$.

LEMMA 2. Assume that the hypotheses (H_1)-(H_4) are satisfied. Then I is an invariant interval of (1).

PROOF. If $I = \{\bar{x}\}$ or $I = (0, \infty)$, the proof is easy. The remaining case is when $I = [a,b]$, where $0 < a < \bar{x} < b < \infty$, $a = f(b,b,b)$ and $b = f(a,a,a)$. Let $x_{-1}, x_0, x_1 \in [a,b]$. Then

$$a = f(b,b,b) \leq x_2 = f(x_1, x_0, x_{-1}) \leq f(a,a,a) = b,$$

$$a = f(b,b,b) \leq x_3 = f(x_2, x_1, x_0) \leq f(a,a,a) = b,$$

and

$$a = f(b,b,b) \leq x_4 = f(x_3, x_2, x_1) \leq f(a,a,a) = b.$$

If $x_{k-1}, x_k, x_{k+1} \in [a,b]$, then by induction,

$$a = f(b,b,b) \leq x_{k+2} = f(x_{k+1}, x_k, x_{k-1}) \leq f(a,a,a) = b.$$

The proof is complete.

THEOREM 2. Assume that the hypotheses (H_1)-(H_5) are satisfied. Then $\bar{x}$ is a global attractor for solutions of (1) originated from I.

PROOF. The case where $I = \{\bar{x}\}$ is trivial, so we will assume $I \neq \{\bar{x}\}$. Let $x_{-1}, x_0, x_1 \in I$. Then the solution $\{x_n\}$ is bounded. So

$$0 < \lambda = \lim_{n\to\infty} \inf x_n \leq \bar{x} \leq \mu = \lim_{n\to\infty} \sup x_n < \infty.$$

Clearly,

$$\lambda, \mu \in I, h(\lambda) \geq \lambda, h(\mu) \leq \mu.$$

Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ such that

$$\lim_{i\to\infty} x_{n_i+1} = \mu.$$

Then for every $\epsilon > 0$, there exists an integer N_0 such that $x_{n_i-1}, x_{n_i}, x_{n_i+1} > \lambda - \epsilon$, and $x_{n_i+2} = f(x_{n_i+1}, x_{n_i}, x_{n_i-1}) < f(\lambda-\epsilon, \lambda-\epsilon, \lambda-\epsilon)$. Hence $\mu \le f(\lambda-\epsilon, \lambda-\epsilon, \lambda-\epsilon)$ for every $\epsilon > 0$, which implies $\mu \le f(\lambda, \lambda, \lambda)$. Similarly, we may show that

$$\lambda \ge f(\mu, \mu, \mu).$$

In view of the fact that $H^2(x) > x$ for $0 < x < \bar{x}$, we have

$$H(\mu) = f(\mu, \mu, \mu) \le \lambda \le \bar{x} \le \mu \le f(\lambda, \lambda, \lambda) = H(\lambda). \tag{5}$$

It is easy to see that $\lambda = \mu = \bar{x}$ for $\lambda = \bar{x}$. Hence, we can assume that $\lambda < \bar{x}$. By (5), the properties of $H(x)$ and (H_5), we have

$$H^2(\mu) \ge H(\lambda) > \bar{x} > \lambda \ge H(\mu) \ge H^2(\lambda) > \lambda.$$

This is a contradiction. Therefore, $\lambda = \bar{x}$ and $\lambda = \mu = \bar{x}$. The proof is complete.

COROLLARY 2. Assume that the hypotheses (H_1)-(H_4) are satisfied. Let

$$(h(x) - x)(x - \bar{x}) \le 0, x \in (0, \infty),$$

where h is defined by (4). Then $\bar{x}$ is a global attractor for Ω.

EXAMPLE 2. Consider the equation

$$x_{n+2} = \frac{1}{\sqrt{x_{n+1}} + \sqrt{x_n} + \sqrt{x_{n-1}}}, \; n = 0, 1, 2, \ldots .$$

Let $M_1 = M_2 = M_3 = 3\sqrt[3]{3}$, $h(x) = H^2(x) = \frac{\sqrt[2]{3}}{3} x^{\frac{3}{4}}$, $f(u, v, w) = \frac{1}{\sqrt{u}+\sqrt{v}+\sqrt{w}}$. We can check that the hypotheses of Theorem 2 are satisfied. Thus, $\bar{x}$ is an attractor of all solutions $\{x_n\}$ with initial conditions $x_{-1}, x_0, x_1 \in I$. In fact, $\bar{x} = \frac{\sqrt[3]{3}}{3}$ is a global attractor for all the solutions $\{x_n\}_{n=1}^{\infty}$ with initial conditions $(x_{-1}, x_0, x_1) \in [0, \infty)^3 \backslash \{(0, 0, 0)\}$.

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