Applied Mathematics E-Notes, 4(2004), 40-47 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

Hermitian Nonnegative-Definite And Positive-Definite Solutions Of The Matrix Equation $AXB = C$ *

Xian Zhang †‡

Received 8 May 2003

Abstract

We propose necessary and sufficient conditions for the existence of Hermitian nonnegative-definite or positive-definite solutions to the matrix equation $AXB = C$. A representation of these solutions is also given.

1 Introduction

Many authors have studied various solutions to the matrix equation

$$AXB = C \tag{1}$$

with unknown matrix X. For instance, Kuo [7] has studied the structure of the solutions to (1) by using the tensor product representation of (1). Dai [3] and Khatri and Mitra [6] have obtained different sufficient and necessary conditions for the existence of Hermitian (symmetric) solutions and given the general expressions of all Hermitian (symmetric) solutions to (1). The Hermitian nonnegative-definite solutions to the special case $B = A^*$ of (1) have been studied by Baksalary [2], Dai and Lancaster [4], Groß [5], Khatri and Mitra [6] and also Zhang and Cheng [8]. Furthermore, Khatri and Mitra [6] investigated a necessary and sufficient condition for the existence of Hermitian nonnegative definite solutions to (1), and deduced a representation of all Hermitian nonnegative-definite solutions to (1) if such solutions exist.

The purpose of this paper is to provide a new approach which gives a representation of all Hermitian nonnegative-definite (respectively, positive-definite) solution to (1). The proposed approach is different from that of Khatri and Mitra [6] which lies on the generalized inverses of matrices.

Let $\mathbf{C}^{m\times n}$ be the set of $m \times n$ complex matrices. We denote by $\mathbf{H}_n^>$, $\mathbf{H}_n^\geq$, $\mathbf{U}_n$ and $\mathbf{GL}_n$ the subsets of $\mathbf{C}^{n\times n}$ consisting of Hermitian positive-definite matrices, Hermitian nonnegative-definite matrices, unitary matrices and nonsingular matrices, respectively. For $X \in \mathbf{C}^{m\times n}$, let X^*, X^+ and $\mathcal{R}(X)$ be, respectively, the conjugate transpose, the Moore-Penrose inverse and the column space of X. The notation $\oplus$ denotes the direct

*Mathematics Subject Classifications: 15A06, 15A24.

†Department of Mathematics, Heilongjiang University, Harbin, 150080, P. R. China

‡Presently at School of Mechanical and Manufacturing Engineering, The Queen's University of Belfast, Ashby Building, Stranmillis Road, Belfast, BT9 5AH, U.K.

sum. We denote by I_k and $O_{m\times n}$ the $k\times k$ identity matrix and the $m\times n$ zero matrix, respectively; We also write them as I and O respectively when their dimensions are clear.

Let $A\in\mathbf{C}^{m\times n}$, $B\in\mathbf{C}^{n\times p}$ and $C\in\mathbf{C}^{m\times p}$. Since

$$\mathcal{R}(C)\subseteq\mathcal{R}(A),\ \mathcal{R}(C^*)\subseteq\mathcal{R}(B^*) \tag{2}$$

are necessary conditions for the existence of solutions to (1), in the following we will always assume that (2) is satisfied. Furthermore, we have the following two observations.

NOTE 1. If $A=O$ or $B=O$, then there exists a Hermitian nonnegative-definite (respectively, positive-definite) solution to (1) if and only if $C=O$. In this case, any matrix in $\mathbf{H}_n^{\geq}$ (respectively, $\mathbf{H}_n^{>}$) is a Hermitian nonnegative-definite (respectively, positive-definite) solution to (1).

NOTE 2. If rank$A<m$, then $A=P_A\left[\begin{array}{cc} A_1^T & O\end{array}\right]^T$, where $P_A\in\mathbf{U}_m$ and A_1 is of full-row rank. Let

$$P_A^*C=\left[\begin{array}{cc} C_1^T & C_2^T\end{array}\right]^T, \tag{3}$$

where C_1 and C_2 have appropriate sizes. Combining (2) and (3) yields $C_2=O$. This implies that the pair of matrix equations (1) and $A_1XB=C_1$ have the same solutions. Therefore, we may assume that rank$A=m$. Similarly, we may also assume that rank$B=p$.

Based on the above two notes, in the rest of this paper we assume

$$0<m=\mathrm{rank}A\leq\mathrm{rank}B=p\leq n. \tag{4}$$

Obviously, (2) holds if (4) is satisfied. Therefore, we are now concerned with the following problem.

PROBLEM 1. Given matrices $A\in\mathbf{C}^{m\times n}$, $B\in\mathbf{C}^{n\times p}$ and $C\in\mathbf{C}^{m\times p}$ satisfying (4). Determine a necessary and sufficient condition for the existence of Hermitian nonnegative-definite (respectively, positive-definite) solutions to (1). Furthermore, give a representation of all Hermitian nonnegative-definite (respectively, positive-definite) solutions to (1) when it has such solutions.

Now we introduce two special equivalent decompositions of matrices. For an arbitrary but fixed $E\in\mathbf{C}^{m\times n}$ with rank$E=r$, let

$$E=U_1(\Sigma\oplus O)V_1 \tag{5}$$

be a singular value decomposition of E, where $U_1\in\mathbf{U}_m$, $V_1\in\mathbf{U}_n$ and $\Sigma\in\mathbf{H}_r^{>}$ is diagonal. Denote $V=(\Sigma\oplus I_{n-r})V_1$ and $U=U_1(\Sigma\oplus I_{m-r})$. Then

$$E=U_1(I_r\oplus O)V=U(I_r\oplus O)V_1 \tag{6}$$

are two equivalent decompositions of E. For the sake of convenience, we stipulate in the following that any mentioned equivalent decomposition possesses one of the two forms in (6). Note that the calculations of $E^+=V_1^*(\Sigma^{-1}\oplus O)U_1^*$, U^{-1} and V^{-1} are transformed into calculating Σ^{-1} and matrix product. Therefore, this convenience can ensure good numerical reliability. Furthermore, (5) can be easily obtained by using the function `svd` in MATLAB.

In order to solve problem 1, some preliminary results are given in the next section. We solve Problem 1 in Section 3. An example is presented in Section 4 to illustrate our approach.

2 Preliminary results

We first introduce the following lemma.

LEMMA 1. ([1]) Given matrix $\Psi = \begin{bmatrix} M & Y \\ Y^* & N \end{bmatrix}$ with $M \in \mathbf{C}^{n_1 \times n_1}$, $N \in \mathbf{C}^{n_2 \times n_2}$ and $Y \in \mathbf{C}^{n_1 \times n_2}$. Then

(i) $\Psi \in \mathbf{H}^{\geq}_{n_1+n_2}$ if and only if $M \in \mathbf{H}^{\geq}_{n_1}$, $N - Y^* M^+ Y \in \mathbf{H}^{\geq}_{n_2}$ and $\mathcal{R}(Y) \subseteq \mathcal{R}(M)$.

(ii) $\Psi \in \mathbf{H}^{>}_{n_1+n_2}$ if and only if $M \in \mathbf{H}^{>}_{n_1}$ and $N - Y^* M^{-1} Y \in \mathbf{H}^{>}_{n_2}$.

To solve Problem 1, we need the following algorithm which produces an integer s and three matrices $P \in \mathbf{GL}_m$, $T \in \mathbf{GL}_p$ and $Q \in \mathbf{GL}_n$ based on two matrices $A \in \mathbf{C}^{m \times n}$ and $B \in \mathbf{C}^{n \times p}$ satisfying (4).

ALGORITHM 1.

Step 1 Find $P_1 \in \mathbf{GL}_m$ and $Q_1 \in \mathbf{U}_n$ satisfying the equivalent decomposition $A = P_1 \begin{bmatrix} I_m & O \end{bmatrix} Q_1$.

Step 2 If $m = n$ (this implies from (4) that $m = p = n$ and B is nonsingular), let $s = 0$, $P = P_1$, $Q = Q_1$ and $T = Q_1 B$, and then terminate the algorithm. Otherwise, continue with the following steps.

Step 3 Calculate the matrices $B_1 \in \mathbf{C}^{m \times p}$ and $B_2 \in \mathbf{C}^{(n-m) \times p}$ according to

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = Q_1 B. \tag{7}$$

Step 4 Calculate the integer s according to $s = \mathrm{rank} B_2$.

Step 5 Find $P_2 \in \mathbf{U}_{n-m}$ and $Q_2 \in \mathbf{GL}_p$ satisfying the equivalent decomposition

$$B_2 = P_2 (I_s \oplus O) Q_2. \tag{8}$$

Step 6 Calculate the matrices $B_{11} \in \mathbf{C}^{m \times s}$ and $B_{12} \in \mathbf{C}^{m \times (p-s)}$ according to

$$\begin{bmatrix} B_{11} & B_{12} \end{bmatrix} = B_1 Q_2^{-1}. \tag{9}$$

Step 7 If $s = p$, let $P_3 = I_m$ and Q_3 be the 0×0 empty matrix. Otherwise, find $P_3 \in \mathbf{U}_m$ and $Q_3 \in \mathbf{GL}_{p-s}$ satisfying the equivalent decomposition $B_{12} =$

$P_3 \begin{bmatrix} I_{p-s} \\ O \end{bmatrix} Q_3$. (*Note that: by (4), (7), (8) and (9),* $\mathrm{rank} B_{12} = p - s$ *can be deduced as follows.*)

$$\begin{aligned} \mathrm{rank} B_{12} = \ & \mathrm{rank} \begin{bmatrix} B_{11} & B_{12} \\ I_s & O \\ O & O \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} B_{11} & B_{12} \end{bmatrix} Q_2 \\ P_2 \begin{bmatrix} I_s & O \\ O & O \end{bmatrix} Q_2 \end{bmatrix} - s \\ = \ & \mathrm{rank} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} - s = \mathrm{rank} B - s = p - s. \end{aligned}$$

Step 8 Calculate the matrices $P \in \mathbf{GL}_m$, $Q \in \mathbf{GL}_n$ and $T \in \mathbf{GL}_p$ according to

$$P = P_1 P_3, \ T = \begin{bmatrix} O & Q_3 \\ I_s & O \end{bmatrix} Q_2$$

and

$$Q = \begin{bmatrix} P_3^* & O & O \\ B_{11}^* & I_s & O \\ O & O & I_{n-m-s} \end{bmatrix} \begin{bmatrix} I_m & O \\ O & P_2^* \end{bmatrix} Q_1.$$

To some extent the following lemma reveals the sense of the above algorithm.

LEMMA 2. Given matrices $A \in \mathbf{C}^{m\times n}$ and $B \in \mathbf{C}^{n\times p}$ satisfying (4). Let the integer s and the matrices P, Q and T be obtained by Algorithm 1. Then

$$A = P \begin{bmatrix} I_m & O \end{bmatrix} Q, \ B = Q^* \begin{bmatrix} I_{p-s} & O & O & O \\ O & O_{s\times(m-p+s)} & I_s & O \end{bmatrix}^T T. \tag{10}$$

PROOF. It can be easily completed by following the steps of Algorithm 1.

REMARK 1. It follows from (10) that $\mathrm{rank} \begin{bmatrix} A^* & B \end{bmatrix} = m + s$. This implies that the integer s can be direct obtained from A and B.

3 The Solution to Problem 1

Based on Algorithm 1 and Lemmas 1 and 2 in Section 2, the solution to Problem 1 can be stated as follows.

THEOREM 1. Given matrices $A \in \mathbf{C}^{m\times n}$, $B \in \mathbf{C}^{n\times p}$ and $C \in \mathbf{C}^{m\times p}$ satisfying (4). Furthermore, assume the integer s and the matrices P, Q and T are obtained by Algorithm 1, and let

$$P^{-1} C T^{-1} = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}, \ C_1 \in \mathbf{C}^{(p-s)\times(p-s)}. \tag{11}$$

Then

(i) the matrix equation (1) has at least a Hermitian nonnegative-definite solution if and only if

$$C_1 \in \mathbf{H}^{\geq}_{p-s}, \ \mathcal{R}\left(\begin{bmatrix} C_3^* & C_2 \end{bmatrix}\right) \subseteq \mathcal{R}\left(C_1\right). \tag{12}$$

(ii) when (12) is met, a representation of all Hermitian nonnegative-definite solutions to (1) is given by

$$X = Q^{-1} \begin{bmatrix} C_1 & C_3^* & C_2 & X_{14} \\ C_3 & X_{22} & C_4 & X_{24} \\ C_2^* & C_4^* & X_{33} & X_{34} \\ X_{14}^* & X_{24}^* & X_{34}^* & X_{44} \end{bmatrix} (Q^*)^{-1}, \tag{13}$$

where X_{14}, X_{22}, X_{24}, X_{33}, X_{34} and X_{44} are parameter matrices with appropriate sizes which satisfy

$$\begin{cases} \mathcal{R}(X_{14}) \subseteq \mathcal{R}(C_1) \\ Y_1 \in \mathbf{H}_{m-p+s}^{\geq} \\ \mathcal{R}\left(\begin{bmatrix} C_0 & Y_2 \end{bmatrix}\right) \subseteq \mathcal{R}(Y_1) \\ Y_3 - C_0^* Y_1^+ C_0 \in \mathbf{H}_s^{\geq} \\ \mathcal{R}(Z) \subseteq \mathcal{R}\left(Y_3 - C_0^* Y_1^+ C_0\right) \\ Y_5 - Y_2^* Y_1^+ Y_2 - Z^* \left(Y_3 - C_0^* Y_1^+ C_0\right)^+ Z \in \mathbf{H}_{n-m-s}^{\geq} \end{cases} \tag{14}$$

with

$$\begin{cases} C_0 = C_4 - C_3 C_1^+ C_2 \\ Y_1 = X_{22} - C_3 C_1^+ C_3^* \\ Y_2 = X_{24} - C_3 C_1^+ X_{14} \\ Y_3 = X_{33} - C_2^* C_1^+ C_2 \\ Y_4 = X_{34} - C_2^* C_1^+ X_{14} \\ Y_5 = X_{44} - X_{14}^* C_1^+ X_{14} \\ Z = Y_4 - C_0^* Y_1^+ Y_2 \end{cases}. \tag{15}$$

PROOF. (i) *The "if" part.* Denote

$$X_0 = Q^{-1} \begin{bmatrix} C_1 & C_3^* & C_2 & O \\ C_3 & C_3 C_1^+ C_3^* + C_0 C_0^* & C_4 & O \\ C_2^* & C_4^* & C_2^* C_1^+ C_2 + C_0^* \left(C_0 C_0^*\right)^+ C_0 & O \\ O & O & O & O \end{bmatrix} (Q^*)^{-1}, \tag{16}$$

where C_0 is defined in (15). Firstly, in view of (i) of Lemma 1 and (12), it is easy to verify $X_0 \in \mathbf{H}_n^{\geq}$. Secondly, by (11) and Lemma 2, we can deduce that X_0 defined in (16) is a solution to (1). Because of the above two aspects, X_0 is a Hermitian nonnegative-definite solution to (1).

The "only if" part. Suppose $\tilde{X}$ is a Hermitian nonnegative-definite solution to (1). Then

$$A\tilde{X}B = C. \tag{17}$$

Let

$$Q\tilde{X}Q^* = \begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{12}^* & X_{22} & X_{23} & X_{24} \\ X_{13}^* & X_{23}^* & X_{33} & X_{34} \\ X_{14}^* & X_{24}^* & X_{34}^* & X_{44} \end{bmatrix}, \tag{18}$$

where $X_{11} \in \mathbf{H}_{p-s}^{\geq}$, $X_{22} \in \mathbf{H}_{m-p+s}^{\geq}$ and $X_{33} \in \mathbf{H}_{s}^{\geq}$. Using Lemma 2, (17) and (18) yields

$$C = P \begin{bmatrix} X_{11} & X_{13} \\ X_{12}^* & X_{23} \end{bmatrix} T.$$

This, together with (11), implies

$$Q\tilde{X}Q^* = \begin{bmatrix} C_1 & C_3^* & C_2 & X_{14} \\ C_3 & X_{22} & C_4 & X_{24} \\ C_2^* & C_4^* & X_{33} & X_{34} \\ X_{14}^* & X_{24}^* & X_{34}^* & X_{44} \end{bmatrix}. \tag{19}$$

Combining (19), (i) of Lemma 1 and $\tilde{X} \in \mathbf{H}_n^{\geq}$ deduces (12).

(ii) It follows from (i) and (12) that the matrix equation (1) has at least a Hermitian nonnegative-definite solution. Suppose X is an arbitrary but fixed Hermitian nonnegative-definite solution to (1). Then, by a similar argument to the proof of (19), the matrix X possesses the form (13). Now, to complete the proof, it suffices to show that $X \in \mathbf{H}_n^{\geq}$ is equivalent to (14). Indeed, by (i) of Lemma 1, (12) and (13), we have that $X \in \mathbf{H}_n^{\geq}$ if and only if

$$\mathcal{R}\left(X_{14}\right) \subseteq \mathcal{R}\left(C_1\right), \begin{bmatrix} Y_1 & C_0 & Y_2 \\ C_0^* & Y_3 & Y_4 \\ Y_2^* & Y_4^* & Y_5 \end{bmatrix} \in \mathbf{H}_{n-p+s}^{\geq}, \tag{20}$$

where C_0 and Y_i, $i = 1, \cdots, 5$, are defined in (15). Again applying (i) of Lemma 1 to the second relation in (20), we see that (20) is equivalent to

$$\begin{cases} \mathcal{R}\left(X_{14}\right) \subseteq \mathcal{R}\left(C_1\right) \\ Y_1 \in \mathbf{H}_{m-p+s}^{\geq} \\ \mathcal{R}\left(\begin{bmatrix} C_0 & Y_2 \end{bmatrix}\right) \subseteq \mathcal{R}\left(Y_1\right) \\ \begin{bmatrix} Y_3 - C_0^* Y_1^+ C_0 & Y_4 - C_0^* Y_1^+ Y_2 \\ Y_4^* - Y_2^* Y_1^+ C_0 & Y_5 - Y_2^* Y_1^+ Y_2 \end{bmatrix} \in \mathbf{H}_{n-m}^{\geq} \end{cases}. \tag{21}$$

Similarly, (21) is equivalent to (14). The proof is complete.

Using (ii) of Lemma 1 and an argument similar to the proof of Theorem 1, we can easily deduce the following theorem concerning the positive-definite solutions (i.e., the nonnegative-definite solution with rank n) to (1).

THEOREM 2. Suppose the hypothesis of Theorem 1 is satisfied. Then the matrix equation (1) has at least a Hermitian positive-definite solution if and only if

$$C_1 \in \mathbf{H}_{p-s}^{>}. \tag{22}$$

When this condition is met, a representation of all Hermitian positive-definite solutions to (1) is given by (13), where X_{14}, X_{22}, X_{24}, X_{33}, X_{34} and X_{44} are parameter matrices with appropriate sizes which satisfy

$$\begin{cases} Y_1 \in \mathbf{H}_{m-p+s}^{>} \\ Y_3 - C_0^* Y_1^+ C_0 \in \mathbf{H}_s^{>} \\ Y_5 - Y_2^* Y_1^+ Y_2 - Z^* \left(Y_3 - C_0^* Y_1^+ C_0\right)^+ Z \in \mathbf{H}_{n-m-s}^{>} \end{cases}$$

with (15).

4 An Example

Consider a matrix equation in the form of (1) with the parameter matrices:

$$A = \begin{bmatrix} 0 & 5 & 9 & 3 \\ 2 & 2 & 1 & 3 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 2 & 1 \\ 6 & 2 & -2 \\ 3 & 1 & -1 \end{bmatrix}, \ C = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 2 \end{bmatrix}.$$

Obviously, $m = 2$, $n = 4$, $p = 3$ and (4) is satisfied. Following the steps in Algorithm 1, it is easy to show that

$$s = 2, \ Q = \begin{bmatrix} 0.0964 & 0.4979 & 0.7709 & 0.3854 \\ 0.5903 & 0.1559 & -0.4868 & 0.6247 \\ -0.1667 & 0.3152 & -1.3266 & -0.6633 \\ -1.4985 & -1.4020 & -1.7901 & -0.8950 \end{bmatrix},$$

$$P = \begin{bmatrix} 10.5837 & -1.7278 \\ 3.1157 & 2.8797 \end{bmatrix}, \ T = \begin{bmatrix} 0.1061 & -0.0496 & 0.2410 \\ -4.2002 & 0.3163 & 1.9140 \\ -0.1933 & -1.3730 & -0.1974 \end{bmatrix}.$$

Therefore, $n = m + s$, $C_1 = 1.1078$, $C_2 = \begin{bmatrix} -0.0034 & 0.0177 \end{bmatrix}$, $C_3 = 1.6457$ and $C_4 = \begin{bmatrix} -0.0023 & -0.1233 \end{bmatrix}$. Using Theorem 1, a representation of all Hermitian nonnegative-definite solutions to the matrix equation is given by

$$X = Q^{-1} \begin{bmatrix} 1.1078 & 1.6457 & -0.0034 & 0.0177 \\ 1.6457 & a & -0.0023 & -0.1233 \\ -0.0034 & -0.0023 & b & c \\ 0.0177 & -0.1233 & \bar{c} & d \end{bmatrix} (Q^*)^{-1}, \tag{23}$$

where $a, b, c, d \in \mathbf{C}$ are parameters satisfying $a > 2.4448$ and $\begin{bmatrix} b_1 & c_1 \\ \bar{c}_1 & d_1 \end{bmatrix} \in \mathbf{H}_2^{\geq}$ with

$$\begin{cases} b_1 = b - 10^{-3} \times 0.0106 - \frac{0.0000}{a - 2.4448} \\ c_1 = c + 10^{-3} \times 0.0546 + \frac{0.0004}{a - 2.4448} \\ d_1 = d - 10^{-3} \times 0.2824 - \frac{0.0004}{a - 2.4448} \end{cases}.$$

Furthermore, using Theorem 2, a representation of all Hermitian positive-definite solutions to the matrix equation is given by (23), where $a, b, c, d \in \mathbf{C}$ are parameters satisfying $a > 2.4448$, $b_1 > 0$ and $d_1 > \bar{c}_1 b_1^+ c_1$.

Acknowledgment. This work was supported in part by the NSF of Heilongjiang Province under Grant No. A01-07 and the Fund of Heilongjiang Education Committee for Overseas Scholars. I also wish to thank the referee and the editors for their helpful comments.

References

[1] A. Albert, Condition for positive and nonnegative definite in terms of pseudoinverse, SIAM J. Appl. Math., 17(1969), 434–440.

[2] J. K. Baksalary, Nonnegative definite and positive definite solutions to the matrix equation $AXA^* = B$, Linear and Multilinear Algebra, 16(1984), 133–139.

[3] H. Dai, On the symmetric solutions of linear matrix equations, Linear Algebra Appl., 131(1990), 1–7.

[4] H. Dai and P. Lancaster, Linear matrix equations from an inverse problem of vibration theory, Linear Algebra Appl., 246(1996), 31–47.

[5] J. Groß, Nonnegative-define and positive-definite solutions to the matrix equation $AXA^* = B$— revisited, Linear Algebra Appl., 321(2000), 123–129.

[6] C. G. Khatri and S. K. Mitra, Hermitian and nonnegative definite solutions of linear matrix equations, SIAM J. Appl. Math., 31(1976), 579–585.

[7] Y. E. Kuo, The use of tensor products for solving matrix equations, Matrix Tensor Quart., 26(4)(1975/76), 148–151.

[8] X. Zhang and M. Y. Cheng, The rank-constrained Hermitian nonnegative-definite and positive-definite solutions to the matrix equation $AXA^* = B$, Linear Algebra Appl., 370(2003), 163–174.