

Common Fixed Points Under Contractive Conditions In Symmetric Spaces *

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Abstract

In this paper several common fixed point theorems for selfmappings of a symmetric space are proved. These mappings are assumed to satisfy a new property which generalize the notion of noncompatible maps in the setting of symmetric spaces.

1 Introduction

It is well known that the Banach contraction principle is a fundamental result in fixed point theory, which has been used and extended in many different directions. However, it has been observed [9] that some of the defining properties of the metric are not needed in the proofs of certain metric theorems. Motivated by this fact, Hicks [9] established some common fixed point theorems in symmetric spaces and proved that very general probabilistic structures admit a compatible symmetric or semi-metric. Recall that a symmetric on a set X is a nonnegative real valued function d on $X \times X$ such that (i) $d(x, y) = 0$ if, and only if, $x = y$, and (ii) $d(x, y) = d(y, x)$.

Let d be a symmetric on a set X and for $r > 0$ and any $x \in X$, let $B(x, r) = \{y \in X : d(x, y) < r\}$. A topology $t(d)$ on X is given by $U \in t(d)$ if, and only if, for each $x \in U$, $B(x, r) \subset U$ for some $r > 0$. A symmetric d is a semi-metric if for each $x \in X$ and each $r > 0$, $B(x, r)$ is a neighborhood of x in the topology $t(d)$. Note that $\lim_{n \rightarrow \infty} d(x_n, x) = 0$ if and only if $x_n \rightarrow x$ in the topology $t(d)$.

The following two axioms were given by Wilson [11]. Let (X, d) be a symmetric space.

(W.3) Given $\{x_n\}, x$ and y in X , $\lim_{n \rightarrow \infty} d(x_n, x) = 0$ and $\lim_{n \rightarrow \infty} d(x_n, y) = 0$ imply $x = y$.

(W.4) Given $\{x_n\}, \{y_n\}$ and x in X , $\lim_{n \rightarrow \infty} d(x_n, x) = 0$ and $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$ imply that $\lim_{n \rightarrow \infty} d(y_n, x) = 0$.

It is easy to see that for a semi-metric d , if $t(d)$ is Hausdorff, then (W.3) holds. On the one hand, the notion of the weak commutativity is introduced by Sessa [10]

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as follows: Two selfmappings A and B of a metric space (X, d) are said to be weakly commuting if

$$d(ABx, BAx) \leq d(Ax, Bx), \quad \forall x \in X.$$

Jungck [3] extended this concept in the following way: Let A and B be two self-mappings of a metric space (X, d) . A and B are said to be compatible if

$$\lim_{n \rightarrow \infty} d(ABx_n, BAx_n) = 0$$

whenever (x_n) is a sequence in X such that

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Bx_n = t$$

for some $t \in X$.

Obviously, two weakly commuting mappings are compatibles but the converse is not true as is shown in [3]. Recently, Jungck introduced the concept of weakly compatible maps as follows: Two selfmapping T and S of a metric space X are said to be weakly compatible if they commute at there coincidence points, i.e., if $Tu = Su$ for some $u \in X$, then $TSu = STu$.

It is easy to see that two compatible maps are weakly compatible but the converse is not true.

All these concepts were frequently used to prove existence theorems in common fixed point theory. However, the study of common fixed points of noncompatible maps is also very interesting [6, 7].

On the other hand, in [1], the authors of the present paper have established some new common fixed point theorems under strict contractive conditions on a metric space for mappings satisfying the property (E.A) defined as follows: Let S and T be two selfmappings of a metric space (X, d) . We say that T and S satisfy the property (E.A) if there exists a sequence (x_n) such that

$$\lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} Sx_n = t$$

for some $t \in X$.

The main purpose of this paper is to give some common fixed points theorems for selfmappings of a symmetric space under a generalized contractive condition. These selfmappings are assumed to satisfy a new property, introduced recently in [1] on a metric space, which generalize the notion of noncompatible maps in the setting of a symmetric space.

2 Main results

In the sequel, we need a function $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying the condition $0 < \phi(t) < t$ for each $t > 0$. For example, we could let $\phi(t) = \alpha t$ for some $\alpha \in (0, 1)$, or $t/(t + 1)$.

DEFINITION 2.1. Let A and B be two selfmappings of a symmetric space (X, d) . A and B are said to be compatible if

$$\lim_{n \rightarrow \infty} d(ABx_n, BAx_n) = 0$$

whenever (x_n) is a sequence in X such that

$$\lim_{n \rightarrow \infty} d(Ax_n, t) = \lim_{n \rightarrow \infty} d(Bx_n, t) = 0$$

for some $t \in X$.

DEFINITION 2.2. Two selfmappings A and B of a symmetric space (X, d) are said to be weakly compatible if they commute at their coincidence points.

DEFINITION 2.3. Let A and B be two selfmappings of a symmetric space (X, d) . We say that A and B satisfy the property (E.A) if there exists a sequence (x_n) such that

$$\lim_{n \rightarrow \infty} d(Ax_n, t) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} d(Bx_n, t) = 0$$

for some $t \in X$.

EXAMPLE 2.1.

1. Let $X = [0, +\infty[$. Let d be a symmetric on X defined by

$$d(x, y) = e^{|x-y|} - 1, \quad \text{for all } x, y \text{ in } X$$

Define $A, B : X \rightarrow X$ as follows

$$Ax = 2x + 1 \quad \text{and} \quad Bx = x + 2, \quad \forall x \in X$$

Note that the function d is not a metric. Consider the sequence $x_n = \frac{1}{n} + 1$, $n = 1, 2, \dots$. Clearly

$$\lim_{n \rightarrow \infty} d(Ax_n, 3) = \lim_{n \rightarrow \infty} d(Bx_n, 3) = 0$$

Then A and B satisfy (E.A).

2. Let $X = \mathbb{R}$ with the above symmetric function d . It is easy to see that the condition (W.3) holds. Define $A, B : X \rightarrow X$ by

$$Ax = x + 1 \quad \text{and} \quad Bx = x + 2, \quad \forall x \in X$$

Suppose that property (E.A) holds, then there exists in X a sequence (x_n) satisfying

$$\lim_{n \rightarrow \infty} d(Ax_n, t) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} d(Bx_n, t) = 0$$

for some $t \in X$. Therefore

$$\lim_{n \rightarrow \infty} d(x_n, t - 1) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} d(x_n, t - 2) = 0.$$

In view of (W.3), we conclude that $t - 1 = t - 2$, which is a contradiction. Hence A and B do not satisfy the property (E.A).

REMARK 2.1. It is clear from the above Definition 2.1, that two selfmappings S and T of a symmetric space (X, d) will be noncompatible if there exists at least one sequence (x_n) in X such that

$$\lim_{n \rightarrow \infty} d(Sx_n, t) = \lim_{n \rightarrow \infty} d(Tx_n, t) = 0, \quad \text{for some } t \in X$$

but $\lim_{n \rightarrow \infty} d(STx_n, TSx_n)$ is either non-zero or does not exist.

Therefore, two noncompatible selfmappings of a symmetric space (X, d) satisfy the property (E.A).

DEFINITION 2.4. Let (X, d) be a symmetric space. We say that (X, d) satisfies the property (H_E) if given $\{x_n\}, \{y_n\}$ and x in X , $\lim_{n \rightarrow \infty} d(x_n, x) = 0$ and $\lim_{n \rightarrow \infty} d(y_n, x) = 0$ imply $\lim_{n \rightarrow \infty} d(y_n, x_n) = 0$.

EXAMPLE 2.2.

- (i) Every metric space (X, d) satisfies the property (H_E) .
- (ii) Let $X = [0, +\infty)$ with the symmetric function d defined by

$$d(x, y) = e^{|x-y|} - 1, \quad \text{for all } x, y \text{ in } X$$

It is easy to see that the symmetric space (X, d) satisfies the property (H_E) . Note that (X, d) is not a metric space.

THEOREM 2.1. Let d be a symmetric for X that satisfies $(W.3)$ and (H_E) . Let A and B be two weakly compatible selfmappings of (X, d) such that (1) $d(Ax, Ay) \leq \phi(\max\{d(Bx, By), d(Bx, Ay), d(Ay, By)\})$ for all $(x, y) \in X^2$, (2) A and B satisfy the property (E.A), and (3) $AX \subset BX$. If the range of A or B is a complete subspace of X , then A and B have a unique common fixed point.

PROOF. Since A and B satisfy the property (E.A), there exists a sequence (x_n) in X such that $\lim_{n \rightarrow \infty} d(Ax_n, t) = \lim_{n \rightarrow \infty} d(Bx_n, t) = 0$ for some $t \in X$. Therefore, by (H_E) , we have $\lim_{n \rightarrow \infty} d(Ax_n, Bx_n) = 0$.

Suppose that BX is a complete subspace of X . Then $t = Bu$ for some $u \in X$. We claim that $Au = Bu$. Indeed, by (1), we have

$$\begin{aligned} d(Au, Ax_n) &\leq \phi(\max\{d(Bu, Bx_n), d(Bu, Ax_n), d(Bx_n, Ax_n)\}) \\ &< \max\{d(Bu, Bx_n), d(Bu, Ax_n), d(Bx_n, Ax_n)\} \end{aligned}$$

Letting $n \rightarrow \infty$, we have $\lim_{n \rightarrow \infty} d(Au, Ax_n) = 0$. Hence, by $(W.3)$, we have $Au = Bu$. The weak compatibility of A and B implies that $ABu = BAu$ and then $AAu = ABu = BAu = BBu$.

Let us show that Au is a common fixed point of A and B . Suppose that $AAu \neq Au$. In view of (1), it follows

$$\begin{aligned} d(Au, AAu) &\leq \phi(\max\{d(Bu, BAu), d(Bu, AAu), d(BAu, AAu)\}) \\ &\leq \phi(\max\{d(AAu, Au), d(AAu, Au)\}) \\ &\leq \phi(d(AAu, Au)) \\ &< d(AAu, Au), \end{aligned}$$

which is a contradiction. Therefore $Au = AAu = BAu$ and Au is a common fixed point of A and B . The proof is similar when AX is assumed to be a complete subspace of X since $AX \subset BX$. If $Au = Bu = u$ and $Av = Bv = v$, and $u \neq v$, then (1) gives

$$\begin{aligned} d(u, v) &= d(Au, Av) \\ &\leq \phi(\max\{d(Bu, Bv), d(Bu, Av), d(Bv, Av)\}) \\ &\leq \phi(d(u, v)) \\ &< d(u, v), \end{aligned}$$

which is a contradiction. Therefore $u = v$ and the common fixed point is unique.

Since two noncompatible selfmappings of a symmetric space (X, d) satisfy the property (E.A), we get the following result.

COROLLARY 2.1. Let d be a symmetric for X that satisfies (W.3) and (H_E) . Let A and B be two noncompatible weakly compatible selfmappings of (X, d) such that (1) $d(Ax, Ay) \leq \phi(\max\{d(Bx, By), d(Bx, Ay), d(Ay, By)\})$ for all $(x, y) \in X^2$, and (2) $AX \subset BX$. If the range of A or B is a complete subspace of X , then A and B have a unique common fixed point.

THEOREM 2.2. Let d be a symmetric for X that satisfies (W.3), (W.4) and (H_E) . Let A, B, T and S be selfmappings of (X, d) such that (1) $d(Ax, By) \leq \phi(\max\{d(Sx, Ty), d(Sx, By), d(Ty, By)\})$ for all $(x, y) \in X^2$, (2) (A, T) and (B, S) are weakly compatibles, (3) (A, S) or (B, T) satisfies the property (E.A), and (4) $AX \subset TX$ and $BX \subset SX$. If the range of the one of the mappings A, B, T or S is a complete subspace of X , then A, B, T and S have a unique common fixed point.

PROOF. Suppose that (B, T) satisfies the property (E.A). Then there exists a sequence (x_n) in X such that $\lim_{n \rightarrow \infty} d(Bx_n, t) = \lim_{n \rightarrow \infty} d(Tx_n, t) = 0$, for some $t \in X$. Since $BX \subset SX$, there exists in X a sequence (y_n) such that $Bx_n = Sy_n$. Hence $\lim_{n \rightarrow \infty} d(Sy_n, t) = 0$. Let us show that $\lim_{n \rightarrow \infty} d(Ay_n, t) = 0$. Indeed, in view of (1), we have

$$\begin{aligned} d(Ay_n, Bx_n) &\leq \phi(\max\{d(Sy_n, Tx_n), d(Sy_n, Bx_n), d(Tx_n, Bx_n)\}) \\ &\leq \phi(\max\{d(Bx_n, Tx_n), 0, d(Tx_n, Bx_n)\}) \\ &\leq \phi(d(Tx_n, Bx_n)) \\ &< d(Tx_n, Bx_n) \end{aligned}$$

Therefore, by (H_E) , one has $\lim_{n \rightarrow \infty} d(Ay_n, Bx_n) = 0$. By (W.4), we deduce that $\lim_{n \rightarrow \infty} d(Ay_n, t) = 0$. Suppose that SX is a complete subspace of X . Then $t = Su$ for some $u \in X$. Subsequently, we have

$$\lim_{n \rightarrow \infty} d(Ay_n, Su) = \lim_{n \rightarrow \infty} d(Bx_n, Su) = \lim_{n \rightarrow \infty} d(Tx_n, Su) = \lim_{n \rightarrow \infty} d(Sy_n, Su) = 0$$

Using (1), it follows

$$d(Au, Bx_n) \leq \phi(\max\{d(Su, Tx_n), d(Su, Bx_n), d(Tx_n, Bx_n)\}).$$

Letting $n \rightarrow \infty$, we have $\lim_{n \rightarrow \infty} d(Au, Bx_n) = 0$. By (W.3), we have $Au = Su$. The weak compatibility of A and S implies that $ASu = SAu$ and then $AAu = ASu = SAu = SSAu$.

On the other hand, since $AX \subset TX$, there exists $v \in X$ such that $Au = Tv$. We claim that $Tv = Bv$. If not, condition (1) gives

$$\begin{aligned} d(Au, Bv) &\leq \phi(\max\{d(Su, Tv), d(Su, Bv), d(Tv, Bv)\}) \\ &\leq \phi(\max\{d(Au, Bv), d(Au, Bv)\}) \\ &\leq \phi(d(Au, Bv)) \\ &< d(Au, Bv), \end{aligned}$$

which is a contradiction. Hence $Au = Su = Tv = Bv$. The weak compatibility of B and T implies that $BTv = TBv$ and $TTv = TBv = BTv = BBv$.

Let us show that Au is a common fixed point of A, B, T and S . Suppose that $AAu \neq Au$. We have

$$\begin{aligned} d(Au, AAu) &= d(AAu, Bv) \\ &\leq \phi(\max\{d(SAu, Tv), d(SAu, Bv), d(Tv, Bv)\}) \\ &\leq \phi(\max\{d(AAu, Au), d(AAu, Au)\}) \\ &\leq \phi(d(AAu, Au)) \\ &< d(Au, AAu), \end{aligned}$$

which is a contradiction. Therefore $Au = AAu = SAu$ and Au is a common fixed point of A and S . Similarly, we prove that Bv is a common fixed point of B and T . Since $Au = Bv$, we conclude that Au is a common fixed point of A, B, T and S . The proof is similar when TX is assumed to be a complete subspace of X . The cases in which AX or BX is a complete subspace of X are similar to the cases in which TX or SX respectively is complete since $AX \subset TX$ and $BX \subset SX$. If $Au = Bu = Tu = Su = u$ and $Av = Bv = Tv = Sv = v$ and $u \neq v$, then (1) gives

$$\begin{aligned} d(u, v) &= d(Au, Bv) \\ &\leq \phi(\max\{d(Su, Tv), d(Su, Bv), d(Tv, Bv)\}) \\ &\leq \phi(d(u, v)) \\ &< d(u, v), \end{aligned}$$

which is a contradiction. Therefore $u = v$ and the common fixed point is unique.

COROLLARY 2.2. ([1, Theorem 2]) Let A, B, T and S be selfmappings of a metric space (X, d) such that (1) $d(Ax, By) \leq \phi(\max\{d(Sx, Ty), d(Sx, By), d(Ty, By)\})$ for all $(x, y) \in X^2$, (2) (A, S) and (B, T) are weakly compatibles, (3) (A, S) or (B, T) satisfies the property (E.A), and (4) $AX \subset TX$ and $BX \subset SX$. If the range of the one of the mappings A, B, T or S is a complete subspace of X , then A, B, T and S have a unique common fixed point.

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