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Oscillation Of First Order Neutral Delay Difference Equations *

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Abstract

Sufficient conditions are established for the oscillation of all solutions of first order neutral delay difference equations. Our approach is to reduce the oscillation of neutral delay difference equation to the non-existence of positive solutions of delay difference inequalities. The results obtained here extend and improve several known results in the literature.

1 Introduction

Consider the neutral delay difference equation of the form

$$\Delta(y_n + h_n y_{n-k}) + \delta q_n y_{n-\ell}^{\alpha} = 0, \quad n = 0, 1, 2, ..., \tag{1}$$

where $\delta = \pm 1$, α is a ratio of odd positive integers, Δ is the forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$ and the following conditions (c_1) and (c_2) are assumed to hold:

(c_1) $\{h_n\}$ is a positive real sequence and k is a positive integer;

(c_2) $\{q_n\}$ is a positive real sequence and ℓ is any integer.

If $\alpha = 1$, then equation (1) reduces to the linear equation

$$\Delta(y_n + h_n y_{n-k}) + \delta q_n y_{n-\ell} = 0. \tag{2}$$

By a solution of equation (1), we mean a real sequence $\{y_n\}$ which satisfies equation (1) for $n \geq N$ for some integer $N > 0$. A solution $\{y_n\}$ of equation (1) is said to be oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise.

In recent years there has been much interest in studying the oscillation of first order neutral difference equations. For recent results and for references see the monograph

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by Agarwal [1]. In particular Thandapani and Sundaram [6] have shown that equation (1) is oscillatory if and only if

$$\sum_{n=n_0}^{\infty} q_n = \infty,$$

under one of the following conditions (a) and (b):

(a) $\delta = +1, 0 < \alpha < 1, 1 < \mu \le h_n \le \lambda, \ell > k > 0$;

(b) $\delta = -1, \alpha > 1, 0 \le h_n < \lambda < 1, k > 0,\ \ell < 0$.

However, very little is known about sufficient conditions for equation (1) to be oscillatory without the restrictions, viz. $h_n \le \lambda$. For equation (2) with some restriction on $\{h_n\}$ many oscillation criteria have been established, see for example [1, 2] and the references cited therein.

The purpose of this paper is to establish sufficient conditions for the oscillation of all solutions of equation (1) without the restrictive condition on $\{h_n\}$ or the periodicity condition on $\{q_n\}$ as assumed in some papers. In Section 2, we reduce the oscillation of the neutral difference equation (1) to the nonexistence of eventually positive solutions of non-neutral difference inequalities of the form

$$\delta \Delta z_n + p_n z_{n-m}^{\alpha} \le 0 \tag{3}$$

where m is an integer and $\{p_n\}$ is a positive real sequence. Sufficient conditions for (3) to have no eventually positive solution have been established by many authors. For example see [1, 2] and [4]. By combining these results with the results obtained in Section 2, we derive oscillation criteria for equation (1) in Section 3. The results obtained here extend and improve some of the results obtained in [7] and [5].

2 Non-Neutral Difference Inequalities

In this section we consider the equation

$$\Delta(y_n + h_n y_{n-k}) + \delta q_n f(y_{n-\ell}) = 0 \tag{4}$$

where $\delta = \pm 1$ and conditions (c_1), (c_2) and the following conditions $(c_3) - (c_5)$ hold:

(c_3) $f : R \to R$ is continuous, nondecreasing and $uf(u) > 0$ for $u \ne 0$;

(c_4) there exists a continuous function $\varphi : R \to R$ such that $\varphi(u)$ is nondecreasing in $u \in R$, $u\varphi(u) > 0$ for $u \ne 0$ and

$$|\varphi(u+v)| \le |f(u) + f(v)| \text{ for } uv > 0;$$

(c_5) there exists a continuous function $\omega : (0,\infty) \to (0,\infty)$ such that $|f(uv)| \le \omega(u)|f(v)|$ for $u > 0$ and $v \in R$.

REMARK 1. For the case $f(u) = u^\alpha$ where $\alpha > 0$, we can choose $\varphi(u) = \min\{1, 2^{1-\alpha}\}u^\alpha$ and $\omega(u) = u^\alpha$.

THEOREM 1. Let $\delta = +1$. Suppose that (c_1)-(c_5) hold. Then every solution of equation (4) is oscillatory if there exists a positive real sequence $\{\lambda_n\}$ such that $0 < \lambda_n < 1$ for $n \geq n_0$ and the difference inequality

$$\{\Delta z_n + Q_n \varphi(z_{n+k-\ell})\}\mathrm{sgn} z_n \leq 0, \tag{5}$$

does not have any nonoscillatory solution where

$$Q_n = \min\left\{\lambda_n q_n, \frac{[1-\lambda_{n-k}]q_{n-k}}{\omega(h_{n-\ell})}\right\}. \tag{6}$$

PROOF. Assume that there is a nonoscillatory solution $\{y_n\}$ for the equation (4). We may assume that $y_n > 0$ eventually since the case $\{y_n\}$ eventually negative can be treated similarly. Let $x_n = y_n + h_n y_{n-k}$. Then by equation (4), $x_n > 0$ and x_n is decreasing for all large $n \geq n_0$. Summing (4) from n to ∞ , we have

$$x_n = \lim_{n\to\infty} x_n + \sum_{s=n}^{\infty} q_s f(y_{s-\ell}) \geq \sum_{s=n}^{\infty} q_s f(y_{s-\ell})$$

for all large $n \geq n_0$. We see that

$$\begin{aligned} x_n &\geq \sum_{s=n}^{\infty} \lambda_s q_s f(y_{s-\ell}) + \sum_{s=n}^{\infty} (1-\lambda_s) q_s f(y_{s-\ell}) \\ &= \sum_{s=n}^{\infty} \lambda_s q_s f(y_{s-\ell}) + \sum_{s=n+k}^{\infty} (1-\lambda_{s-k}) q_{s-k} f(y_{s-k-\ell}) \\ &\geq \sum_{s=n+k}^{\infty} Q_s f(y_{s-\ell}) + \sum_{s=n+k}^{\infty} Q_s \omega(h_{s-\ell}) f(y_{s-k-\ell}) \\ &\geq \sum_{s=n+k}^{\infty} Q_s f(y_{s-\ell}) + \sum_{s=n+k}^{\infty} Q_s f(h_{s-\ell} y_{s-k-\ell}) \\ &= \sum_{s=n+k}^{\infty} Q_s \left[f(y_{s-\ell}) + f(h_{s-\ell} y_{s-k-\ell})\right] \\ &\geq \sum_{s=n+k}^{\infty} Q_s \varphi\left(y_{s-\ell} + h_{s-\ell} y_{s-k-\ell}\right) \\ &= \sum_{s=n+k}^{\infty} Q_s \varphi\left(x_{s-\ell}\right) \end{aligned}$$

for all $n \geq n_0$. Set

$$z_n = \sum_{s=n}^{\infty} Q_s \varphi(x_{s-\ell}) > 0.$$

Then $x_n \geq z_{n+k}$ eventually. We see that

$$\Delta z_n = -Q_n \varphi(x_{n-\ell}) \leq -Q_n \varphi(z_{n-\ell+k})$$

for all $n \geq n_0$ so that $\{z_n\}$ is an eventually positive solution of (5). This contradiction completes the proof of the theorem.

THEOREM 2. Let $\delta = -1$. Assume that conditions $(c_1) - (c_5)$ hold. Then every solution of equation (4) is oscillatory if there exists a positive real sequence $\{\lambda_n\}$ such that $0 < \lambda_n < 1$ for $n \geq n_0$ and the difference inequality

$$\{-\Delta z_n + Q_n \varphi(z_{n-\ell})\} \operatorname{sgn} z_n \leq 0$$

does not have any nonoscillatory solution where $\{Q_n\}$ is the same as defined by (6).

PROOF. Let $\{y_n\}$ be an eventually positive solution of equation (4). Let $x_n = y_n + h_n y_{n-k} > 0$. Summation of (4) from N to $n-1$ yields,

$$x_n \geq \sum_{s=N}^{n-1} q_s f(y_{s-\ell}), \quad n \geq N,$$

where N is sufficiently large positive integer. Now by the same arguments as in the proof of Theorem 1, we see that

$$x_n \geq \sum_{s=N+k}^{n-1} Q_s \varphi\left(x_{s-\ell}\right) \equiv z_n > 0$$

for all $n > N+k$. Then we have

$$\Delta z_n = Q_n \varphi(x_{n-\ell}) \geq Q_n \varphi(z_{n-\ell})$$

for all $n > N+k$, which is a contradiction. Now assume that $\{y_n\}$ is eventually negative. Arguing as above we obtain

$$-\Delta z_n + Q_n \varphi(z_{n-\ell}) \geq 0,$$

does not have any eventually negative solution. This completes the proof.

Applying Theorems 1 and 2 to equation (1), we obtain the following corollaries.

COROLLARY 3. Let $\delta = +1$ and $\alpha > 0$. Assume that the conditions (c_1) and (c_2) hold. Then every solution of equation (1) is oscillatory if the difference inequality

$$\Delta z_n + P_n z_{n-\ell+k}^{\alpha} \leq 0$$

does not have any eventually positive solution, where

$$P_n = \min\left\{1, 2^{1-\alpha}\right\} \min\left\{\frac{q_n}{1+h_{n-\ell+k}^{\alpha}}, \frac{q_{n-k}}{1+h_{n-\ell}^{\alpha}}\right\}. \tag{7}$$

COROLLARY 4. Let $\delta = -1$ and $\alpha > 0$. Assume that the conditions (c_1) and (c_2) hold. Then every solution of equation (1) is oscillatory if the difference inequality

$$\Delta z_n - P_n z_{n-\ell}^{\alpha} \geq 0$$

does not have any eventually positive solution, where $\{P_n\}$ is same as defined by (7).

PROOF of Corollaries 3 and 4: We know that

$$\delta\Delta z_n + p_n z_{n-m}^{\alpha} \leq 0 \tag{8}$$

does not have any eventually positive solution if and only if

$$\{\delta\Delta z_n + p_n z_{n-m}^{\alpha}\}\mathrm{sgn} z_n \leq 0 \tag{9}$$

does not have any nonoscillatory solution, where m is any integer and $\{p_n\}$ is any real sequence. In fact , if $\{\omega_n\}$ is an eventually negative solution of (9), then $z_n = -\omega_n$ is an eventually positive solution of (8). Therefore, the conclusions of Corollaries 3 and 4 follow by applying Theorems 1 and 2 to equation (1) and by choosing $\varphi(u) = \min\{1, 2^{1-\alpha}\}u^{\alpha}, \omega(u) = u^{\alpha}$ and $\lambda_n = 1/(1 + h_{n-\ell+k}^{\alpha})$ (see Remark 1).

3 Oscillation Theorems

In this section we establish oscillation theorems for the equation (1). First we consider the case $\alpha \neq 1$. We need the following results proved in [4] for our subsequent discussion.

LEMMA 1. Let $0 < \alpha < 1$ and ℓ be a positive integer. Assume that (c_2) holds. Then the difference inequality

$$\Delta u_n + q_n u_{n-\ell}^{\alpha} \leq 0, \quad n = 0, 1, 2, \cdots$$

does not have any eventually positive solution if

$$\sum_{n=0}^{\infty} q_n = \infty. \tag{10}$$

LEMMA 2. Let $\alpha > 1$ and ℓ be a negative integer. Assume that (c_2) holds. Then the difference inequality

$$\Delta u_n - q_n u_{n-\ell}^{\alpha} \geq 0, \quad n = 0, 1, 2, \cdots$$

does not have any eventually positive solution if (10) holds.

Combining Corollaries 3 and 4 with Lemmas 1 and 2, we derive the following theorems.

THEOREM 5. Assume that (c_1) and (c_2) are satisfied and one of the following cases holds:

(i) $\delta = +1, 0 < \alpha < 1$ and $\ell > k$;

(ii) $\delta = -1, \alpha > 1$ and ℓ is a negative integer.

If

$$\sum_{n=n_0}^{\infty} \min\left\{\frac{q_n}{(1+h^{\alpha}_{n-\ell+k})}, \frac{q_{n-k}}{(1+h^{\alpha}_{n-\ell})}\right\} = \infty, \tag{11}$$

then every solution of equation (1) is oscillatory.

COROLLARY 6. Suppose that conditions (c_1) and (c_2) are satisfied and either (i) or (ii) of Theorem 5 holds. Further assume that $\{h_n\}$ is bounded and

$$\sum_{n=n_0}^{\infty} \min\{q_n, q_{n-k}\} = \infty. \tag{12}$$

Then every solution of equation (1) oscillates.

PROOF. If $\{h_n\}$ is bounded for $n \geq n_0$, then it is easy to see that (12) implies (11). Hence, the conclusion follows from Theorem 3.

Next we consider the equation (2). To prove our results we need the following lemmas which can be found in [2, 3].

LEMMA 3. Assume condition (c_2) holds and $\ell > 0$. If

$$\liminf_{n\to\infty} \sum_{s=n-\ell}^{n-1} q_s > \left(\frac{\ell}{\ell+1}\right)^{\ell+1},$$

then the difference inequality

$$\Delta u_n + q_n u_{n-\ell} \leq 0$$

does not have any eventually positive solution .

LEMMA 4. Assume that condition (c_2) holds and $\ell > 0$. If

$$\liminf_{n\to\infty} \sum_{s=n}^{n+\ell-1} q_s > \left(\frac{\ell}{\ell+1}\right)^{\ell+1},$$

then the difference inequality

$$\Delta v_n - q_n v_{n+\ell} \geq 0$$

does not have any eventually positive solution.

From Corollaries 3 and 4, Lemmas 3 and 4, we have the following results.

THEOREM 7. Let $\delta = +1$ and $\ell > 0$ and $\ell - k > 1$. Assume that conditions (c_1) and (c_2) hold. If

$$\liminf_{n\to\infty} \sum_{s=n-\ell+k}^{n-1} \min\left\{\frac{q_n}{(1+h_{n-\ell+k})}, \frac{q_{n-k}}{(1+h_{n-\ell})}\right\} > \left(\frac{\ell}{\ell+1}\right)^{\ell+1}$$

holds, then every solution of equation (2) is oscillatory.

THEOREM 8. Let $\delta = -1$ and $\ell > 0$. Assume that conditions (c_1) and (c_2) hold. If

$$\liminf_{n\to\infty} \sum_{s=n}^{n+\ell-1} \min\left\{\frac{q_n}{(1+h^{\alpha}_{n-\ell+k})}, \frac{q_{n-k}}{(1+h^{\alpha}_{n-\ell})}\right\} > \left(\frac{\ell}{\ell+1}\right)^{\ell+1}$$

holds, then every solution of equation

$$\Delta(y_n + h_n y_{n-k}) - q_n y_{n+\ell} = 0$$

is oscillatory.

REMARK 2. Note that restrictive conditions on $\{h_n\}$ such as $h_n \leq \lambda$ is not assumed in Theorems 5, 7 and 8. Theorem 7 improves some of the results obtained in [5, 7].

REMARK 3. Recently Zhang [8] considered equation of type (1) and obtained oscillation results for the case $\{h_n\}$ is a negative constant in $(-1, 0]$ where as we obtain results for the case $\{h_n\}$ is a positive real sequences.

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