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On Some Types Of Continuous Fuzzy Functions*

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Abstract

In this paper, by using operations, some characterizations and some properties of fuzzy continuous functions and its weaker and stronger forms including fuzzy weakly continuous, fuzzy θ-continuous, fuzzy strongly θ-continuous, fuzzy almost strongly θ-continuous, fuzzy weakly θ-continuous, fuzzy almost continuous, fuzzy super continuous, fuzzy δ-continuous, are presented.

1 Introduction

Several types of fuzzy continuous functions and its weaker and stronger forms occur in the literature. The aim of this paper is to give some characterizations and some properties of fuzzy continuous functions and its weaker and stronger forms including fuzzy weakly continuous, fuzzy θ-continuous, fuzzy strongly θ-continuous, fuzzy almost strongly θ-continuous, fuzzy weakly θ-continuous, fuzzy almost continuous, fuzzy super continuous, fuzzy δ-continuous.

The class of fuzzy sets on a universe X will be denoted by I^X and fuzzy sets on X will be denoted by Greek letters as μ, ρ, η, etc. A family τ of fuzzy sets in X is called a fuzzy topology for X iff (1) $\emptyset$, $X \in \tau$, (2) $\mu \wedge \rho \in \tau$ whenever $\mu, \rho \in \tau$ and (3) $\bigcup\{\mu_\alpha : \alpha \in I\} \in \tau$ whenever each $\mu_\alpha \in \tau$ $(\alpha \in I)$. Moreover, the pair (X, τ) is called a fuzzy topological space. Every member of τ is called an open fuzzy set [8].

A fuzzy set in X is called a fuzzy point iff it takes the value 0 for all $y \in X$ except one, say, $x \in X$. If its value at x is α $(0 < \alpha \leq 1)$ we denote this fuzzy point by x_α, where the point x is called its support [8]. For any fuzzy point x_ε and any fuzzy set μ, we write $x_\varepsilon \in \mu$ iff $\varepsilon \leq \mu(x)$.

Let $f : X \to Y$ be a fuzzy function from a fuzzy topological space (X, τ) to a fuzzy topological space (Y, υ). The function f is called fuzzy continuous iff for each $x_\varepsilon \in X$ and each fuzzy open set ρ containing $f(x_\varepsilon)$, there exists a fuzzy open set μ containing x_ε such that $f(\mu) \leq \rho$ [9].

By $int(\mu)$ and $cl(\mu)$, we mean the interior of μ and the closure of μ.

Let $f : X \to Y$ be a fuzzy function from a fuzzy topological space X to a fuzzy topological space Y. Then the function $g : X \to X \times Y$ defined by $g(x_\varepsilon) = (x_\varepsilon, f(x_\varepsilon))$ is called the graph function of f and it will be denoted by grf [1].

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2 Some Types of Continuous Fuzzy Functions

There are some useful definitions.

DEFINITION 1. Let (X, τ) be a fuzzy topological space. A mapping $\alpha : I^X \to I^X$ is called an operation on I^X if for each $\mu \in I^X \backslash \{\emptyset\}$, $int(\mu) \leq \mu^\alpha$ and $\emptyset^\alpha = \emptyset$ where μ^α denotes the value of α in μ [5].

DEFINITION 2. Let (X, τ) be a fuzzy topological space and let α be an operation on I^X. α is called a monotonous operation if for each μ, $\rho \in I^X$ and $\mu \leq \rho$, then $\mu^\alpha \leq \rho^\alpha$ [5].

DEFINITION 3. Let (X, τ) and (Y, υ) be fuzzy topological spaces and let φ, ψ be operations on I^X, I^Y respectively. A function f from (X, τ) into (Y, υ) is called fuzzy $\varphi\psi$-continuous if for each $x_\varepsilon \in X$ and each fuzzy open set ρ containing $f(x_\varepsilon)$, there exists a fuzzy open set μ containing x_ε such that $f(\mu^\varphi) \leq \rho^\psi$.

The following table provides us a list of fuzzy $\varphi\psi$-continuous function with operations φ and ψ.

Operations	Fuzzy $\varphi\psi$-continuity
1. $\varphi = \psi = i$	f. continuity [9]
2. $\varphi = i,\ \psi = cl$	f. weakly continuity [1]
3. $\varphi = \psi = cl$	f. θ-continuity [3, 7]
4. $\varphi = cl,\ \psi = i$	f. strongly θ-continuity [4, 6]
5. $\varphi = cl,\ \psi = int \circ cl$	f. almost strongly θ-continuity [7]
6. $\varphi = int \circ cl,\ \psi = cl$	f. weakly θ-continuity [7]
7. $\varphi = i,\ \psi = int \circ cl$	f. almost continuity [1]
8. $\varphi = int \circ cl,\ \psi = i$	f. super continuity [6]
9. $\varphi = \psi = int \circ cl$	f. δ-continuity [2, 10]

DEFINITION 4. Let (X, τ) be a fuzzy topological space and let $(x^\alpha_{\varepsilon_\alpha})$ be a net in X. $(x^\alpha_{\varepsilon_\alpha})$ is called φ-converges to x_ε if for each open set μ containing x_ε, there exists an index $\alpha_0 \in J$ such that $x^\alpha_{\varepsilon_\alpha} \in \mu^\varphi$ for all $\alpha \geq \alpha_0$. We will denote by $x^\alpha_{\varepsilon_\alpha} \xrightarrow{\varphi} x_\varepsilon$.

DEFINITION 5. Suppose that (X, τ) is a fuzzy topological space and φ is an operation on I^X. Let (X, τ) be a fuzzy topological space and let $(x^\alpha_{\varepsilon_\alpha})$ be a net in X. Then $(x^\alpha_{\varepsilon_\alpha})$ is called φ-eventually in the fuzzy set $\mu \leq X$ if there exists an index $\alpha_0 \in J$ such that $x^\alpha_{\varepsilon_\alpha} \in \mu^\varphi$ for all $\alpha \geq \alpha_0$.

The following theorem gives us the characterizations of fuzzy $\varphi\psi$-continuous function.

THEOREM 1. Suppose that (X, τ) and (Y, υ) are fuzzy topological spaces and φ, ψ are operations on I^X, I^Y respectively. For a function $f : (X, \tau) \to (Y, \upsilon)$, the following statements are equivalent.

i-) f is fuzzy $\varphi\psi$-continuous.

ii-) For each $x_\varepsilon \in X$ and for each net $(x^\alpha_{\varepsilon_\alpha})$ in X, if $x^\alpha_{\varepsilon_\alpha} \xrightarrow{\varphi} x_\varepsilon$, then $f(x^\alpha_{\varepsilon_\alpha}) \xrightarrow{\psi} f(x_\varepsilon)$.

iii-) For each $x_\varepsilon \in X$ and for each net $(x^\alpha_{\varepsilon_\alpha})$ in X, if $x^\alpha_{\varepsilon_\alpha} \xrightarrow{\varphi} x_\varepsilon$, then the net $f(x^\alpha_{\varepsilon_\alpha})$ is ψ-eventually in ρ for all fuzzy open set ρ containing $f(x_\varepsilon)$.

PROOF. (i)⇒(ii). Let $f(x_\varepsilon) \in \rho \in \upsilon$. Since f is fuzzy $\varphi\psi$-continuous, there exists an open set μ containing x_ε such that $f(\mu^\varphi) \le \rho^\psi$. Since $x^\alpha_{\varepsilon_\alpha} \xrightarrow{\varphi} x_\varepsilon$, there exists an index $\alpha_0 \in J$ such that $x^\alpha_{\varepsilon_\alpha} \in \mu^\varphi$ for all $\alpha \ge \alpha_0$. Thus $f(x^\alpha_{\varepsilon_\alpha}) \in f(\mu^\varphi) \le \rho^\psi$ and $f(x^\alpha_{\varepsilon_\alpha}) \in \rho^\psi$ for all $\alpha \ge \alpha_0$. We obtain that $f(x^\alpha_{\varepsilon_\alpha}) \xrightarrow{\psi} f(x_\varepsilon)$.

(ii)⇒(iii). Obvious.

(iii)⇒(i). Suppose that (i) is not true. There would then exist a point x_ε and an open set ρ containing $f(x_\varepsilon)$ such that $\mu^\varphi \nleq f^{-1}(\rho^\psi)$ for each $\mu \in \tau$ where $x_\varepsilon \in \mu$. Let $x_{\varepsilon_\mu} \in \mu^\varphi$ and $x_{\varepsilon_\mu} \notin f^{-1}(\rho^\psi)$ for each $\mu \in \tau$ where $x_\varepsilon \in \mu$. Then for the neighborhood net (x_{ε_μ}), $x_{\varepsilon_\mu} \xrightarrow{\varphi} x_\varepsilon$, but $(f(x_{\varepsilon_\mu}))$ is not ψ-eventually in ρ. This is a contradiction.

EXAMPLE 1. Suppose that (X,τ) and (Y,υ) are fuzzy topological spaces. For a function $f : (X,\tau) \to (Y,\upsilon)$, the following statements are equivalent with operations $\varphi = i$, $\psi = cl$.

i-) f is fuzzy weakly continuous.

ii-) For each $x_\varepsilon \in X$ and for each net $x^\alpha_{\varepsilon_\alpha}$ in X, if $x^\alpha_{\varepsilon_\alpha} \to x_\varepsilon$, then for each open set μ containing $f(x_\varepsilon)$, there exists an index $\alpha_0 \in J$ such that $f(x^\alpha_{\varepsilon_\alpha}) \in cl(\mu)$ for all $\alpha \ge \alpha_0$.

iii-) For each $x_\varepsilon \in X$ and for each net $x^\alpha_{\varepsilon_\alpha}$ in X, if $x^\alpha_{\varepsilon_\alpha} \to x_\varepsilon$, then the net $f(x^\alpha_{\varepsilon_\alpha})$ is eventually in $cl(\rho)$ for all fuzzy open set ρ containing $f(x_\varepsilon)$.

THEOREM 2. Let $f : X \to Y$ be a fuzzy function from fuzzy topological space (X,τ) to fuzzy topological space (Y,υ) and let φ, ψ be operations on I^X, I^Y, respectively. If f is fuzzy $\varphi\psi$-continuous and φ is a monotonous operation on I^X, then the restriction function $f\mid_\mu : \mu \to Y$ for any fuzzy set $\mu \le X$ is a fuzzy $\varphi\psi$-continuous.

PROOF. Let $x_\varepsilon \in \mu$ and $f\mid_\mu (x_\varepsilon) \in \rho \in \upsilon$. Since f is fuzzy $\varphi\psi$-continuous, it follows that there exists a fuzzy open set η containing x_ε such that $f(\eta^\varphi) \le \rho^\psi$. From here we obtain that $f(\eta^\varphi) \wedge \mu \le \rho^\psi \wedge \mu$. Since $f\mid_\mu (\mu^\varphi) = f(\mu^\varphi) \wedge \mu$, $f\mid_\mu (\mu^\varphi) \le \rho^\psi \wedge \mu \le \rho^\psi$. Since φ is a monotonous operation on I^X, it follows that $f\mid_\mu ((\eta \wedge \mu)^\varphi) \le f\mid_\mu (\eta^\varphi) \le \rho^\psi$. Thus, we obtain that the restriction function $f\mid_\mu$ is fuzzy $\varphi\psi$-continuous.

THEOREM 3. Suppose that (X,τ) and (Y,υ) are fuzzy topological spaces and φ, ψ are operations on I^X, I^Y respectively. Let $f : X \to Y$ be any fuzzy function. If $\{\mu_\alpha : \alpha \in J\}$ is an open cover of X and $f_\alpha = f\mid_{\mu_\alpha}$ is fuzzy $\varphi\psi$-continuous for each $\alpha \in J$, then f is fuzzy $\varphi\psi$-continuous.

PROOF. Let $x_\varepsilon \in X$, $f(x_\varepsilon) \in \rho \in \upsilon$. Since $\{\mu_\alpha : \alpha \in J\}$ is an open cover of X, there exists an index α such that $x_\varepsilon \in \mu_\alpha$ and $f_\alpha(x_\varepsilon) \in \rho \in \upsilon$. Since each f_α is fuzzy $\varphi\psi$-continuous, there exists an open set $x_\varepsilon \in \eta$ such that $(\eta \wedge \mu_\alpha)^\varphi \le f_\alpha^{-1}(\rho^\psi)$ and hence $(\eta \wedge \mu_\alpha)^\varphi \le f_\alpha^{-1}(\rho^\psi) = f^{-1}(\rho^\psi) \wedge \mu_\alpha$. Since $\{\mu_\alpha : \alpha \in J\}$ is an open cover of X, $(\eta \wedge \mu_\alpha)^\varphi \le f^{-1}(\rho^\psi)$ and $x_\varepsilon \in \eta \wedge \mu_\alpha \in \tau$. Thus, f is fuzzy $\varphi\psi$-continuous.

DEFINITION 6. Let $(\prod_{\alpha\in J} X_\alpha, \tau)$ be a product space and let ψ be an operation on $I^{\prod_{\alpha\in J} X_\alpha}$ and on I^{X_α} for all $\alpha \in J$. ψ is called a productive operation if $(\prod_{\alpha\in J} \mu_\alpha)^\psi \le \prod_{\alpha\in J} \mu_\alpha^\psi$ for all $\prod_{\alpha\in J} \mu_\alpha \le \prod_{\alpha\in J} X_\alpha$, $\mu_\alpha \le X_\alpha$.

DEFINITION 7. Let (Y,υ) be a fuzzy topological space and let ψ be an operation on I^Y. (Y,υ) is called fuzzy ψ-hyperconnected space if $\mu^\psi = Y$ for all fuzzy open set $\mu \neq \emptyset$.

EXAMPLE 2. Let $(\prod_{\alpha\in J} X_\alpha, \tau)$ be a product space. Take $\psi = cl$. It is known that the operation $\psi = cl$ is productive, since $cl(\prod_{\alpha\in J}\mu_\alpha) \leq \prod_{\alpha\in J} cl(\mu_\alpha)$ for all $\prod_{\alpha\in J}\mu_\alpha \leq \prod_{\alpha\in J} X_\alpha$, $\mu_\alpha \leq X_\alpha$.

Let the fuzzy topological space (Y, υ) be ψ-hyperconnected. It means that $cl(\mu) = Y$ for all fuzzy open set $\mu \neq \emptyset$.

THEOREM 4. Suppose that (X, τ) and (Y, υ) are fuzzy topological spaces and φ, ψ are operations on I^X, I^Y, respectively. Let $f : X \to Y$ be any fuzzy function and let $grf : (X, \tau) \to (X \times Y, \tau^{\shortmid})$ be graph function of f and ψ be a productive operation on $I^{X\times Y}$. If grf is $\varphi\psi$-continuous, then f is $\varphi\psi$-continuous.

PROOF. Let $x_\varepsilon \in X$ and $f(x_\varepsilon) \in \rho \in \upsilon$. Then $grf(x_\varepsilon) = (x_\varepsilon, f(x_\varepsilon)) \in X \times \rho \in \tau^{\shortmid}$. Since grf is $\varphi\psi$-continuous and ψ is a productive operation, there exists an open set fuzzy set η containing x_ε such that $grf(\eta^\varphi) = \eta^\varphi \times f(\eta^\varphi) \leq (X \times \rho)^\psi \leq X \times \rho^\psi$ and hence $f(\eta^\varphi) \leq \rho^\psi$. Thus, f is $\varphi\psi$-continuous.

THEOREM 5. Suppose that $f : X \to Y$ is a fuzzy function from fuzzy topological space (X, τ) to fuzzy topological space (Y, υ) and φ, ψ are operations on I^X, I^Y, respectively. Let $grf : (X, \tau) \to (X \times Y, \tau^{\shortmid})$ be graph function of f and ψ be a productive operation on $I^{X\times Y}$ and let $X \times Y$ be fuzzy ψ-hyperconnected space. grf is fuzzy $\varphi\psi$-continuous function if and only if f is fuzzy $\varphi\psi$-continuous function.

PROOF. $\Rightarrow$: Obvious from the above theorem.

$\Leftarrow$: Let $x_\varepsilon \in X$ and let $\bigvee_{j\in J}(\mu_j \times \eta_j) \leq X \times Y$ be a fuzzy open set such that $x_\varepsilon \in (grf)^{-1}(\bigvee_{j\in J}(\mu_j \times \eta_j))$. Since $X \times Y$ is fuzzy ψ-hyperconnected space, it follows that $(\bigvee_{j\in J}(\mu_j \times \eta_j))^\psi = X \times Y$. Hence for all open fuzzy set ρ containing x_ε, $\rho^\varphi \leq (grf)^{-1}((\bigvee_{j\in J}(\mu_j \times \eta_j))^\psi) = (grf)^{-1}(X \times Y) = X$. Thus, grf is fuzzy $\varphi\psi$-continuous function.

THEOREM 6. Suppose that (X_α, τ_α) is fuzzy topological space and ψ is an operation on I^{X_α} for all α. Let $(\prod_{\alpha\in J} X_\alpha, \tau^{\shortmid})$ be a product space and let ψ be a productive operation on $I^{\prod_{\alpha\in J} X_\alpha}$. Let (X, τ) be a fuzzy topological space, let φ be an operation on I^X and let $f : (X, \tau) \to (\prod_{\alpha\in J} X_\alpha, \tau^{\shortmid})$ be any fuzzy function. If f $\varphi\psi$-continuous, then $p_\alpha \circ f$ is fuzzy $\varphi\psi$-continuous where p_α is projection function for each $\alpha \in J$.

PROOF. Let $x_\varepsilon \in X$ and $(p_\alpha \circ f)(x_\varepsilon) \in \rho_\alpha \in \tau_\alpha$. Then $f(x_\varepsilon) \in p_\alpha^{-1}(\rho_\alpha) = \rho_\alpha \times (\prod_{\beta\neq\alpha} X_\beta) \in \tau^{\shortmid}$. Since f is $\varphi\psi$-continuous, there exists an open set μ containing x_ε such that $f(\mu^\varphi) \leq (\rho_\alpha \times \prod_{\beta\neq\alpha} X_\beta)^\psi$. Since ψ is a productive operation, $f(\mu^\varphi) \leq \rho_\alpha^\psi \times (\prod_{\beta\neq\alpha} X_\beta)^\psi = \rho_\alpha^\psi \times \prod_{\beta\neq\alpha} {X_\beta}^\psi = \rho_\alpha^\psi \times \prod_{\beta\neq\alpha} X_\beta = p_\alpha^{-1}(\rho_\alpha^\psi)$ and hence $\mu^\varphi \leq (p_\alpha \circ f)^{-1}(\rho_\alpha^\psi)$ and we obtain that $p_\alpha \circ f$ is fuzzy $\varphi\psi$-continuous for each $\alpha \in J$.

THEOREM 7. Suppose that (X, τ) and (Y, υ) are fuzzy topological spaces and φ, ψ are operations on I^X, I^Y respectively. Let $f : X \to Y$ be any fuzzy function and let ß be a base of υ and let ψ be a monotonous operation on I^Y. f is fuzzy $\varphi\psi$-continuous iff for each $x_\varepsilon \in X$ and each $\rho \in$ ß containing $f(x_\varepsilon)$, there exists an open set μ containing x_ε such that $f(\mu^\varphi) \leq \rho^\psi$.

PROOF. ($\Rightarrow$:) Obvious.

($\Leftarrow$:) Let $x_\varepsilon \in X$ and $f(x_\varepsilon) \in \eta \in \upsilon$. Since ß is a base of υ, there exists an open set $\xi \in$ ß containing $f(x_\varepsilon)$ such that $\xi \leq \eta$. Now from hypothesis, there exists an open set

γ containing x_ε such that $f(\gamma^\varphi) \leq \xi^\psi$. Since ψ is a monotonous operation on I^Y and $\xi \leq \eta$, $f(\gamma^\varphi) \leq \xi^\psi \leq \eta^\psi$. Hence, f is $\varphi\psi$-continuous.

References

[1] K. K. Azad, On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity, J. Math. Anal. Appl., 82(1981), 14-32.

[2] S. Ganguly and S. Saha, A note on δ-continuity and δ-connected sets in fuzzy set theory, Simon Stevin (A Quarterly Journal of Pure and Applied Mathematics), 62(1988)(2), 127-141.

[3] A. Kandil, E. E. Kerre, A. A. Nouh and M. E. El-Shaffi, Fuzzy θ-perfect irreducible mappings and fuzzy θ-proximity spaces, Fuzzy Sets and Systems, to appear.

[4] A. Kandil, E. E. Kerre, M. E. El-Shaffi and A. A. Nouh, Fuzzy strongly θ-continuous mappings, Proc. Assiut First Intern. Conf., Part VIII (1990), 97-113.

[5] E. E. Kerre, A. A. Nouh and A. Kandil, Operations on the class of all fuzzy sets on a universe endowed with a fuzzy topology, Proceedings IFSA, Brussels, Belgium (1991), 109-113, .

[6] M. N. Mukherjee and S. P. Sinha, On some strong forms of fuzzy continuous mappings on fuzzy topological spaces, Fuzzy Sets and Systems, 38(1990), 375-387.

[7] M. N. Mukherjee and S. P. Sinha, On some near-fuzzy continuous functions between fuzzy topological spaces, Fuzzy Sets and Systems, 34(1990), 245-254.

[8] P. M. Pu and Y. M. Liu, Fuzzy Topology I Neighborhood structure of a fuzzy point and Moore-Smith convergence, J. Math. Anal. Appl., 76(1980), 571-599.

[9] P. M. Pu and Y. M. Liu, Fuzzy Topology II Product and quotient spaces, J. Math. Anal. Appl., 77(1980), 20-37.

[10] S. Saha, Fuzzy δ-continuous mappings, J. Math. Anal. Appl., 126(1987), 130-142.