

Exact Linearization Of Stochastic Dynamical Systems By State Space Coordinate Transformation And Feedback I – g -Linearization *

Ladislav Sládeček †

Received 26 November 2002

Abstract

Given a dynamical system, the task of exact feedback linearization by coordinate transformation of the state vector is to look for a combination of coordinate transformation and feedback which will make the system linear and controllable.

This paper studies linearization methods for stochastic SISO affine dynamical systems represented by vectorfield triplets in Euclidean space.

The paper is divided into two self-contained parts. In this first part the problem is defined for both the Itô and the Stratonovich systems and the difference between complete and incomplete linearizations is emphasized.

1 Introduction

The theory of exact linearization of deterministic dynamical systems has been thoroughly studied since the seventies with many applications in control and optimization. See e.g. [1], [5], [9], and the references contained therein. Recently there have been attempts to apply some of the results to stochastic systems. In this paper we extend some of these results to linearization by state space transformation. We define the problem of $g\sigma$ -linearization (also called *complete linearization*) which linearizes both the control and the dispersion part of the system and the problem of g -linearization which linearizes only the control part. One of our main goals is to emphasize the differences between these two classes of problems.

Our paper consists of two parts: in the first part we will define several classes of stochastic dynamical systems, two transformations of such systems and their linearity and controllability. Then we will study Itô transformations and prove useful invariance properties of the correcting term which is the main point of the first part. Then we will discuss the problem of g -linearization.

In the second part we will study in deeper detail the more useful problem — the $g\sigma$ -linearization. Finally, the results will be illustrated with a numerical example — control of a crane under influence of noise.

*Mathematics Subject Classifications: 93B18, 93E03.

†Řrkovice 18, CZ 751 18, Czech Republic

1.1 Previous Works

The problem of feedback g -linearization of SISO dynamical system defined in the Itô formalism has been studied by Lahdhiri and Alouani [3]. The authors derive equations corresponding to (10), (11). These equations are combined and then reduced to a set of PDEs of a single unknown function T_1 . Because there is no commuting relation similar to the Leibniz rule, the equations contain partial derivatives of T_1 up to the $2n$ -th order. Next, the authors propose a lemma (Lemma 1) that identifies the linearity conditions with non-singularity and involutiveness of $\{\text{ad}_f^i g, 0 \leq i \leq n-2\}$. However, it can easily be verified that for $\sigma = 0$ this statement does not correspond to the conditions known for deterministic systems (see [9]), because the deterministic case requires non-singularity up to the $(n-1)$ -th bracket, not only up to the $(n-2)$ -th one. Furthermore, although the method of finding T_1 was given (solving PDE), we do not think that the existence of T_1 was proved as claimed.

The recent works of Pan [7] and [6] build on the idea *invariance under transformation rule* which is equivalent to our Theorem 1 (we speak of the *correcting term*). In the article [6], Pan defines and solves the problem of *feedback complete linearization of stochastic nonlinear systems*. In our terminology, this problem is equivalent to feedback MIMO input-output Itô $g\sigma$ -linearization. The *deterministic uncertain systems* considered by Pan can be identified with Stratonovich stochastic systems. In [7] Pan examines three other canonical forms of stochastic nonlinear systems, namely the noise-prone strict feedback form, zero dynamics canonical form and observer canonical form.

1.2 Dynamical systems

From now on, let us assume that all objects are smooth and bounded on $U \in \mathbb{R}^n$.

DEFINITION 1. A stochastic dynamical system $\Theta := (f(x), g(x), \sigma(x), U, x_0)$ is defined to be a triplet of vector fields f , g and σ defined on an open neighborhood U of a point $x_0 \in \mathbb{R}^n$. We usually call $U \in \mathbb{R}^n$ the state space, f the drift vector field, g the control vector field, and σ the dispersion vector field.

It is customary to study exact linearization problems for dynamical systems defined at equilibrium, i.e., we require that $f(x_0) = 0$ which can be linearized into a linear system $\dot{x} = Ax + Bu$ without a constant term (see [5] for details). This can be assumed without any loss of generality because the non-equilibrium case can easily be handled by extending the linear model with a constant term $\dot{x} = Ax + Bu + A_0$. Moreover we will require that all transformations preserve this condition.

The definition may be interpreted as follows: there is a stochastic process x_t defined on \mathbb{R}^n which is a strong solution of the stochastic differential equation $dx_t = f(x_t) dt + g(x_t)u(t) dt + \sigma(x_t) dw_t$, with initial condition x_0 , where $u(t)$ is a smooth function with bounded derivatives and w_t is a one-dimensional Brownian motion. The differential dw_t is just a notational shortcut for the stochastic integral.

For MIMO systems with m control inputs and k -dimensional noise the symbols g and σ stand for $n \times m$ ($n \times k$ respectively) matrix of vector fields having its rank equal to m (k respectively). The class of all deterministic n -dimensional dynamical

systems with m inputs will be called $X_D(n, m)$ and the class of stochastic systems with k -dimensional noise will be denoted with $X(n, m, k)$.

Theory of stochastic processes offers several alternative definitions of the stochastic integral, among them the *Itô integral* and the *Stratonovich integral*; each of them is used to model different physical problems. Consequently there are two classes of differential equations and two alternative definitions of a stochastic dynamical system — Itô dynamical systems defined by Itô integrals and Stratonovich systems defined by Stratonovich integrals. Itô and Stratonovich dynamical systems will be distinguished by a subscript: $\Theta_I \in X_I(n, m, k)$ and $\Theta_S \in X_S(n, m, k)$.

Serious differences between these integrals exist but from our point of view there is a single important one: *the rules for coordinate transformations of dynamical systems defined by Itô stochastic integral are quite different from the transformation rules which are valid for Stratonovich systems.*

1.3 Transformations

Furthermore, we will study two transformations of dynamical systems: the coordinate transformation \mathcal{T}_T and the feedback $\mathcal{F}_{\alpha, \beta}$. The definition of these transformation should be in accord with their common interpretation. This can be illustrated on the definition of the coordinate transformation of a deterministic dynamical system $\mathcal{T}_T : X_D(n, m) \rightarrow X_D(n, m)$ which is induced by a diffeomorphism $T : U \rightarrow \mathbb{R}^n$ between two coordinate systems on an open set $U \subset \mathbb{R}^n$. The mapping \mathcal{T}_T is defined by: $\mathcal{T}_T(f(x), g(x), U, x_0) := (T_*f, T_*g, T(U), T(x_0))$. Recall that the symbol T_* stands for the contravariant transformation $(T_*f)_i = \sum_{j=0}^n f_j \partial T_i / \partial x_j$.

Note that the words “coordinate transformation” are used in two different meanings: first as the diffeomorphism $T : U \rightarrow \mathbb{R}^n$ between coordinates; second as the mapping between systems $\mathcal{T}_T : X_D(n, m) \rightarrow X_D(n, m)$.

Coordinate transformation for stochastic systems distinguishes between the Itô and the Stratonovich systems. One of the major complications of the linearization problems for Itô systems is the second-order term in the transformation rules for Itô systems:

DEFINITION 2. Let $U \in \mathbb{R}^n$ be an open set and let $T : U \rightarrow \mathbb{R}^n$ be a diffeomorphism from U to \mathbb{R}^n such that $T(x_0) = 0$. The mapping $\mathcal{T}_T : X_I(n, m, k) \rightarrow X_I(n, m, k)$ will be called a coordinate transformation of an Itô system induced by diffeomorphism T if the systems $\Theta_1 := (f(x), g(x), \sigma(x), U, x_0)$ and $\Theta_2 := (\tilde{f}, \tilde{g}, \tilde{\sigma}, T(U), x_0)$; $\Theta_2 = \mathcal{T}_T(\Theta_1)$ are related by: $\tilde{f} = T_*f + P_\sigma T$, $\tilde{g}_i = T_*g_i$ and $\tilde{\sigma}_j = T_*\sigma_j$ for $1 \leq j \leq k$ and $1 \leq i \leq m$. We require that the transformation maps the equilibrium of the dynamical systems into the origin, i.e., $T(x_0) = 0$.

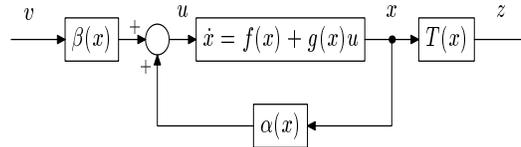
The symbol $P_\sigma T$ stands for the Itô term which is a second order linear operator defined by the following relation for the m -th component of $P_\sigma T$, $1 \leq m \leq n$,

$$P_\sigma T_m := \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 T_m}{\partial x_i \partial x_j} \sum_{l=1}^k \sigma_{il} \sigma_{jl}. \quad (1)$$

The transformation rules for Stratonovich system $T_T : X_S(n, m, k) \rightarrow X_S(n, m, k)$,

$(f, g, \sigma, U, x_0) \mapsto (T_*f, T_*g, T_*\sigma, T(U), T(x_0))$ are equivalent to rules valid for the deterministic systems; only the rules $\tilde{\sigma}_j = T_*\sigma_j$ for the drift vector field must be added.

Another important transformation of dynamical systems is the regular feedback transformation. A feedback transformation is determined by two smooth nonlinear functions $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $\beta : \mathbb{R}^n \rightarrow \mathbb{R}^m \times \mathbb{R}^m$ defined on U with β nonsingular for every $x \in U$ (see the following figure). Usually, α is written as a column $m \times 1$ matrix and β as a square $m \times m$ matrix.



Regular State Feedback

DEFINITION 3. Let $\Theta = (f(x), g(x), \sigma(x), U, x_0) \in \mathbf{X}(n, m, k)$ be a stochastic dynamical system. A regular state feedback is the transformation $\mathcal{F}_{\alpha, \beta} : \mathbf{X}(n, m, k) \rightarrow \mathbf{X}(n, m, k)$, $(f, g, \sigma, U, x_0) \mapsto (f + g\alpha(x), g\beta(x), \sigma(x), U, x_0)$. Moreover we require that the transformation preserves the equilibrium: $\alpha(x_0) = 0$, and $\beta(x_0)$ is nonsingular.

The symbol $\mathcal{J}_{T, \alpha, \beta}$ is used to indicate the combination of coordinate transformation with feedback $\mathcal{J}_{T, \alpha, \beta} := \mathcal{F}_{\alpha, \beta} \circ \mathcal{I}_T$. Note that the order of the composition can be interchanged.

1.4 Linearity

The definition of linearity is straightforward in the deterministic case. In contrast, the stochastic case is more complex, because there are two “input” vector fields and thereby several degrees of linearity can be specified.

DEFINITION 4. The stochastic dynamical system $\Theta = (f, g, \sigma, U, 0)$ is g -linear if the mapping $f(x) = Ax$ is linear without constant term and $g(x) = B$ is constant on U . Θ is σ -linear if the mapping $f(x) = Ax$ is linear without constant term and $\sigma(x) = S$ is constant on U . Θ is $g\sigma$ -linear if it is both g -linear and σ -linear. Here, A is a square $n \times n$ matrix, B is a $n \times m$ matrix and S is a $n \times k$ matrix.

For stochastic system we distinguish: g -linearizing transformation which transforms Θ into a g -linear systems and $g\sigma$ -linearizing transformation which transforms Θ into a $g\sigma$ -linear system.

2 Transformations of Itô Dynamical Systems

The transformation rules of Itô systems are motivated by the Itô differential rule (see e.g. [10, Section 3.3]), which defines the influence of nonlinear coordinate transformations on Itô stochastic processes.

The Itô differential rule applies to the situation where a scalar valued stochastic process x_t defined by a stochastic differential equation $dx_t = f(x_t)dt + \sigma(x_t)dw_t$ ($f : \mathbb{R} \rightarrow \mathbb{R}$ and $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ are smooth real functions and w_t is a Brownian motion)

is transformed by a diffeomorphic coordinate transformation $T : \mathbb{R} \rightarrow \mathbb{R}$. Then the stochastic process $z_t := T(x_t)$ exists and is an Itô process. Further, the process z_t is the solution of the stochastic differential equation

$$dz_t = \frac{\partial T}{\partial x} f(x_t) dt + \frac{\partial T}{\partial x} \sigma(x_t) dw_t + \frac{1}{2} \sigma^2 \frac{\partial^2 T}{\partial x^2} dt. \quad (2)$$

All details together with a proof are available for example in [2].

The Itô rule can also be derived for the multidimensional case: for the m -th component of an n -dimensional stochastic process the Itô rule can be expressed as follows:

$$dz_m = \sum_{i=1}^n \frac{\partial T_m}{\partial x_i} f_i dt + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^k \frac{\partial T_m}{\partial x_i} \sigma_{ij} dw_j + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 T_m}{\partial x_i \partial x_j} \sum_{l=1}^k \sigma_{il} \sigma_{jl} dt, \quad (3)$$

$$dz_m = \mathcal{L}_f T_m dt + \sum_{j=1}^k \mathcal{L}_{\sigma_j} T_m dw_j + P_\sigma T_m dt. \quad (4)$$

The operator $P_\sigma T_m$ is sometimes written using matrix notation as:

$$P_\sigma T_m = \frac{1}{2} \text{trace} \left(\sigma^T \sigma \frac{\partial^2 T_m}{\partial x^2} \right).$$

Generally, P_σ vanishes for linear T or zero σ .

2.1 The Correcting Term

In this section we introduce an extremely useful equivalence between Itô and Stratonovich systems, which allows us to use some Stratonovich linearization techniques for Itô problems. The motivation is following: let $\Theta_I = (f(x), g(x), \sigma(x), U, x_0) \in X_I(n, m, k)$ be an Itô system. We are looking for a Stratonovich system $\Theta_S = (\vec{f}(x), \vec{g}(x), \vec{\sigma}(x), U, x_0)$ such that the trajectories of Θ_I and Θ_S are identical. The aim is to find equations relating the quantities \vec{f} , \vec{g} and $\vec{\sigma}$ with f , g and σ .

DEFINITION 5. Let $\Theta_{1I} = (f(x), g(x), \sigma(x), U, x_0) \in X_I(n, m, k)$ be an n -dimensional Itô dynamical system with k -dimensional Brownian motion w . The vector field $\text{corr}_\sigma(x)$ whose r -th coordinate is equal to

$$(\text{corr}_\sigma(x))_r = -\frac{1}{2} \sum_{i=1}^n \sum_{j=1}^k \frac{\partial \sigma_{rj}}{\partial x_i} \sigma_{ij} \quad \text{for } 1 \leq r \leq n \quad (5)$$

is called the correcting term. Note that the derivative is always evaluated in the corresponding coordinate system. Further, let us define the correcting mapping $\text{Corr}_\sigma : X_I(n, m, k) \rightarrow X_S(n, m, k)$ by $\text{Corr}_\sigma(f, g, \sigma, U, x_0) := (f + \text{corr}_\sigma(x), g, \sigma, U, x_0)$.

The general treatment of the subject can be found for example in [10, p.160] or in [8]. The following theorem describes the behavior of the correcting term under the coordinate transformation. The usage of this relation for exact linearization of stochastic system was first published in [6].

THEOREM 1. Let $\Theta_I = (f(x), g(x), \sigma(x), U, x_0) \in X_I(n, m, k)$ be an Itô dynamical system. Let T be a diffeomorphism defined on U and the symbols \mathcal{T}_T^I and \mathcal{T}_T^S denote a Itô coordinate transformation and a Stratonovich coordinate transformation induced by the same diffeomorphism T and $\tilde{\sigma} = T_*\sigma$. Then $\mathcal{T}_T^I = \text{Corr}_{\tilde{\sigma}}^{-1} \circ \mathcal{T}_T^S \circ \text{Corr}_{\sigma}$ and $\mathcal{T}_T^S = \text{Corr}_{\sigma}^{-1} \circ \mathcal{T}_T^I \circ \text{Corr}_{\tilde{\sigma}}$. The notation $\text{Corr}_{\sigma}^{-1}$ is used to denote the inverse mapping $\text{Corr}_{\sigma}^{-1}(f, g, \sigma, U, x_0) := (f - \text{corr}_{\sigma}(x), g, \sigma, U, x_0)$.

The proof consist of evaluation of Θ_{2I} in both ways. For details see for example [8], [4] or [10].

Theorem 1 is valid also for combined transformations:

COROLLARY 1. Let $\Theta_I = (f(x), g(x), \sigma(x), U, x_0) \in X_I(n, 1, 1)$, T , \mathcal{T}_T^I and \mathcal{T}_T^S have the same meaning as in Theorem 1. Then for arbitrary regular feedback $\mathcal{F}_{\alpha, \beta} : \mathcal{F}_{\alpha, \beta} \circ \mathcal{T}_T^I = \text{Corr}_{\tilde{\sigma}}^{-1} \circ \mathcal{F}_{\alpha, \beta} \circ \mathcal{T}_T^S \circ \text{Corr}_{\sigma}$.

PROOF. We want to prove equivalence of Θ_{4I} and $\text{Corr}_{T_*\sigma}^{-1}\Theta_{4S}$. The control and dispersion vector fields of Θ_{4I} and Θ_{4S} are identical and they are not influenced by the correcting mapping. The effect of feedback is purely additive and both the systems are equal.

When Theorem 1 is used for exact linearization of Itô systems we require that the Stratonovich system obtained by the correcting term is at equilibrium: $\vec{f}(x_0) = 0$. Therefore the Itô systems require an additional condition $f(x_0) + \text{corr}_{\sigma}(x_0) = 0$.

Finally note that there are many physical systems in which the correcting term vanishes. This happens when the drift vector σ is perpendicular to the gradient of σ , for example on a pendulum like system (see e.g. the crane of Section 3 of the second part of this article). Moreover, one can always find a coordinate system in which the correcting term vanishes by straightening-out of σ (see Flow-box Theorem [5, p.48]) without any loss of generality.

3 Itô g -linearization

The Itô g -linearization problem is probably the most complicated variant of exact linearization studied in this paper. The dispersion vector field of an Itô dynamical system transformed by a coordinate transformation \mathcal{T}_T consists of two terms: the transformed vector field T_*f and the Itô term P_{σ} . We require that the sum of these terms is linear, thus the nonlinearity of the drift T_*f must compensate for the Itô term. Since the Itô term behaves to T as a second order differential operator, this problems generates a set of second order partial differential equations.

3.1 Canonical Form — n unknowns

The canonical form for the g -linearization is the integrator chain with a nonlinear drift

$$\tilde{f}_i(x) = x_{i+1}, \quad 1 \leq i \leq n-1 \quad (6)$$

$$\tilde{f}_n(x) = 0 \quad (7)$$

$$\tilde{g}_i(x) = 0, \quad 1 \leq i \leq n-1 \quad (8)$$

$$\tilde{g}_n(x) = 1. \quad (9)$$

Assume that there is a g -linear system $\Theta_I = (Ax, B, \sigma(x), U, x_0)$. Then the drift part of Θ_I can be transformed by a *linear* transformation into the integrator chain. This is because the Itô term of a linear transformation vanishes.

The equations which define T can be obtained by comparing this canonical form with the equations of $\tilde{\Theta}$.

THEOREM 2. Let $\Theta_I = (f(x), g(x), \sigma(x), U, x_0) \in X_I(n, 1, 1)$ be an Itô dynamical system with $f(x_0) + \text{corr}_\sigma(x_0) = 0$. There is a g -linearizing transformations $\mathcal{J}_{T, \alpha, \beta}$ of the system Θ_I into a g -controllable linear system if, and only if, there is a solution $T_i : \mathbb{R}^n \rightarrow r$, $1 \leq i \leq n$, to the set of partial differential equations defined on U :

$$T_{i+1} = \mathcal{L}_f T_i + P_\sigma T_i, \quad 1 \leq i \leq n-1 \quad (10)$$

$$\mathcal{L}_g T_i = 0, \quad 1 \leq i \leq n-1 \quad (11)$$

$$\mathcal{L}_g T_n \neq 0. \quad (12)$$

The symbol P_σ denotes the Itô operator. The feedback can be constructed as:

$$\alpha = -\frac{(\mathcal{L}_f T_n + P_\sigma T_n)}{\mathcal{L}_g T_n}, \quad \beta = \frac{1}{\mathcal{L}_g T_n}.$$

Indeed, the partial differential equations (10), (11) and (12) are obtained by comparing the equations of Definition 2 with the equations (6)-(9).

One can attempt to reduce the equations (10), (11) and (12) to a set of equations of a single unknown in spirit of the deterministic case. In general, the equations of the system are of an order up to $2n$ and cannot be reduced to a lower order. Because the commutator of two second order operators is of third order as can be easily checked by direct computation.

4 Conclusion

In this part of the article we defined the exact linearization problem for the state space transformations. The main difficulty is the Itô term, which is a second order operator. Unfortunately, for the $g\sigma$ case we have not found any easy method to eliminate the Itô term and the a set of second order partial differential equations must be solved to get the linearizing transformation.

References

- [1] A. Isidori, *Nonlinear Control Systems*, Springer-Verlag, third edition, 1989.
- [2] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, Graduate Texts in Mathematics, 113, Springer, 1987.
- [3] T. Lahdhiri and A. T. Alouani, An introduction to the theory of exact stochastic feedback linearization for nonlinear stochastic systems, In Proc. 1996 IFAC 13 the Triennial World Congress, 515-520.

- [4] P. Malliavin, *Stochastic Analysis*, Springer, 1997.
- [5] H. Nijmeijer and A. J. van der Schaft, *Nonlinear Dynamical Control Systems*, Springer-Verlag, New York, 1994, corrected 2nd printing.
- [6] Z. G. Pan, Differential geometric condition for feedback complete linearization of stochastic nonlinear system. *Automatica*, 37(2001), 145–149.
- [7] Z. G. Pan, Canonical forms for stochastic nonlinear systems. *Automatica*, 38(2002), 1163–1170.
- [8] A. P. Sage and J. L. Melsa, *Estimation Theory with Applications to Communications and Control*, McGraw Hill, 1971.
- [9] M. Vidyasagar, *Nonlinear Systems Analysis*. Prentice-Hall, second edition, 1993.
- [10] E. Wong and B. Hajek, *Stochastic Processes in Engineering Systems*, Springer, 1984.