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Attractivity In A Nonlinear Delay Difference Equation *

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Abstract

In this paper, we study the global stability and periodic character of the positive solution of the difference equation $x_{n+1} = (a - bx_{n-k})/(A + x_n)$, where $a \geq 0$, $b, A > 0$ and $k \in \{1, 2, \cdots\}$, and initial conditions $x_{-k}, \cdots, x_0$ are arbitrary real numbers. We show that the positive equilibrium of the equation is a global attractor with a basin that depends on certain conditions posed on the coefficients.

1 Introduction

The global asymptotic stability of the rational recursive relation

$$x_{n+1} = (\alpha - \beta x_n)/(\gamma + x_{n-k}),\ n = 0, 1, ..., \tag{1}$$

and

$$x_{n+1} = (\alpha - \beta x_n)/(\gamma - x_{n-k}),\ n = 0, 1, ..., \tag{2}$$

is investigated when α, β, γ are nonnegative real numbers and $k \in \{1, 2, ...\}$, and sufficient conditions for the global attractivity of the positive equilibriums of (1) and (2) are obtained, see [1, 3, 7]. Also, Yan et al. [8] studied the rational recursive equation

$$x_{n+1} = (\alpha + \beta x_n)/(\gamma - x_{n-1}),\ n = 0, 1, ..., \tag{3}$$

where $\alpha \geq 0$, $\beta, \gamma > 0$ are real numbers, and obtained the global attractivity of positive equilibrium of (3).

Other related results can be found in [2, 4, 5, 6].

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Our aim in this paper is study the global attarctivity and periodic character of positive solution of the rational recursive relation

$$x_{n+1} = \frac{a - bx_{n-k}}{A + x_n}, \ n = 0, 1, ..., \tag{4}$$

where $a \geq 0$, $A, b > 0$ are real numbers and the initial values $x_{-k}, ..., x_0$ are arbitrary real numbers. We show that the nonnegative equilibrium point of the equation is a global attractor with a basin that depends on certain conditions of the coefficients.

We first recall some results which will be useful in the sequel.

Let I be some real interval and let F be a continuous function defined on I^{k+1}. Then, for initial conditions $x_{-k}, ..., x_0 \in I$, it is easy to see that the difference equation

$$x_{n+1} = F(x_n, ..., x_{n-k}), \ n = 0, 1, ..., \tag{5}$$

has a unique solution $\{x_n\}$.

A point $\overline{x}$ is called an equilibrium of (5) if $\overline{x} = F(\overline{x}, ..., \overline{x})$. That is, $x_n = \overline{x}$ for $n \geq 0$ is a solution of (5), or equivalently, is fixed point of F.

An interval $J \subset I$ is called an invariant interval of (5) if

$$x_{-k}, ..., x_0 \in J \Rightarrow x_n \in J, \ n > 0.$$

That is, every solution of Eq.(5) with initial conditions in J remains in J.

DEFINITION 1.1. The difference equation (5) is said to be permanent, if there exist numbers P and Q with $0 < P \leq Q < \infty$ such that for any initial conditions $x_{-k}, ..., x_0$ there exists a positive integer N which depends on the initial conditions such that $P \leq x_n \leq Q$ for $n \geq N$.

The linearized equation associated with (5) about the equilibrium $\overline{x}$ is

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial F}{\partial u_i}(\overline{x}, ..., \overline{x}) y_{n-i}, \ n = 0, 1, ... \ . \tag{6}$$

Its characteristic equation is

$$\lambda^{n+1} = \sum_{i=0}^{k} \frac{\partial F}{\partial u_i}(\overline{x}, ..., \overline{x}) \lambda^{n-i}. \tag{7}$$

THEOREM A [5]. Assume that F is a C^1 function and let $\overline{x}$ be an equilibrium of (5). Then the following statements are true:

(a) If all the roots of the equation (7) lie in the open unit disk $| \lambda |< 1$, then the equilibrium $\overline{x}$ of (5) is asymptotically stable.

(b) If at least one root of (5) has absolute value greater than one, then the equilibrium $\overline{x}$ of (5) is unstable.

THEOREM B [2, 5]. Assume that $p, q \in R$ and $k \in \{1, 2, ...\}$. Then

$$|p| + |q| < 1 \tag{8}$$

is a sufficient condition for asymptotic stability of the difference equation

$$x_{n+1} - px_n + qx_{n-k} = 0, \;\; n = 0, 1, \dots \; . \tag{9}$$

Suppose in addition that one of the following two cases holds: (a) k is odd and $q < 0$, or, (b) k is even and $pq < 0$. Then (8) is also a necessary condition for asymptotic stability of (9).

2 The Case $a > 0$

In this section, we discuss the periodic character and global attractivity of positive solutions of (4).

Consider the difference equation (4) with

$$a > 0 \text{ and } A, b > 0. \tag{10}$$

The unique positive equilibrium point of (4) is

$$\overline{x} = \frac{-(A+b) + \sqrt{(A+b)^2 + 4a}}{2}.$$

The linearized equation associated with (4) about the equilibrium $\overline{x}$ is

$$y_{n+1} + \frac{\overline{x}}{A+\overline{x}} y_n + \frac{b}{A+\overline{x}} y_{n-k} = 0, \;\; n = 0, 1, \dots \; .$$

Its characteristic equation is

$$\lambda^{k+1} + \frac{-(A+b) + \sqrt{(A+b)^2 + 4a}}{A - b + \sqrt{(A+b)^2 + 4a}} \lambda^k + \frac{2b}{A - b + \sqrt{(A+b)^2 + 4a}} = 0.$$

By using Theorem B, we have the following result.

LEMMA 2.1. The following statements are true.

(i) Assume that k is even. Then the positive equilibrium $\overline{x}$ of (4) is locally asymptotically stable if and only if $A > b$.

(ii) Assume that k is odd. Then the positive equilibrium $\overline{x}$ of (4) is locally asymptotically stable if $A > b$.

In the following, we always assume that

$$a > 0 \text{ and } A > b > 0. \tag{11}$$

Set $f(u, v) = (a - bv)/(A + u)$, then it is easy to see that $f(u, v)$ satisfies the following properties.

LEMMA 2.2. Assume that (11) holds. Then the following statements are true.

(i) $0 < \overline{x} < \frac{a}{A} < \frac{a}{b}$.

(ii) $f(x, x)$ is a strictly decreasing function in $[0, \infty)$.

(iii) If $(u, v) \in [0, \infty] \times (-\infty, a/b)$, then $f(u, v)$ is a strictly decreasing function in each of its arguments.

THEOREM 2.1. Assume that (11) holds. Then Eq.(4) has no positive solution with prime period two for all $a \in [0, \infty)$.

PROOF. Assume for the sake of contradiction that there exist distinctive positive real numbers ϕ and ψ, such that

$$..., \phi, \psi, \phi, \psi, ...$$

forms a period-two solution of Eq.(4). There are two cases to consider.

Case (a) k is odd.

In this case $x_{n+1} = x_{n-k}$, ϕ and ψ satisfy the system

$$\phi(A + \psi) = a - b\phi \text{ and } \psi(A + \phi) = a - b\psi.$$

Subtracting these equations, we get $(A + b)(\phi + \psi) = 0$. Since $\phi \neq \psi$, then we have $A + b = 0$, this is a contradiction.

Case (b) k is even.

In this case $x_n = x_{n-k}$, ϕ and ψ satisfy the system

$$\phi(A + \psi) = a - b\psi \text{ and } \psi(A + \phi) = a - b\phi.$$

Subtracting these equations, we obtain $(A-b)(\phi-\psi) = 0$, so $\phi = \psi$, which contradicts the hypothesis $\phi \neq \psi$. The proof is complete.

THEOREM 2.2. Assume that (11) holds, and let initial conditions $x_{-k}, ..., x_0 \in [0, a/b]$. Then Eq.(4) is permanent, that is, there exist constants P and Q with $0 < P \leq Q < \infty$ such that $P \leq x_n \leq Q$, for $n \geq 0$.

PROOF. Set $Q = f(0, 0), P = f(Q, Q)$. Then we have

$$0 < P < Q = f(0, 0) = a/A < a/b.$$

By part (iii) of Lemma 2.1, we have

$$\begin{aligned} 0 &= f(a/b, a/b) \leq x_1 = f(x_0, x_{-k}) \leq f(0, 0) = Q, \\ 0 &= f(Q, a/b) \leq x_2 = f(x_1, x_{-k+1}) \leq f(0, 0) = Q, \end{aligned}$$

and

$$0 < P = f(Q, Q) \leq x_2 = f(x_1, x_{-k+1}) \leq f(0, 0) = Q.$$

Hence, the result follows by induction. The proof is complete.

By Theorem 2.2, we know that the interval $[0, a/b]$ is an invariant interval of Eq.(4).

THEOREM 2.3. Assume that (11) holds. Then the positive equilibrium $\overline{x}$ of Eq.(4) is a global attractor with the basin $S = [0, a/b]^{k+1}$.

PROOF. Let $\{x_n\}$ be a solution of Eq.(4) with initial condition $(x_{-k}, \cdots, x_0) \in S$. Then, by part (iii) of Lemma 2.1, for any $u, v \in [0, a/b]$, we have

$$0 < f(u, v) = \frac{a - bv}{A + u} < a/b.$$

Hence, $f \in C([0,a/b]^2,[0,a/b])$ and is strictly decreasing in each of its arguments.

Let $\lambda = \liminf_{n\to\infty} x_n$, $\Lambda = \limsup_{n\to\infty} x_n$, and let $\varepsilon > 0$ such that $\varepsilon < \min\{a/b-\Lambda, \lambda\}$. Then there exist $n_0 \in N$ such that $\lambda - \varepsilon \le x_n \le \Lambda + \varepsilon$. Thus

$$\frac{a-b(\Lambda+\varepsilon)}{A+(\Lambda+\varepsilon)} < x_{n+1} < \frac{a-b(\lambda-\varepsilon)}{A+(\lambda-\varepsilon)},\ n \ge n_0+1.$$

Then we get the following inequality

$$\frac{a-b(\Lambda+\varepsilon)}{A+(\Lambda+\varepsilon)} \le \lambda \le \Lambda \le \frac{a-b(\lambda-\varepsilon)}{A+(\lambda-\varepsilon)}.$$

This inequality yields

$$\frac{a-b\Lambda}{A+\Lambda} \le \lambda \le \Lambda \le \frac{a-b\lambda}{A+\lambda},$$

which implies that $a-b\Lambda-A\lambda \le \lambda\Lambda \le a-b\lambda-A\Lambda$. In view of $A > b$, $\Lambda \le \lambda$. Hence $\lambda = \Lambda = \overline{x}$, that is $\lim_{n\to\infty} x_n = \overline{x}$. This completes the proof.

3 The Case $a = 0$

In the section, we study the asymptotic stability for the difference equation

$$x_{n+1} = \frac{-bx_{n-k}}{A+x_n},\ n = 0,1,..., \tag{12}$$

where

$$b, A \in (0,\infty),\ k \in \{1,2,...\}, \tag{13}$$

and the initial condition $x_{-k},...,x_0$ are arbitrary real numbers.

By putting $x_n = by_n$, Eq.(4) yields

$$y_{n+1} = \frac{-y_{n-k}}{C+y_n},\ n = 0,1,..., \tag{14}$$

where $C = A/b > 0$. Eq.(14) has two equilibria $\overline{y}_1 = 0$ and $\overline{y}_2 = -(C+1)$. The linearized equations of the Eq.(14) about the equilibria $\overline{y}_1$ and $\overline{y}_2$ are

$$Z_{n+1} + \frac{\overline{y}_i}{C+\overline{y}_i}Z_n + \frac{1}{C+\overline{y}_i}Z_{n-k} = 0,\ i = 1,2,\ n = 0,1,\dots\ .$$

For $\overline{y}_2 = -(C+1)$, by Theorem A we can see that it is unstable. For $\overline{y}_1 = 0$, we have

$$Z_{n+1} + \frac{1}{C}Z_{n-k} = 0,\ n = 0,1,\dots\ . \tag{15}$$

The characteristic equation of Eq.(15) is $\lambda^{k+1} + 1/C = 0$. Hence, by Theorem A , we have

(i) if $A > b$, then $\overline{y}_1$ is locally asymptotically stable.
(ii) if $A < b$, then $\overline{y}_1$ is unstable.
(iii) if $A = b$, then linearized stability analysis fails.

In the sequel, we discuss the global attractivity of the zero equilibrium of Eq.(14). So, we assume that $A > b$, namely, $C > 1$.

LEMMA 3.1. Assume that the initial conditions $y_{-k}, ..., y_0 \in [-C+1, C-1]$. Then $y_n \in [-C+1, C-1]$ for $n \geq -1$.

PROOF. Suppose $y_{-k}, \cdots, y_0 \in [-C+1, C-1]$. Then we have

$$-C+1 = \frac{-C+1}{C-C+1} \leq \frac{-C+1}{C+y_0} \leq y_1 = \frac{-y_{-k}}{C+y_0} \leq \frac{C-1}{C+y_0} \leq \frac{C-1}{C-C+1} = C-1,$$

and

$$-C+1 = \frac{-C+1}{C-C+1} \leq y_2 = \frac{-y_{-k+1}}{C+y_1} \leq \frac{C-1}{C+y_1} \leq \frac{C-1}{C-C+1} = C-1.$$

Our result now follows by induction.

By Lemma 3.1, we know that the interval $[-C+1, C-1]$ is an invariant interval of Eq.(14). Also, Lemma 3.1 implies that the following is true.

THEOREM 3.1. The equilibrium $\overline{y}_1 = 0$ of Eq.(14) is a global attractor with a basin $S = [-C+1, C-1]^{k+1}$.

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