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Applications of Langenhop Inequality to Difference Equations: Lower Bounds and Oscillation *

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Abstract

By employing a discrete analogue of Langenhop's inequality, we establish lower bounds on the norm of solutions of the difference system

$$\Delta z_n = f(n, z_n), \quad n \in N(l) = \{l, l+1, \ldots\},$$

and derive an oscillation criterion for higher order delay difference equations of the form

$$\Delta^m x_n + p_n \Delta^{m-1} x_n + q_n |x_{\sigma(n)}|^{\alpha-1} x_{\sigma(n)} = 0, \quad 0 < \alpha < 1.$$

1 Introduction

There is no doubt that the Gronwall inequality and its generalization the Bihari inequality in continuous and discrete cases have been the most powerful tools in studying the qualitative behavior of differential and difference equations. These inequalities have been applied very successfully to investigate the global existence, uniqueness, stability, boundedness and other properties of solutions of various nonlinear differential and difference equations.

In 1960, Langenhop [1] proved the following theorem.

THEOREM 1.1. Let $g(x)$ be a continuous and nondecreasing function for $x \geq 0$ and $g(x) > 0$ for $x > 0$. If $u(t)$ and $v(t)$ are continuous nonnegative functions satisfying

$$u(t) \geq u(s) - \int_s^t v(r) g(u(r))\, dr \quad \text{for all } t, s \in [t_0, T], \tag{1.1}$$

then

$$u(t) \geq G^{-1}\left(G(u(s)) - \int_s^t v(r)\, dr\right) \tag{1.2}$$

for all $t, s \in [t_0, T]$ for which $G(u(s)) - \int_s^t v(r)\, dr$ is in the domain of G^{-1}, where

$$\frac{d}{du} G(u) = \frac{1}{g(u)}.$$

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If $g(u) = u$ in (1.1), then (1.2) reduces to

$$u(t) \geq u(s) \exp(-\int_s^t v(r)\,dr) \quad \text{for all } t, s \in [t_0, T]. \tag{1.3}$$

We note that the above conclusions remain valid as s tends to t_0, but if s is fixed as t_0 in (1.1), then as was shown by Langenhop they are no longer true.

Langenhop type inequalities have also been used quite successfully in studying the qualitative behavior of differential equations [2, 3].

In this paper by using a discrete analogue of Theorem 1.1, we establish lower bounds on the norm of a solution of a general difference equation, and obtain sufficient conditions for oscillation of solutions of higher order delay difference equations.

In what follows we denote by $N(a)$ the set of integers greater than or equal to a, where $a \geq 0$ is a given integer. For all $m > n$, $n \in N(a)$ and any sequence $\{b(n)\}$ defined for $n \in N(a)$, we shall use the usual conventions that

$$\sum_{i=m}^{n} b(i) = 0 \quad \text{and} \quad \prod_{i=m}^{n} b(i) = 1.$$

As usual, Δ denotes the forward difference operator defined by $\Delta u_n = u_{n+1} - u_n$.

2 Discrete Langenhop Inequality

We begin with the following result.

THEOREM 2.1. Let $\{u_n\}$ and $\{v_n\}$ be nonnegative sequences defined for $n \in N(l)$ and $g(u)$ a nondecreasing function for $u \geq 0$ with $g(u) > 0$ for $u > 0$. If

$$u_n \geq u_k - \sum_{i=k}^{n-1} v_i g(u_i), \quad \text{for all } k, n \in N(l) \tag{2.4}$$

then

$$u_n \geq G^{-1}\left(G(u_k) - \sum_{i=k}^{n-1} v_i \right) \tag{2.5}$$

for all $k, n \in N(l)$ for which $G(u_k) - \sum_{i=k}^{n-1} v_i$ is in the domain of G^{-1}, where G is defined by

$$\Delta G(u_n) = \frac{\Delta u_n}{g(u_n)}.$$

PROOF. For a fixed n, $n \in N(l)$, we define for $l \leq k \leq n$

$$w_k = u_n + \sum_{i=k}^{n-1} v_i g(u_i). \tag{2.6}$$

It is clear that

$$\Delta w_k + v_k g(u_k) = 0 \tag{2.7}$$

and

$$w_n = u_n. \tag{2.8}$$

Moreover, by (2.4) and (2.6), we have

$$w_k \geq u_k, \quad \text{for } k \in N(l). \tag{2.9}$$

Using (2.9) in (2.7), we obtain

$$\frac{\Delta w_k}{g(w_k)} + v_k \geq 0. \tag{2.10}$$

Summation of (2.10) from k to $n-1$ leads to

$$G(w_n) - G(w_k) + \sum_{i=k}^{n-1} v_i \geq 0. \tag{2.11}$$

Using (2.4), (2.6), (2.8), and the monotonicity of G, it follows from (2.11) that

$$G(u_n) \geq G(u_k) - \sum_{i=k}^{n-1} v_i. \tag{2.12}$$

Since G^{-1} is nondecreasing, we see from (2.12) that (2.5) is satisfied for those $k, n \in N(l)$ for which $G(u_k) - \sum_{i=k}^{n-1} v_i$ is in the domain of G^{-1}. Thus, the proof is complete.

THEOREM 2.2. Let $\{u_n\}$ and $\{v_n\}$ be nonnegative sequences defined for $n \in N(l)$. If

$$u_n \geq u_k - \sum_{i=k}^{n-1} v_i u_i, \quad \text{for all } k, n \in N(l),$$

then

$$u_n \geq u_k \prod_{i=k}^{n-1} (1 - v_i), \quad \text{for all } k, n \in N(l).$$

PROOF. We proceed as in the proof of Theorem 2 until inequality (2.10) is obtained. That is,

$$\frac{w_{k+1}}{w_k} + v_k - 1 \geq 0. \tag{2.13}$$

It is easy to see from (2.13) that

$$\frac{w_n}{w_k} \geq \prod_{i=k}^{n-1} (1 - v_i). \tag{2.14}$$

By using (2.8) and (2.9) in (2.14), we complete the proof.

3 Bounds on the Norm of Solutions

Let $\{z_n\}$ be a sequence with terms $z_n \in C^m$ and $f(n, z_n)$ a C^m valued sequence. We consider the first order difference system

$$\Delta z_n = f(n, z_n), \quad n \in N(l), \tag{3.15}$$

where Δ is the forward difference operator, $\Delta z_n = z_{n+1} - z_n$.

Let us assume that for some norm in C^m, which we shall denote by $|.|$, the function f satisfies

$$|f(n, z_n)| \leq v_n g(|z_n|), \quad n \geq N(l), \tag{3.16}$$

where

(*i*) $\{v_n\}$ is a sequence of nonnegative real numbers, and

(*ii*) $g(u)$ is nondecreasing for $u \geq 0$ and strictly positive for $u > 0$.

It follows from (3.15) and (3.16) that

$$|z_n| \leq |z_k| + \sum_{i=k}^{n-1} v_i g(|z_i|) \tag{3.17}$$

and

$$|z_n| \geq |z_k| - \sum_{i=k}^{n-1} v_i g(|z_i|) \tag{3.18}$$

for all $k, n \in N(l)$.

The main results of this section are as follows.

THEOREM 3.1. If z_n is solution of (3.15), then

$$|z_n| \leq G^{-1}\left(G(|z_l|) + \sum_{i=l}^{n-1} v_i\right) \tag{3.19}$$

and

$$|z_n| \geq G^{-1}\left(G(|z_l|) - \sum_{i=k}^{n-1} v_i\right) \tag{3.20}$$

for all $n \in N(l)$. for which $G(|z_l|) \pm \sum_{i=l}^{n-1} v_i$ is in the domain of G^{-1}, where G is defined by

$$\Delta G(u_n) = \frac{\Delta u_n}{g(u_n)}. \tag{3.21}$$

THEOREM 3.2. Let $g(u) = u$. If z_n is solution of (3.15), then

$$|z_n| \leq |z_l| \prod_{i=l}^{n-1} (1 + v_i) \tag{3.22}$$

and

$$|z_n| \geq |z_l| \prod_{i=l}^{n-1}(1 - v_i) \tag{3.23}$$

for all $n \in N(l)$.

Upper bounds in (3.19) and (3.22) can be obtained from (3.17) by applying the discrete Bihari and Gronwall inequalities [5], respectively. Lower bounds (3.20) and (3.23), however, are new and follow from (3.18) on using Theorem 2.1 and Theorem 2.2, respectively.

4 Oscillation of Higher Order Difference Equations

We shall consider the following m-th order delay difference equation

$$\Delta^m x_n + p_n \Delta^{m-1} x_n + q_n |x_{\sigma(n)}|^{\alpha} \operatorname{sgn}(x_{\sigma(n)}) = 0, \quad 0 < \alpha < 1, \quad n \in N(n_0), \tag{4.24}$$

where $\Delta^m x_n$ means $\Delta(\Delta^{m-1} x_n)$ for $m > 1$. We shall assume that the sequences $\{p_n\}$ and $\{q_n\}$ are nonnegative having infinitely many nonzero terms, $p_n < 1$ for $n \in N(n_0)$, $\sigma(n) \in N(n_0)$, and $\sigma(n) \leq n$ with $\lim_{n\to\infty} \sigma(n) = \infty$.

A solution $\{x_n\}$ of (4.24) is called oscillatory if for a given integer $n_1 \geq n_0$ there exists a $k \in N(n_1)$ such that $x_k\, x_{k+1} \leq 0$; otherwise the solution is said to be nonoscillatory.

Recently, the present author [4], proved that

$$\sum^{\infty} [\sigma(n)]^{\alpha(n-1)} q_n = \infty \tag{4.25}$$

is a necessary and sufficient condition for every solution of

$$\Delta^m x_n + q_n |x_{\sigma(n)}|^{\alpha} \operatorname{sgn}(x_{\sigma(n)}) = 0$$

to be oscillatory when m is even, and to be either oscillatory or $\lim_{n\to\infty} \Delta^j x_n = 0$ for $j = 0, 1, 2, \ldots, m-1$ when m is odd. Here $n^{(s)}$ denotes the usual factorial function; that is, $n^{(s)} = n(n-1)\ldots(n-s+1)$, $n^{(0)} = 1$, and $n^{\alpha(s)} = [n^{(s)}]^{\alpha}$.

It is of both theoretical and practical interest to know the effect of a middle term on the oscillatory character of the solutions. We will show that the above conclusion still remains valid for the solutions of (4.24), if, in addition to (4.25), condition (4.26) is also satisfied. First we present a lemma which we will make use of in the proof of our oscillation theorem. The proof will be accomplished by the help of Theorem 2.2.

LEMMA 4.1. Suppose that $\{x_n\}$ is a nonoscillatory solution of (4.24) and for any fixed $M \in N(n_0)$,

$$\lim_{n\to\infty} \sum_{j=M}^{n-1} \prod_{i=M}^{j-1} (1 - p_i) = \infty. \tag{4.26}$$

Then there exists $n_1 \in N(n_0)$ such that for $n \in N(n_1)$,

$$x_n \Delta^{m-1} x_n > 0. \tag{4.27}$$

PROOF. Let $\{x_n\}$ be an eventually positive solution of (4.24). Suppose that $\{\Delta^{m-1}x_n\}$ is oscillatory. Then, given any $K \in N(n_0)$ there exists $i \geq K$ such that

$$\Delta^{m-1}x_i < 0 \quad \text{and } \Delta^{m-1}x_{i+1} > 0. \tag{4.28}$$

From (4.24),

$$\Delta^m x_i + p_i \Delta^{m-1} x_i \leq 0$$

or

$$\Delta^{m-1}x_{i+1} \leq (1-p_i)\Delta^{m-1}x_i < 0$$

which clearly contradicts (4.28). Therefore $\{\Delta^{m-1}x_n\}$ must be nonoscillatory. We shall show that $\Delta^{m-1}x_n$ is eventually positive. Assume on the contrary that there exists $k_1 \in N(n_0)$ such that $\Delta^{m-1}x_n < 0$ for all $n \in N(k_1)$. From (4.24), we have

$$\Delta^{m-1}x_n\Delta^m x_n + p_n(\Delta^{m-1}x_n)^2 \geq 0, \quad n \in N(k_1). \tag{4.29}$$

In view of

$$\Delta(\Delta^{m-1}x_n)^2 = (\Delta^m x_n)^2 + 2\Delta^{m-1}x_n\Delta^m x_n$$

and

$$(\Delta^m x_n)^2 \geq p_n^2(\Delta^{m-1}x_n)^2,$$

it follows from (4.29) that

$$\Delta(\Delta^{m-1}x_n)^2 \geq (p_n^2 - 2p_n)(\Delta^{m-1}x_n)^2, \quad n \in N(k_1). \tag{4.30}$$

Summing (4.30) from k_1 to $n-1$ leads to

$$(\Delta^{m-1}x_n)^2 \geq (\Delta^{m-1}x_{k_1})^2 - \sum_{i=k_1}^{n-1}(2\,p_i - p_i^2)(\Delta^{m-1}x_i)^2. \tag{4.31}$$

Employing Theorem 2.2, we obtain

$$(\Delta^{m-1}x_n)^2 \geq (\Delta^{m-1}x_{k_1})^2 \prod_{i=k_1}^{n-1}(1-2p_i+p_i^2),$$

and hence

$$\Delta^{m-1}x_n \leq \Delta^{m-1}x_{k_1}\prod_{i=k_1}^{n-1}(1-p_i). \tag{4.32}$$

In view of (4.26), we may conclude from (4.32) that

$$\lim_{n\to\infty}\Delta^{m-2}x_n = -\infty,$$

which clearly is a contradiction with $\{x_n\}$ being eventually positive.

A similar proof can easily be given if $\{x_n\}$ is eventually negative. Thus, the proof is complete.

THEOREM 4.1. Assume that (4.25) and (4.26) are satisfied, and $0 \leq p_n < 1$. Then every solution $\{x_n\}$ of (4.24) is oscillatory when m is even, and is either oscillatory or satisfies $\lim_{n\to\infty} \Delta^j x_n = 0$ for $j = 0, 1, 2, \ldots, m-1$ when m is odd.

PROOF. Let $\{x_n\}$ be a nonoscillatory solution of (4.24). We may assume that $\{x_n\}$ is eventually positive. The proof when $\{x_n\}$ is eventually negative is similar. Because of Lemma 4.1, $\{\Delta^{m-1}x_n\}$ must also be eventually positive. This implies that there exists $n_1 \geq n_0$ such that for $n \geq n_1$,

$$\Delta^m x_n + q_n|x_{\sigma(n)}|^\alpha \mathrm{sgn}(x_{\sigma(n)}) \leq 0. \tag{4.33}$$

The remainder of the proof follows easily from (4.33) by using the arguments developed in [4].

The above result can easily be generalized to obtain an oscillation theorem for neutral type difference equations of the form

$$\Delta^m z_n + p_n\Delta^{m-1}z_n + q_n|x_{\sigma(n)}|^\alpha \mathrm{sgn}(x_{\sigma(n)}) = 0, \quad 0 < \alpha < 1, \quad n \in N(n_0), \tag{4.34}$$

where $z_n = x_n + a_n x_{n-r}$ with $0 \leq a_n < 1$ and $r > 0$.

THEOREM 4.2. Let (4.26) be satisfied, and for any fixed $L \geq 0$

$$\sum_{n=L}^{\infty} [\sigma(n)]^{\alpha(n-1)}[1 - a_{n-r}]^\alpha q_n = \infty.$$

If m is even, then every solution $\{x_n\}$ of (4.34) is oscillatory.

PROOF. Let $\{x_n\}$ be an eventually positive solution of (4.34). In view of (4.26), as in Lemma 4.1, one can show that $\{\Delta^{m-1}z_n\}$ is eventually positive. Since m is even, it follows that $\{\Delta z_n\}$ is also eventually positive, see [4, Lemma 1]. Then,

$$z_n = x_n + a_n x_{n-r} \leq x_n + a_n z_n$$

and so

$$x_{n-k} \geq (1 - a_{n-k})z_{n-k}.$$

It follows that

$$\Delta^m z_n + p_n\Delta^{m-1}z_n + q_n(1 - a_{n-k})^\alpha z_{n-k}^\alpha \leq 0.$$

The remainder of the proof is similar to that of Theorem 4.1.

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