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Existence Of Nonoscillatory Solutions Of Some Higher Order Difference Equations *

Małgorzata Migda†

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Abstract

We consider the linear difference equation

$$\Delta^m x_n + \delta a_n x_{n+1} = 0$$

where $m \geq 2$, $\delta = \pm 1$ and $\{a_n\}$ is a positive real sequence. We study the existence of its nonoscillatory solutions.

1 Introduction

In this paper we consider the m-th order linear difference equation

$$\Delta^m x_n + \delta a_n x_{n+1} = 0, \quad n \in N \tag{1}$$

where $m \geq 2$, $\delta = \pm 1$ and $\{a_n\}$ is a positive real sequence. For any function $x : N \to R$ we define the forward difference operator Δ as usual:

$$\Delta^0 x_n = x_n,\ \Delta x_n = x_{n+1} - x_n, \quad \Delta^k x_n = \Delta(\Delta^{k-1} x_n) \text{ for } k \geq 1.$$

Here by N we denote the set of positive integers and by R the set of the real numbers. For all $k \in N$ we use the usual factorial notation $n^{(k)} = n(n-1)...(n-k+1)$ with $n^{(0)} = 1$. Moreover, $\sum_{j=k}^{k-1} a_j = 0$.

A nontrivial solution $\{x_n\}$ of (1) is said to be oscillatory if for every $n_0 \in N$ there exists an $n \geq n_0$ such that $x_n x_{n+1} \leq 0$. Otherwise it is called nonoscillatory. A sequence $\{x_n\}$ is termed quickly oscillatory if and only if $x_n = (-1)^n u_n$, where $\{u_n\}$ is a sequence of positive numbers or negative numbers. Since (1) is linear, we can assume without loss of generality that all nonoscillatory solutions of (1) are eventually positive.

Our main interest in this paper is to study the existence of nonoscillatory solutions of (1). Similar problems for linear difference equation of third and fourth order were considered in [3], [7], [9]-[12]. The linear difference equations of second order have been investigated by a number of authors (see e.g. [2], [4]-[6], [8]). Several sufficient conditions for the oscillation of all solutions of (1) where m is odd and $\delta > 0$ can be found in [14].

For the sake of convenience, we will denote (1) by (1_-) if $\delta = -1$, and by (1_+) if $\delta = +1$.

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†Institute of Mathematics, Poznań University of Technology, Piotrowo 3A, 60-965 Poznań, Poland

2 Main results

We begin by classifying the nonoscillatory solutions of (1) on the basis of discrete analogue of Kiquradze's Lemma (see, for example [1]).

LEMMA 1. Let $\{x_n\}$ be a sequence of real numbers and let x_n and Δx_n be of constant sign with $\Delta^m x_n$ not eventually identically zero. If eventually

$$\delta x_n \Delta^m x_n < 0,$$

then there exists integers $l \in \{0, 1, 2, \ldots, m\}$ and $N' > 0$ such that $(-1)^{m-l-1}\delta = 1$ and

$$\begin{array}{c} x_n \Delta^j x_n > 0 \text{ for } j = 0, 1, \ldots, l \\ (-1)^{j-l} x_n \Delta^j x_n > 0 \text{ for } j = l+1, \ldots, m \end{array} \tag{2}$$

for $n \geq N'$.

A sequence $\{x_n\}$ satisfying (2) is called a sequence of degree l. Let $\mathcal{N}_l$ denote the set of solutions of degree l of (1). If we denote by $\mathcal{N}$ the set of all nonoscillatory solutions of (1) then by Lemma 1 we have

$$\begin{array}{l} \mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2 \cup \cdots \cup \mathcal{N}_{m-1} \text{ for } \delta = 1 \text{ and } m \text{ odd;} \\ \mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_3 \cup \cdots \cup \mathcal{N}_{m-1} \text{ for } \delta = 1 \text{ and } m \text{ even;} \\ \mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_3 \cup \cdots \cup \mathcal{N}_{m} \text{ for } \delta = -1 \text{ and } m \text{ odd;} \\ \mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2 \cup \cdots \cup \mathcal{N}_{m} \text{ for } \delta = -1 \text{ and } m \text{ even.} \end{array}$$

Our first result shows that for (1_-), we have $\mathcal{N}_m \neq 0$. We need the following lemma which is proved in [12].

LEMMA 2. Let $\{x_n\}$ be a sequence. Suppose $k \geq 1$ and that $\Delta^k x_n > 0$ and $\Delta^{k+1} x_n > 0$ for all $n \geq M$, then

$$\lim_{n\to\infty} \Delta^{k-1} x_n = \lim_{n\to\infty} \Delta^{k-2} x_n = \ldots = \lim_{n\to\infty} x_n = \infty.$$

We will use the initial values to construct nonoscillatory, unbounded solutions of (1).

THEOREM 1. There exists a nontrivial solution $\{x_n\}$ of (1_-) satisfying

$$\Delta^i x_n > 0 \text{ for all } n \in N \text{ and } i = 0, 1, \ldots, m-1, \tag{3}$$

in addition

$$\lim_{n\to\infty} \Delta^i x_n = \infty, \ i = 0, 1, \ldots, m-2. \tag{4}$$

PROOF. Let $\{x_n\}$ be a nontrivial solution of (1_-) satisfying the initial conditions $\Delta^i x_1 > 0$ for $i = 0, 1, \ldots, m-1$. Suppose $\Delta^i x_k > 0$ for some positive integer k and $i = 0, 1, \ldots, m-1$. From the identities

$$\begin{aligned} x_{k+1} &= \Delta x_k + x_k \\ \Delta x_{k+1} &= \Delta^2 x_k + \Delta x_k \\ &\vdots \\ \Delta^{m-2} x_{k+1} &= \Delta^{m-1} x_k + \Delta^{m-2} x_k \end{aligned}$$

we have $\Delta^i x_{k+1} > 0$ for $i = 0, 1, ..., m-2$. Since $\Delta^{m-1} x_{k+1} = \Delta^m x_k + \Delta^{m-1} x_k$, from (1), we get

$$\Delta^{m-1} x_{k+1} = a_k x_{k+1} + \Delta^{m-1} x_k > 0$$

and (3) follows by mathematical induction. Now, from (3) we have $x_n > 0$ for all $n \in N$ and so $\Delta^m x_n = a_n x_{n+1} > 0$ for all $n \in N$. Therefore use of Lemma 1 completes the proof.

REMARK 1. Theorem 1 can be extended to nonlinear equation of the form

$$\Delta^m x_n = a_n f(x_{n+k})$$

where $f : R \to R$ is a continuous function such that $xf(x) > 0$ for $x \neq 0$ and $k = 0, 1, \ldots, m-1$.

To prove our next theorem we need the following lemmas.

LEMMA 3. (see [1, Corollary 1.7.13]). Let $\{x_n\}$ be a bounded sequence and $x_n > 0$ with $\Delta^k x_n \leq 0$ and not identically zero. Then

$$\lim_{n\to\infty} \Delta^i x_n = 0 \text{ for } i = 1, 2, ..., k-1.$$

LEMMA 4. Let m be an odd [even] number. If $\{x_n\}$ is a nontrivial solution of (1_+) $[(1_-)]$ satisfying

$$(-1)^i \Delta^i x_r \geq 0 \text{ for } i = 0, 1, ..., m-2 \text{ and } \Delta^{m-1} x_r > 0 \quad [< 0] \tag{5}$$

for some $r > 1$ then

$$(-1)^i \Delta^i x_n > 0 \text{ for all } n = 1, 2, ..., r-1 \text{ and } i = 0, 1, ..., m-1. \tag{6}$$

PROOF. Let m be an odd number. We show the lemma true for $n = r-1$. From (5) we have $\Delta^m x_{r-1} = -a_{r-1} x_r \leq 0$. Hence $\Delta^{m-1} x_r \leq \Delta^{m-1} x_{r-1}$ and we get $\Delta^{m-1} x_{r-1} > 0$. Similarly, $\Delta^{m-1} x_{r-1} > 0$ implies $\Delta^{m-2} x_{r-1} < 0$, which implies $\Delta^{m-3} x_{r-1} > 0$ and step by step we get (6) for $n = r-1$. Repeating this process for each $n = r-2, r-3, ..., 1$ proves the lemma. For even m the proof is similar.

The next theorem shows that for (1_+) with m odd and for (1_-) with even m we have $\mathcal{N}_0 \neq 0$.

THEOREM 2. Let m be an odd [even] number. Then there exists a solution $\{x_n\}$ of (1_+) $[(1_-)]$ such that

$$(-1)^i \Delta^i x_n > 0 \text{ for all } n \in N, \quad i = 0, 1, 2, \ldots, m-1. \tag{7}$$

In addition

$$\lim_{n\to\infty} \Delta^i x_n = 0 \text{ for } i = 1, 2, ..., m-1, [m-2]. \tag{8}$$

PROOF. We shall apply a construction similar to that given in [11]. Suppose that $x_1, x_2, ..., x_m$ is a fundamental system of solutions of (1). For every integer k we define a sequence $\tilde{x_k}$ as follows

$$\tilde{x_k}(n) = C_{1,k}x_1(n) + C_{2,k}x_2(n) + ... + C_{m,k}x_m(n), \quad n \in N \tag{9}$$

where the numbers $C_{1,k}, C_{2,k}, ..., C_{m,k}$ are chosen in such a way that

$$\tilde{x_k}(k) = \tilde{x_k}(k+1) = ... = \tilde{x_k}(k+m-2) = 0 \tag{10}$$

and $\sum\limits_{j=1}^{m} C_{j,k}^2 = 1$. Let $(-1)^{m-1}\tilde{x_k}(k+m-1) > 0$. Since $x_1, x_2, ..., x_m$ are linearly independent and $\tilde{x_k}(k+m-1)$ can be arbitrarily chosen, the numbers $C_{1,k}, C_{2,k}, ..., C_{m,k}$ do exist and $\tilde{x_k}(n)$ are nontrivial solutions of (1). Note that, by (10), for every k we have

$$\tilde{x_k}(k) = \Delta\tilde{x_k}(k) = ... = \Delta^{m-2}\tilde{x_k}(k) = 0, \quad (-1)^{m-1}\tilde{x_k}(k) > 0.$$

Therefore, from Lemma 3 we get

$$(-1)^i \Delta^i \tilde{x_k}(n) > 0 \text{ for all } n = 1, 2, ..., k-1; \quad i = 0, 1, ..., m-1. \tag{11}$$

Let us denote $\tilde{C_k} = (C_{1,k}, C_{2,k}, ..., C_{m,k})$. Then $\|\tilde{C_k}\| = 1$ for each $k \in N$. The unit ball is compact in R^n so $(\tilde{C_k})$ has a convergent subsequence $(\tilde{C_{k_i}})$ such that $\tilde{C_{k_i}} \to \tilde{C} = (C_1, C_2, ..., C_m)$ where $\sum\limits_{j=1}^{m} C_j^2 = 1$. Hence, by (9),

$$\lim_{k_i \to \infty} \tilde{x_{k_i}}(n) = C_1x_1(n) + C_2x_2(n) + ... + C_mx_m(n) = x(n)$$

defines a nontrivial solution (x_n) of (1). The inequality (11) implies that

$$(-1)^i \Delta^i x(n) \geq 0 \text{ for all } n \in N \text{ and } i = 0, 1, ..., m-1. \tag{12}$$

Let m be an odd number. If $x_{n_0} = 0$ for some $n = n_0$, then since $\{x_n\}$ is nonincreasing, $x_n = 0$ for all $n \geq n_0$ which contradicts the fact that $\{x_n\}$ is nontrivial. Thus $x_n > 0$ for each $n \geq 1$. Then $\Delta^m x_n = -a_n x_{n+1} < 0$ for all $n \geq 1$. If $\Delta^{m-1}x_{n_0} = 0$ for some $n = n_0$, then since $\{\Delta^{m-1}x_n\}$ is decreasing and nonnegative sequence, $\Delta^{m-1}x_n = 0$ for all $n \geq n_0$. This contradicts the fact that $\{\Delta^{m-1}x_n\}$ is strictly decreasing. Hence $\Delta^{m-1}x_n > 0$ for all $n \geq 1$. In a similar way $\Delta^{m-1}x_n > 0$ implies $\Delta^{m-2}x_n < 0$. Repeating this process we see that (12) holds with strict inequality for all $n \geq 1$.

From (7) we have $x_n > 0$ and $\{x_n\}$ is decreasing for all $n \in N$. Hence $\{x_n\}$ is bounded. By Lemma 2 we get (8). This completes the proof of the theorem.

EXAMPLE 1. The equation

$$\Delta^m x_n + (-1)^{m-1} \frac{m!}{n(n+2)^2(n+3)...(n+m)} x_{n+1} = 0, \quad n \in N$$

satisfies hypotheses of Theorem 2 and has a solution $x_n = 1 + \frac{1}{n}$. An easy calculation shows that $\Delta^i x_n = (-1)^i \frac{i!}{n(n+1)(n+2)\ldots(n+i)}$ and (7), (8) are satisfied.

REMARK 2. Theorem 2 is not true for the equation

$$\Delta^m x_n + a_n x_n = 0, \quad n \in N.$$

It can be showed by the following example.

EXAMPLE 2. Consider the equation

$$\Delta^3 x_n + 8x_n = 0, \quad n \in N.$$

The general solution of this equation is

$$x_n = C_1(-1)^n + C_2 \sin\left(n \arctan\frac{\sqrt{3}}{2}\right) + C_3 \cos\left(n \arctan\frac{\sqrt{3}}{2}\right).$$

Therefore every particular solution is oscillatory.

It would be interesting to know when the solution constructed in the proof of Theorem 2 converge to zero.

THEOREM 3. Let m be an odd [even] number. If $\sum\limits_{n=1}^{\infty} n^{m-1} a_n = \infty$, then every solution of degree $l = 0$ of (1_+) $[(1_-)]$ approaches zero as $n \to \infty$.

PROOF. Assume m is odd. Let $\{x_n\}$ be a solution of (1_+) of degree $l = 0$. By summing (1) from k to $n-1$ we get

$$\Delta^{m-1} x_n - \Delta^{m-1} x_k = -\sum_{j=k}^{n-1} a_j x_{j+1},\ n \geq k.$$

Letting $n \to \infty$ and using (8) we obtain

$$\Delta^{m-1} x_k = \sum_{j=k}^{\infty} a_j x_{j+1}.$$

The summation the above equality over k yields

$$\Delta^{m-2} x_k - \Delta^{m-2} x_s = \sum_{k=s}^{n-1} \sum_{j=k}^{\infty} a_j x_{j+1}$$

and by (8) we get

$$\Delta^{m-2} x_s = \sum_{k=s}^{\infty} \sum_{j=k}^{\infty} a_j x_{j+1} = -\sum_{j=s}^{\infty} (j+1-s) a_j x_{j+1}.$$

Repeating the reasoning (m-times) we obtain

$$x_r = x_n + \sum_{s=r}^{n-1} \sum_{j=s}^{\infty} \frac{(j+m-2-s)^{(m-2)}}{(m-2)!} a_j x_{j+1}. \tag{13}$$

Since $x_n > 0$ and $\Delta x_n < 0$ then there exists a finite limit $g = \lim_{n\to\infty} x_n \geq 0$. Assume $g > 0$, then from (13) we have

$$x_r \geq g + \frac{(-1)^{m-1}}{(m-1)!} g \sum_{j=r}^{\infty} (j+m-1-r)^{(m-1)} a_j. \tag{14}$$

One can observe that for $j \geq r$ holds $(j+m-1-r)^{(m-1)} \geq (j-r)^{m-1} \geq 0$. Therefore we get

$$\sum_{j=r}^{\infty} (j+m-1-r)^{(m-1)} a_j \geq \sum_{j=r}^{\infty} (j-r)^{m-1} a_j > \frac{1}{K} \sum_{j=r}^{\infty} j^{m-1} a_j$$

for some $K > 0$. Hence, by (14)

$$x_r \geq g + \frac{1}{(m-1)!} \frac{g}{K} \sum_{j=r}^{\infty} j^{m-1} a_j = g + \infty,$$

which is not possible. Thus $\lim_{n\to\infty} x_n = 0$. For even m the proof is similar. This completes the proof of the theorem.

From Theorem 1 and Theorem 2 we get following.

COROLLARY 1. If $m = 2$ then (1_-) does not have oscillatory solutions.

THEOREM 4. Assume m is an odd [even] number. Then (1_+) [(1_-)] cannot have a quickly oscillatory solution.

PROOF. Let m be an odd number and let $z_n > 0$ for all $n \in N$. Suppose that $x_n = (-1)^n z_n$ is a solution of (1_+). Then we have

$$\Delta x_n = (-1)^{n+1}(z_{n+1} + z_n)$$
$$\Delta^2 x_n = (-1)^{n+2}(z_{n+2} + 2z_{n+1} + z_n)$$

and one can see that

$$\Delta^m x_n = (-1)^{n+m} \sum_{k=0}^{m} \binom{m}{k} z_{n+k}.$$

Therefore equation (1_+) can be written in the form

$$(-1)^m \sum_{k=0}^{m} \binom{m}{k} z_{n+k} = a_n z_{n+1}.$$

where

$$(-1)^m \sum_{k=0}^{m} \binom{m}{k} z_{n+k} < 0 \text{ and } a_n z_{n+1} > 0$$

This contradiction proves our Theorem for m odd. For even m the proof is similar.

References

[1] R. P. Agarwal, Difference Equations and Inequalities, Marcel Dekker, New York, 1992.

[2] S. Chen and L.M. Erbe, Riccati techniques and discrete oscillations, J. Math. Anal. Appl., 142(1989), 468–487.

[3] S. S. Cheng, On a class of fourth order linear recurrence equations, Internat. J. Math. Math. Sci., 7(1984), 131–149.

[4] S. S. Cheng, Sturmian comparison theorems for three term recurrence equations, J. Math. Anal. Appl., 111(1985), 465–474.

[5] S. S. Cheng, W. T. Patula , Bounded and zero convergent solutions of second-order difference equations, J. Math. Anal. Appl., 141(1989), 463–483.

[6] S. S. Cheng , T. C. Yan and H. J. Li , Oscillation criteria for second order difference equation, Funkcialaj Ekvacioj, 34(1991), 223–239.

[7] J. W. Hooker, W. T. Patula, Growth and oscillation properties of solutions of a fourth order linear difference equation, J. Austral. Math. Soc. Ser. B. 26(1985), 310–328.

[8] M. K. Kwong, J. W. Hooker and W. T. Patula, Riccati Type Transformations for second-order difference equations, J. Math. Anal. Appl., 107(1985), 182–196.

[9] J. Popenda and E. Schmeidel, Nonoscillatory solutions of third order difference equations, Port. Math., 49(1992), 233–239.

[10] B. Smith, Oscillatory and asymptotic behavior in certain third order difference equations, Rocky Mt. J. Math., 17(1987), 597–606.

[11] B. Smith, Oscillation and nonoscillation theorems for third order quasi-adjoint difference equations, Port. Math., 45(1988), 229–243.

[12] B. Smith and W. E. Taylor Jr., Oscillation properties of fourth order linear difference equations, Tamkang J. Math., 18(1987), 89–95.

[13] B. Szmanda, Note on the oscillation of certain difference equations, Glasnik Math., 31(1996), 115–121.

[14] Y. Zhou, Oscillations of higher-order linear difference equations, Comp. Math. Appl. 42(2001), 323–331.