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Oscillation and Asymptotic Behavior of Second Order Difference Equations With Nonlinear Neutral Terms*

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Abstract

In this paper, we derive sufficient conditions for the oscillation of all / bounded solutions of a class of second order nonlinear difference equations with a nonlinear neutral term. Existence criterion is also derived for the eventually positive and asymptotically stable solution of this equation. Examples are provided to illustrate the results.

1 Introduction

Consider the second order nonlinear neutral difference equations of the form

$$\Delta\left(a_n \Delta\left(y_n - p y_{n-k}^{\alpha}\right)\right) + q_n f\left(y_{n+1-\ell}\right) = 0, n \geqslant n_0 \geqslant 0, \tag{1}$$

where p is a real number, $k > 0, \ell \geqslant 0$ are integers, α is a ratio of odd positive integers, Δ is the forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$, $\{a_n\}$ is a positive sequence with $\sum_{n=n_0}^{\infty} \frac{1}{a_n} = \infty$, $\{q_n\}$ is a nonnegative real sequence with a positive subsequence and $f : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing with $uf(u) > 0$ for $u \neq 0$.

When $\alpha = 1$ and $f(u) = u^{\beta}$, then equation (1) reduces to the equation

$$\Delta\left(a_n \Delta\left(y_n - p y_{n-k}\right)\right) + q_n y_{n+1-\ell}^{\beta} = 0, n \geqslant n_0. \tag{2}$$

Furthermore, if $a_n \equiv 1$ and $\beta = 1$, then equation (2) becomes

$$\Delta^2\left(y_n - p y_{n-k}\right) + q_n y_{n+1-\ell} = 0.$$

Such equations have been studied by a number of authors, and some of the related results can be found in [1, 5, 6, 8, 9, 10, 11, 12].

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Since equation (1) can be written in the recurrence form

$$y_{n+2} = F(n, y_n, y_{n+1}, y_{n-k}, y_{n+1-\ell}),$$

it is clear that given y_i and y_{i+1} for $-\max\{k,\ell\} \leqslant i \leqslant 1$, one can successively calculate $y_3, y_4, \cdots$ in a unique manner. Such a sequence $\{y_n\}$ will be called a solution of equation (1). A solution of equation (1) is called oscillatory if its terms are neither eventually positive nor eventually negative, otherwise it is nonoscillatory. In this paper, we are concerned with sufficient conditions for oscillation of all/bounded solutions of equation (1) and for existence of asymptotically stable solutions of equation (1). For related results one may see, for example [4, 7, 13]. Examples are inserted in the text of the paper to illustrate our results.

2 Main Results

In this section, we derive sufficient conditions for oscillation as well as existence of asymptotically stable solution of equation (1). We begin with the following lemma.

LEMMA 1. Let $\{y_n\}$ be a real sequence such that $y_n > 0, \Delta y_n > 0$ and $\Delta^2 y_n \leqslant 0$ for $n \geqslant n_0$ and $\{\sigma_n\}$ is a sequence of positive integers such that $\sigma_n \leqslant n$ and $\sigma_n \to \infty$ as $n \to \infty$. Then for each $\lambda \in (0,1)$ there is an integer $N \geqslant n_0$ such that $y_{\sigma_n} \geqslant \frac{\lambda \sigma_n}{n} y_n$ for all $n \geqslant N$.

PROOF. It is sufficient to consider only those n for which $\sigma_n < n$. Then, we have for $n > \sigma_n \geqslant n_0$, $y_n - y_{\sigma_n} \leqslant \Delta y_{\sigma_n} (n - \sigma_n)$ where we used the monotone property of $\{\Delta y_n\}$. Hence $\frac{y_n}{y_{\sigma_n}} \leqslant 1 + \frac{\Delta y_{\sigma_n}}{y_{\sigma_n}} (n - \sigma_n), n > \sigma \geqslant n_0$. Also $y_{\sigma_n} \geqslant y_{n_0} + \Delta y_{\sigma_n} (\sigma_n - n_0)$ so that for any $0 < \lambda < 1$, there is an integer $N \geqslant n_0$ with $\frac{y_{\sigma_n}}{\Delta y_{\sigma_n}} \geqslant \lambda \sigma_n, n \geqslant N$. Hence $\frac{y_n}{y_{\sigma_n}} \leqslant \frac{n + (\lambda - 1)\sigma_n}{\lambda \sigma_n} \leqslant \frac{n}{\lambda \sigma_n}, n \geqslant N$. The proof is now complete.

THEOREM 1. With respect to the difference equation (1) assume that (i) $p > 0, \ell > k, \Delta a_n \geqslant 0$ for $n \geqslant n_0$ and $\alpha \in (0,1]$; and (ii) there exists β (ratio of odd positive integers) $\in (0,1]$ such that

$$\frac{f(u)}{u^\beta} \geqslant M > 0 \text{ for } u \neq 0. \tag{3}$$

If

$$\limsup_{n\to\infty} \sum_{s=n-\ell+k}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} q_t > 0 \text{ for } \beta < \alpha, \tag{4}$$

or

$$\limsup_{n\to\infty} \sum_{s=n-\ell+k}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} q_t > \frac{p}{M} \text{ for } \beta = \alpha, \tag{5}$$

where $p \in (0,1)$ for $\alpha = 1, p \in (0,\infty)$ for $\alpha \in (0,1)$; and there exists $0 < \mu < M$ such that every solution of the difference equation

$$\Delta (a_n \Delta x_n) + \mu q_n \left(\frac{n+1-\ell}{n+1}\right)^\beta x_{n+1}^\beta = 0, \tag{6}$$

is oscillatory. Then every solution of equation (1) is oscillatory.

PROOF. Assume to the contrary that $\{y_n\}$ is an eventually positive solution of equation (1), say $y_n > 0$ for $n \geqslant N_0 \geqslant n_0$. Set

$$z_n = y_n - p y_{n-k}^{\alpha}. \tag{7}$$

From equation (1), we have $\Delta(a_n \Delta z_n) \leq 0$ for $n \geqslant N_0 + \theta, \theta = \max\{k, \ell\}$. If $\Delta z_n < 0$ eventually, then $\lim_{n\to\infty} z_n = -\infty$. Consequently, $\limsup_{n\to\infty} y_n = \infty$. Thus, there exists a sequence $\{n_j\}$ such that $\lim_{j\to\infty} n_j = \infty$ and $y_{n_j} = \max_{N_0 \leqslant n \leqslant n_j} y_n \to \infty$ as $j \to \infty$. Then

$$z_{n_j} = y_{n_j} - p y_{n_j - k}^{\alpha} \geqslant y_{n_j} - p y_{n_j}^{\alpha} = y_{n_j}\left(1 - p y_{n_j}^{\alpha-1}\right) \to \infty \text{ as } n \to \infty,$$

which is a contradiction. Therefore $\Delta z_n > 0$ for $n \geqslant N_0 + \theta$.

If $z_n < 0$ eventually, then $z_n > -p y_{n-k}^{\alpha}$. Hence

$$y_{n-k} > \left(-\frac{z_n}{p}\right)^{\frac{1}{\alpha}}. \tag{8}$$

From equation (1), (3) and (8), we have

$$\Delta(a_n \Delta z_n) - \frac{M q_n}{p^{\frac{\beta}{\alpha}}} z_{n+1-\ell+k}^{\frac{\beta}{\alpha}} \leqslant 0. \tag{9}$$

Summing (9) from s to $n-1$ for $n > s+1$, we obtain

$$a_n \Delta z_n - a_s \Delta z_s - \frac{M}{p^{\frac{\beta}{\alpha}}} \sum_{t=s}^{n-1} q_t z_{t+1-\ell+k}^{\frac{\beta}{\alpha}} \leqslant 0. \tag{10}$$

Let $\beta < \alpha$. If $\lim\limits_{n\to\infty} z_n = c = 0$, summing (10) from $n-\ell+k$ to $n-1$ for s, we have $z_{n-\ell+k} - z_n \leqslant \frac{M}{p^{\frac{\beta}{\alpha}}} z_{n-\ell+k}^{\frac{\beta}{\alpha}} \sum\limits_{s=n-\ell+k}^{n-1} \frac{1}{a_s} \sum\limits_{t=s}^{n-1} q_t$, or

$$\frac{z_{n-l+k}}{z_{n-l+k}^{\frac{\beta}{\alpha}}} \geq \frac{M}{p^{\frac{\beta}{\alpha}}} \sum_{s=n-l+k}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} q_t. \tag{11}$$

Since $z_{n-l+k}/z_{n-l+k}^{\frac{\beta}{\alpha}} = |z_{n-l+k}|^{1-\frac{\beta}{\alpha}}$ and $1 - \frac{\beta}{\alpha} > 0$, we have $\lim\limits_{n\to\infty} z_{n-l+k}/z_{n-l+k}^{\frac{\beta}{\alpha}} = 0$, and therefore from (11), we obtain

$$\limsup_{n\to\infty} \sum_{s=n-l+k}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} q_t \leq 0, \tag{12}$$

which contradicts (4).

If $\lim\limits_{n\to\infty} z_n = c < 0$, from (4), we claim that

$$\lim_{n\to\infty} \sum_{s=N_1}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} q_t = \infty. \tag{13}$$

In fact, from (4), there exists a subsequence $\{n_i\}$ and $n_{i+1} - n_i \geq l - k$ such that $\lim\limits_{n\to\infty} \sum\limits_{s=n_i-l+k}^{n_i-1} \frac{1}{a_s} \sum\limits_{t=s}^{n_i-1} q_t \geq b > 0$, where b is some positive number. Hence

$$\lim_{n\to\infty} \sum_{s=N_1}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} q_t \geq \lim_{j\to\infty} \sum_{i=1}^{j} \sum_{s=n_i-l+k}^{n_i-1} \frac{1}{a_s} \sum_{t=s}^{n-1} q_t \geq \lim_{j\to\infty} \sum_{i=1}^{j} \sum_{s=n_i-l+k}^{n_i-1} \frac{1}{a_s} \sum_{t=s}^{n_i-1} q_t = \infty,$$

where $n_j = \max\{n_i | n_i \leq n\}$.

From (10), we have

$$\Delta z_s + \frac{M}{p^{\frac{\beta}{\alpha}}} z_n^{\frac{\beta}{\alpha}} \frac{1}{a_s} \sum_{t=s}^{n-1} q_t \geq 0. \tag{14}$$

Summing (14) from N_1 to $n-1$, we obtain

$$z_{N_1} - z_n \leq \frac{M}{p^{\frac{\beta}{\alpha}}} z_n{}^{\frac{\beta}{\alpha}} \sum_{s=N_1}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} q_t,$$

or

$$\frac{p^{\frac{\beta}{\alpha}} z_{N_1}}{M z_n^{\frac{\beta}{\alpha}}} \geq \sum_{s=N_1}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} q_t.$$

In view of $c < 0$, hence $p^{\frac{\beta}{\alpha}} z_{N_1} / M z_n^{\frac{\beta}{\alpha}}$ has an upper bound. So $\lim\limits_{n\to\infty} \sum\limits_{s=N_1}^{\infty} \frac{1}{a_s} \sum\limits_{t=s}^{\infty} q_t < \infty$, which contradicts (13).

Now let $\beta = \alpha$. Then , (11) implies that $\sum\limits_{s=n-\ell+k}^{n-1} \frac{1}{a_s} \sum\limits_{t=s}^{n-1} q_t \leq \frac{p}{M}$, which contradicts (5). Therefore, $z_n > 0$ for all $n \geqslant N_0 + \theta$. Since $\Delta a_n \geqslant 0$ for $n \geqslant N_0 + \theta$ and from $\Delta(a_n \Delta z_n) \leqslant 0$, we have $\Delta^2 z_n \leqslant 0$ for all $n \geqslant N_0 + \theta$. Hence from Lemma 1, for each $\lambda \in (0,1)$ and such that $\mu \leq M\lambda^\beta$, there is an integer $N \geq N_0 + \theta$ such that

$$z_{n-\ell} \geqslant \lambda \frac{(n-\ell)}{n} z_n, \tag{15}$$

for $n \geqslant N$. Substituting (15) into equation (1), we have

$$\Delta\left(a_n \Delta z_n\right) + M\lambda^\beta q_n \left(\frac{n+1-\ell}{n+1}\right)^\beta z_{n+1}^\beta \leqslant 0,$$

or $\Delta\left(a_n\Delta z_n\right)+\mu q_n\left(\frac{n+1-\ell}{n+1}\right)^{\beta} z_{n+1}^{\beta}\leq 0$, for $n\geqslant N$, which implies that (6) has an eventually positive solution. This contradiction completes the proof of the theorem.

REMARK 1. The oscillatory criteria for equation (6) when $\beta\in(0,1]$ are given in [1] and [3].

REMARK 2. Let $a_n\equiv 1$ for all $n\geq n_0$ in equation (1). Then conditions (4) and (5) given in Theorem 1 reduce to

$$\limsup_{n\to\infty}\sum_{s=n-\ell+k}^{n-1}(s-n+\ell-k+1)q_s>0 \text{ for } \beta<\alpha,$$

or

$$\limsup_{n\to\infty}\sum_{s=n-\ell+k}^{n-1}(s-n+\ell-k+1)q_s>\frac{p}{M} \text{ for } \beta=\alpha.$$

The linear equation

$$\Delta^2\left(y_n-py_{n-k}\right)+q_n y_{n+1-\ell}=0, \tag{16}$$

is a special case of equation (1). From the above result, we have the following conclusion.

COROLLARY 1. Let $p\in(0,1)$ and $\ell>k$ holds . If every solution of

$$\Delta^2 x_n+\mu q_n\left(\frac{n+1-\ell}{n+1}\right)x_{n+1}=0,$$

is oscillatory, then every solution of equation (16) is oscillatory.

EXAMPLE 1. Consider the difference equation

$$\Delta^2\left(y_n-6y_{n-1}^{\frac{1}{3}}\right)+28y_{n-1}^{\frac{1}{3}}=0, n\geqslant 2. \tag{17}$$

It is easy to see that condition (5) of Theorem 1 is satisfied. Further, it is known that [3], every solution of

$$\Delta^2 x_n+28\mu\left(\frac{n-1}{n+1}\right)^{\frac{1}{3}}x_{n+1}^{\frac{1}{3}}=0,$$

is oscillatory. Therefore by Theorem 1, every solution of equation (17) is oscillatory. In fact $\{y_n\}=\{(-1)^n\}$ is one such solution of equation (17).

EXAMPLE 2. Consider the difference equation

$$\Delta^2\left(y_n-\frac{1}{6}y_{n-1}^{\frac{1}{3}}\right)+\frac{14}{3}y_{n-1}^{\frac{1}{5}}=0, n\geqslant 2. \tag{18}$$

It is easy to verify that all conditions of Theorem 1 hold. Hence every solution of equation (18) is oscillatory. In fact $\{y_n\}=\{(-1)^n\}$ is one such solution of equation (18).

Next we consider the equation (1) with $q_n \leq 0$ for all $n \geq n_0$. For the sake of convenience, we write equation (1) in the form

$$\Delta\left(a_n \Delta\left(y_n - p y_{n-k}^{\alpha}\right)\right) = Q_n f\left(y_{n+1-\ell}\right), n \geqslant n_0, \tag{19}$$

where $Q_n = -q_n \geq 0$ for all $n \geq n_0$.

THEOREM 2. In addition to condition (3), assume that $p > 0$ and

$$\limsup_{n\to\infty} \sum_{s=n-\ell}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} Q_t > \frac{1}{M}. \tag{20}$$

Then every bounded solution of equation (19) oscillates.

PROOF. Without loss of generality, we may assume that $\{y_n\}$ is a bounded and eventually positive solution of equation (19). Then from equation (19), we have $\Delta(a_n \Delta z_n) \geq 0$ for all $n \geq n_0$. By the boundedness of $\{z_n\}$, we have $\Delta z_n < 0$ eventually. If $z_n > 0$ eventually, summing equation (19) twice, then we have

$$-z_n + z_{n-\ell} \geqslant M z_{n-\ell}^{\beta} \sum_{s=n-\ell}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} Q_t. \tag{21}$$

Let $\lim_{n\to\infty} z_n = c$, then $c \geq 0$, we claim that $\lim_{n\to\infty} z_n = 0$. In fact, if $c > 0$, from (21), we have $\limsup_{n\to\infty} \sum\limits_{s=n-l}^{n-1} \frac{1}{a_s} \sum\limits_{t=s}^{n-1} Q_t \leq 0$, which is a contradiction. Hence there exists an integer $N > n_0 + \theta$ such that $z_{n-\ell} < 1$ for all $n \geq N$. Then from (21), we have

$$z_n + z_{n-\ell}\left[M \sum_{s=n-\ell}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} Q_t - 1\right] \leqslant 0,$$

which is a contradiction. Hence $z_n < 0$ eventually. Then, $z_n < -d$ for some $d > 0$. Thus, $-p y_{n-k}^{\alpha} \leqslant -d$ or $y_{n-k}^{\alpha} \geqslant \frac{d}{p} > 0$. From equation (19), we obtain

$$\Delta\left(a_n \Delta z_n\right) \geqslant M\left(\frac{d}{p}\right)^{\frac{\beta}{\alpha}} Q_n. \tag{22}$$

We note from (20) that $\sum\limits_{n=N}^{\infty} \frac{1}{a_n} \sum\limits_{s=N}^{n-1} Q_s = \infty$. Hence (22) implies that $\lim\limits_{n\to\infty} z_n = \infty$, which is a contrary to the boundedness of $\{z_n\}$. This completes the proof of our theorem.

EXAMPLE 3. Consider the difference equation

$$\Delta\left(n \Delta\left(y_n + \frac{4}{3} y_{n-1}^{3}\right)\right) = \frac{2}{3}\left(2n+1\right) y_{n-1}^{\frac{1}{3}}, n \geqslant 2. \tag{23}$$

It is easy to see that all conditions of Theorem 2 are satisfied and hence every bounded solution of equation (23) oscillates. In fact $\{y_n\} = \{(-1)^n\}$ is one such solution of equation (23).

Finally we consider a special case of equation (19) in the form

$$\Delta^2\left(y_n - p y_{n-k}^{\alpha}\right) = Q_n y_{n+1-\ell}^{\beta}, n \geqslant n_0, \tag{24}$$

and study the asymptotic behavior of positive solution of equation (24).

THEOREM 3. With respect to the difference equation (24) assume that $p > 0, \alpha \geq 1, \beta > 0$ and there exists a constant $\lambda > 0$ such that

$$p\,\alpha 2^{\lambda\alpha k + \lambda n(1-\alpha)} \leqslant L < 1, \tag{25}$$

and

$$p\,2^{\lambda\alpha k + \lambda n(1-\alpha)} + \sum_{s=n}^{\infty}(s-n+1)\,Q_s 2^{\lambda(n-\beta(s-\ell+1))} \leqslant 1, \tag{26}$$

holds eventually. Then equation (24) has a positive solution $\{y_n\}$ which tends to zero as $n \to \infty$.

PROOF. If the equality holds in (26) eventually, then $\{y_n\} = \{2^{-\lambda n}\}$ is a positive solution of equation (24) which tends to zero as $n \to \infty$. Therefore, we may assume that there exists an integer $N \geq n_0$ such that $n-k \geq N$ and $n+1-\ell \geq N$ for $n \geq N$.

$$p\,2^{\lambda\alpha k + \lambda n(1-\alpha)} + \sum_{s=N}^{\infty}(s-n+1)\,Q_s 2^{\lambda(n-\beta(s-\ell+1))} < 1,$$

and (26) holds for all $n \geq N$.

Consider the Banach space $\mathcal{B}_{n_0}$ of all bounded real sequences $\{x_n\}$ with norm $\|x_n\| = \sup_{n\geqslant n_0}|x_n|$. Let $\mathcal{S}$ be the subset of $\mathcal{B}_{n_0}$ defined by

$$\mathcal{S} = \left\{x \in \mathcal{B}_{n_0} : 0 \leqslant x_n \leqslant 1, n \geqslant n_0\right\}.$$

It is easy to see that $\mathcal{S}$ is a closed, bounded and convex subset of $\mathcal{B}_{n_0}$. Define a map $\mathcal{T} : \mathcal{S} \to \mathcal{B}_{n_0}$ by

$$\left(\mathcal{T}x\right)_n = \left(\mathcal{T}_1 x\right)_n + \left(\mathcal{T}_2 x\right)_n,$$

where

$$\left(\mathcal{T}_1 x\right)_n = \begin{cases} p\,2^{\lambda\alpha k + \lambda n(1-\alpha)} x_{n-k}^{\alpha}, & n \geqslant N, \\ \left(\mathcal{T}_1 x\right)_N + \exp\left(\varepsilon\left(N-n\right)\right) - 1, & n_0 \leqslant n \leqslant N, \end{cases}$$

$$\left(\mathcal{T}_2 x\right)_n = \begin{cases} \displaystyle\sum_{s=n}^{\infty}(s-n+1)Q_s\, 2^{\lambda(n-\beta(s-\ell+1))} x_{s+1-\ell}^{\beta}, & n \geqslant N \\ \left(\mathcal{T}_2 x\right)_N, & n_0 \leqslant n \leqslant N \end{cases}$$

and $\varepsilon = \frac{\log 2}{N_1 - N}$.

It is easy to see that for every pair $x, z \in \mathcal{S}, \mathcal{T}_1 x + \mathcal{T}_2 z \in \mathcal{S}$. Further $\mathcal{T}_1$ is a contraction and $\mathcal{T}_2$ is completely continuous. Hence by Krasnoselskii's fixed point theorem [2], $\mathcal{T}$ has a fixed point $x \in \mathcal{S}$. That is,

$$x_n = \begin{cases} p\,2^{\lambda\alpha k + \lambda n(1-\alpha)} x_{n-k}^{\alpha} + \sum_{s=n}^{\infty} (s-n+1) Q_s\, 2^{\lambda(n-\beta(s-\ell+1))} x_{s+1-\ell}^{\beta}, n \geqslant N, \\ x_N + \exp\left(\varepsilon\left(N-n\right)\right) - 1, \qquad n_0 \leqslant n \leqslant N. \end{cases}$$

Since $x_n > 0$ for $n_0 \le n \le N$, it follows that $x_n > 0$ for all $n \ge n_0$. Set $y_n = \frac{x_n}{2^{\lambda n}}$. Then,

$$y_n = p\, y_{n-k}^{\alpha} + \sum_{s=n}^{\infty} \left(s-n+1\right) Q_s y_{s-\ell+1}^{\beta}, n \geqslant n_0,$$

which implies that $\{y_n\}$ is a positive solution of equation (24). It is clear that $y_n \to 0$ as $n \to \infty$. This completes the proof.

We conclude this paper with the following example.

EXAMPLE 4. Consider the difference equation

$$\Delta^2 \left(y_n - \frac{1}{8} y_{n-1}^3 \right) = \left(\frac{1}{4} - \frac{49}{2^{2n+6}} \right) y_n, n \geqslant 1. \tag{27}$$

It is easy to see that all conditions of Theorem 3 are satisfied. Therefore, equation (27) has a positive solution $\{y_n\}$ which tends to zero as $n \to \infty$. In fact $\{y_n\} = \{\frac{1}{2^n}\}$ is such a solution of equation (27).

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