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Auto-Bäcklund Transformation And Exact Analytical Solutions For The Kupershmidt Equation *

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Abstract

The homogenous balance method in [1,2] is improved and applied to the Kupershmidt equation. As a result, some families of exact analytical solutions are derived.

1 Introduction

Wang in [1,2] proposed a homogenous balance method (HBM) for constructing soliton solutions. The purpose of the HBM is to search for exact solution of nonlinear evolution equations or nonlinear partial differential equation. Zhang in [3,4] generalized Wang's result and obtained multi-soliton solutions for some equations.

The HBM has two main step: the first one is to set the order of the highest linear item in the equation in concern equal to the order of the highest nonlinear item. The second one is to set up nonlinear transformations.

For a given nonlinear partial differential equations, say, in two variables x, t

$$P(u, u_t, u_x, u_{xx}, u_{xt}, \cdots) = 0, \tag{1}$$

where P is a polynomial function and the subscripts denote partial derivatives. Wang obtained the following relation from the first step of the homogenous balance method

$$u(x,t) = f(\phi, \phi_t, \phi_x, \phi_{xt}, \phi_{xx}, \cdots) + u_0, \tag{2}$$

$$F_i(\phi_1, \phi_t, \phi_x, \phi_{xt}, \phi_{xx}, \cdots) = 0,\ i = 1, 2, \cdots, n, \tag{3}$$

where $\phi = \phi(x,t)$, f is a known function and $F_1, \cdots, F_n$ are homogenous functions with respect to various derivatives of ϕ. In order to search for the function ϕ, Wang assumed that $\phi(x,t) = 1 + e^{\alpha x + \beta t + \gamma}$. Then solitary wave solutions were derived. However, in view of the homogenous property of (3), we can make more formal assumptions, so that additional exact solutions for (1) can be found. These solutions contain the results of Wang and Zhang.

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2 Exact Solutions for Kupershmidt equation

As illustrative examples, we consider the following Kupershmidt equation [4,5]

$$u_t + uu_x + H_x + \delta u_{xx} = 0, \tag{4}$$

$$H_t + (Hu)_x - \delta H_{xx} = 0. \tag{5}$$

In order to 'balance' the orders of u_{xx} and uu_x, H_{xx} and $(Hu)_x$, suppose equations (4) and (5) have the following transformations

$$H = f''\phi_x^2 + f'\phi_{xx} + H_1(x,t), \tag{6}$$

$$u = f'\phi_x + u_1(x,t), \tag{7}$$

where H_1 and u_1 are the solutions of equations (4) and (5). By taking $H_1 = 0$, $u_1 = d$ where d is a constant, it is clear that $(0, d)$ is also a solution of equations (4) and (5). Therefore equations (6) and (7) are transformed into the following form

$$H = f''\phi_x^2 + f'\phi_{xx}, \tag{8}$$

$$u = f'\phi_x + d. \tag{9}$$

After substituting (8) and (9) into equations (4) and (5), with the help of Maple, we may obtain

$$\begin{aligned} & [f'f'' + (1+\delta)f^{(3)}]\phi_x^3 + (\phi_x\phi_t + d\phi_x^2 + 3\phi_x\phi_{xx} + 3\delta\phi_x\phi_{xx})f'' \\ & +(\phi_{xt} + d\phi_{xx} + \phi_{xxx} + \delta\phi_{xxx})f' + (f')^2\phi_x\phi_{xx} \\ = \ & 0, \end{aligned} \tag{10}$$

$$\begin{aligned} & (f'f^{(3)} - \delta f^{(4)} + f''^2)\phi_x^4 + (\phi_x^2\phi_t + d\phi_x^3 - 4\delta\phi_x^2\phi_{xxx})f^{(3)} \\ & +5\phi_x^2\phi_{xx}f'f'' + (2\phi_x\phi_{xt} + \phi_t\phi_{xx} + 3d\phi_x\phi_{xx} - 3\delta\phi_{xx}^2 - 4\delta\phi_x\phi_{xxx})f'' \\ & +(\phi_{xxt} + d\phi_{xxx} - \delta\phi_{xxxx})f' + (\phi_x\phi_{xxx} + \phi_{xx}^2)(f')^2 \\ = \ & 0. \end{aligned} \tag{11}$$

Setting coefficients of ϕ_x^3 in (10) and ϕ_x^4 in (11) to zero respectively, we get

$$f'f'' + (1+\delta)f''' = 0, \tag{12}$$

$$f'f''' + f''^2 - \delta f^{(4)} = 0. \tag{13}$$

From (12) and (13), we arrive at the special solution

$$f = -2\delta \ln \phi. \tag{14}$$

Therefore equations (4) and (5) have the following Auto-Bäcklund transformation:

$$H = -2\delta \frac{\partial^2}{\partial x^2} \ln \phi = -2\delta \frac{\phi_{xx}}{\phi} + 2\delta \left(\frac{\phi_x}{\phi}\right)^2, \tag{15}$$

$$u = -2\delta \frac{\partial}{\partial x} \ln \phi = -2\delta \frac{\phi_x}{\phi} + d. \tag{16}$$

From (14), we also have

$$f' f'' - \delta f^{(3)} = 0,$$

$$(f')^2 - 2\delta f'' = 0.$$

Then after substituting (14) and above relations into (10) and (11), and setting the coefficients of f'', f' in (10) and f''', f'', f' in (11) to zero, we see that ϕ satisfies the relation

$$\phi_x(\phi_t + d\phi_x - \delta\phi_{xx}) = 0, \tag{17}$$

$$(\phi_t + d\phi_x - \delta\phi_{xx})_x = 0, \tag{18}$$

$$\phi_x^2(\phi_t + d\phi_x - \delta\phi_{xx}) = 0, \tag{19}$$

$$2\phi_x(\phi_t + d\phi_x - \delta\phi_{xx})_x + \phi_x(\phi_t + d\phi_x - \delta\phi_{xx}) = 0, \tag{20}$$

$$(\phi_t + d\phi_x - \delta\phi_{xx})_{xx} = 0, \tag{21}$$

provided that

$$\phi_t + d\phi_x - \delta\phi_{xx} = 0. \tag{22}$$

From (22), we can guess at the form of solutions: (i) exponent function; (ii) product of trignometric function and exponent function; (iii) polynomial function; (iv) product of hyperbolic function and exponent function; (v) linear combination of the above four types of functions.

To find more solutions of (22), we need to discuss the following cases:

Case 1. First suppose (22) has the following formal solution

$$\phi_1(x,t) = l_0 + \Sigma_{i=1}^n l_i e^{(\alpha_i x + \beta_i t + \gamma_i)}, \tag{23}$$

where α_i and β_i are to be determined later; l_i and γ_i are arbitrary constants and $l_j \neq 0$ for $j = 0, 1, 2, \cdots, n$. Substituting (23) into (22) yields

$$\beta_i + d\alpha_i - \delta_i^2 = 0, \ i = 0, 1, 2, \cdots, n. \tag{24}$$

Finally from (23) and the Bäcklund transformation (15)-(16), we can obtain multi-soliton solutions for equations (4) and (5)

$$u(x,t) = -2\delta \frac{\Sigma_{i=1}^n \alpha_i l_i e^{[\alpha_i^2 x + (\delta\alpha_i^2 - d\alpha_i)t + \gamma_i]}}{l_0 + \Sigma_{i=1}^n l_i e^{[\alpha_i x + (\delta\alpha_i^2 - d\alpha_i)t + \gamma_i]}} + d, \tag{25}$$

$$H(x,t) = -2\delta \frac{\Sigma_{i=1}^n \alpha_i^2 l_i e^{[\alpha_i x + (\delta\alpha_i^2 - d\alpha_i)t + \gamma_i]}}{l_0 + \Sigma_{i=1}^n l_i e^{[\alpha_i x + (\delta\alpha_i^2 - d\alpha_i)t + \gamma_i]}} + 2\delta \left[\frac{\Sigma_{i=1}^n \alpha_i^2 l_i e^{[\alpha_i x + (\delta\alpha_i^2 - d\alpha_i)t + \gamma_i]}}{1 + \Sigma_{i=1}^n e^{[\alpha_i x + (\delta\alpha_i^2 - d\alpha_i)t + \gamma]}} \right]^2 \tag{26}$$

which are the solutions obtained by Zhang [5] under the conditions $l_0 = l_1 = \cdots = l_n = 1$. When $n = 1$, two sets of solitary wave solutions of equations (4) and (5) can be written as follows:

$$u_1(x,t) = -\alpha_1\delta \tanh\frac{1}{2}\left[\alpha_1 x + (\delta\alpha_1^2 - d\alpha_1)t + \gamma_1 + \ln\frac{l_1}{l_0}\right] - \frac{1}{2}\alpha_1 + d,\ l_1 l_0 > 0, \quad (27)$$

$$H_1(x,t) = -\frac{1}{2}\alpha_1^2\delta\text{sech}^2\frac{1}{2}\left[\alpha_1 x + (\delta\alpha_1^2 - d\alpha_1)t + \gamma_1 + \ln\frac{l_1}{l_0}\right],\ l_1 l_0 > 0), \quad (28)$$

which are solitary wave solutions obtained by Wang [2], and

$$u_2(x,t) = -\alpha_1\delta\coth\frac{1}{2}\left[\alpha_1 x + (\delta\alpha_1^2 - d\alpha_1)t + \gamma_1 + \ln\left(-\frac{l_1}{l_0}\right)\right] - \frac{1}{2}\alpha_1\delta + d,\ l_1 l_0 > 0, \quad (29)$$

$$H_2(x,t) = -\frac{1}{2}\alpha_1^2\delta\text{csch}^2\frac{1}{2}\left[\alpha_1 x + (\delta\alpha_1^2 - d\alpha_1)t + \gamma_1 + \ln\left(-\frac{l_1}{l_0}\right)\right],\ l_1 l_0 < 0 \quad (30)$$

which are singular solitary wave solutions missing in [2]. They may be called blow-up solutions.

Case 2. We can also find the following formal solutions of (22):

$$\phi_2(x,t) = a_0 + \sum_{i=1}^{n} \sin\sqrt{-(\frac{d}{2\delta})^2 + \frac{c_i}{d}}x \exp\left(\frac{d}{2\delta}x - c_i t + a_i\right), \quad (31)$$

and

$$\phi_3(x,t) = a_0 + \sum_{i=1}^{n} \cos\sqrt{-(\frac{d}{2\delta})^2 + \frac{c_i}{d}}x \exp\left(\frac{d}{2\delta}x - c_i t + a_i\right). \quad (32)$$

Substituting (31) and (32) into Bäcklund transformation (15) and (16) respectively, we have two kinds of new exact solution for equations (4) and (5)

$$u(x,t) = -2\delta\frac{\sum_{i=1}^{n}\left[\Psi_i\cos\Psi_i x + \frac{d}{2\delta}\sin\Psi_i\right]\exp\left(\frac{d}{2\delta}x - c_i t + a_i\right)}{a_0 + \Sigma_{i=1}^{n}\sin\Psi_i x\exp\left(\frac{d}{2\delta}x - c_i t + a_i\right)} + d,$$

$$\begin{aligned} H(x,t) \quad = \quad & -2\delta\frac{\Sigma_{i=1}^{n}\left[(2(\frac{d}{2\delta})^2 - \frac{c_i}{d})\sin\Psi_i x + \frac{d}{2\delta}\Psi_i\cos\Psi_i x\right]\exp\left(\frac{d}{2\delta}x - c_i t + a_i\right)}{a_0 + \Sigma_{i=1}^{n}\sin\Psi_i x\exp\left(\frac{d}{2\delta}x - c_i t + a_i\right)} \\ & +2\delta\frac{\left[\Sigma_{i=1}^{n}\Psi_i\cos\Psi_i x + \frac{d}{2\delta}\sin\Psi_i x\exp(\frac{d}{2\delta} - c_i t + a_i)\right]^2}{(a_0 + \Sigma_{i=1}^{n}\sin\Psi_i x\exp(\frac{d}{2\delta}x - c_i t + a_i))^2}, \end{aligned}$$

$$u(x,t) = -2\delta\frac{\Sigma_{i=1}^{n}[\frac{d}{2\delta}\cos\Psi_i - \Psi_i\sin\Psi_i]\exp(\frac{d}{2\delta}x - c_t + a_i)}{a_0 + \Sigma_{i=1}^{n}\cos\Psi_i x\exp(\frac{d}{2\delta}x - c_i t + a_i)} + d,$$

$$\begin{aligned} H(x,t) \quad = \quad & -2\delta\frac{\Sigma_{i=1}^{n}[(2(\frac{d}{2\delta})^2 - \frac{c_i}{d})\cos\Psi_i - \frac{d}{\delta}\Psi_i\sin\Psi_i]\exp(\frac{d}{2\delta}x - c_i t + d_i)}{a_0 + \Sigma_{i=1}^{n}\cos\Psi_i x\exp(\frac{d}{2\delta}x - c_i t + a_i)} \\ & +2\delta\frac{\left[\Sigma_{i=1}^{n}[\frac{d}{2\delta}\cos\Psi_i x - \Psi_i\sin\Psi_i x]\exp(\frac{d}{2\delta}x - c_i t + a_i)\right]^2}{(a_0 + \Sigma_{i=1}^{n}\cos\Psi_i x\exp(\frac{d}{2\delta}x - c_i t + a_i))^2}, \end{aligned}$$

where

$$\Psi_i = \sqrt{\frac{c_i}{d} - \left(\frac{d}{2\delta}\right)^2}.$$

Case 3. Suppose that ϕ in (22) has the following formal rational analytical solutions

$$\phi_4(x,t) = \sum_{i=1}^{n} k_i(t)x^i, \tag{33}$$

where $k_i(t)$, $i = 0, 1, 2, \cdots, n$, are functions to be determined later.

With the aid of Mathematica or Maple, substituting (33) into (22) and then setting the coefficients of x^i for $i = 0, 1, 2, \cdots, n$ to zero, we obtain a system of differential equations with respect to $k_0, k_1, ..., k_n$. Then k_i, $i = 0, 1, 2, \cdots, n$, can be determined. For convenience, we only consider the case where $n = 3$. Then

$$k_0(t) = c_3 - (2\delta c_1 + dc_2)t + \frac{1}{2}(d^2c_1 + 9\delta dc)t^2 - d^3ct^3,$$

$$k_1(t) = 3d^2ct^2 - (dc_1 + 3\delta c)t + c_2,$$

$$k_2(t) = -3dc + c_1,$$

$$k_3(t) = c,$$

where c, c_1, c_2, c_3 are arbitrary constants. Thus we get a new rational function solution for equations (4) and (5):

$$u(x,t) = -2\delta\frac{\phi_{4x}}{\phi_4} = -2\delta\frac{3d^3ct^2 - (dc_1 + 3\delta c)t + c_2 + (2c_1 - 6dct)x + 3cx^2}{\phi_4},$$

$$H(x,t) = -2\delta\frac{2c_1 - 6dct + 6cx}{\phi_4} + 2\delta\frac{[3d^2ct^2 - (dc_1 + 3\delta c)t + c_2 + (2c_1 - 6dct)x + 3cx^2]}{\phi_4^2},$$

where

$$\begin{aligned}\phi_4 &= [c_3 - (2\delta c_1 + dc_2)t + \frac{1}{2}(d^2c_1 + 9\delta dc)t^2 - d^3ct^3] \\ &\quad +[(3d^2ct^2 - (dc_1 + 3\delta c)t + c_2)x + (-3dct + c_1)x^2 + cx^3].\end{aligned}$$

Case 4. Suppose equation (22) has the following formal solutions

$$\phi_5 = \gamma_0 + \sum_{i=1}^{n} \cosh\sqrt{\left(\frac{d}{2\delta}\right)^2 + \left(\frac{c_i}{d}\right)}x \exp\left(\frac{d}{2\delta}x + c_it + \gamma_i\right), \tag{34}$$

$$\phi_6 = \gamma_0 + \sum_{i=1}^{n} \sinh\sqrt{\left(\frac{d}{2\delta}\right)^2 + \left(\frac{c_i}{d}\right)}x \exp\left(\frac{d}{2\delta}x + c_it + \gamma_i\right). \tag{35}$$

Then after substituting (34) and (35) into Bäcklund transformation (15) and (16), we have another kind of new exact solution for (4) and (5):

$$u(x,y) = -2\delta \frac{\Sigma_{i=1}^n \left(\Omega_i \sinh \Omega_i x + \frac{d}{2\delta}\cosh \Omega_i\right) \exp\left(\frac{d}{2\delta}x + c_i t + \gamma_i\right)}{\gamma_0 + \Sigma_{i=1}^n \cosh \Omega_i x \exp\left(\frac{d}{2\delta}x + c_i t + \gamma_i\right)} + d,$$

$$\begin{aligned} H(x,y) \quad = \quad & -2\delta \frac{\Sigma_{i=1}^n \left[\left(\frac{1}{2}\frac{d^2}{\delta^2} + \frac{c_i}{d}\right)\cosh \Omega_i x + \frac{d}{\delta}\Omega_i\right] \exp\left(\frac{d}{2\delta}x + c_i t + \gamma_i\right)}{\gamma_0 + \Sigma_{i=1}^n \cosh \Omega_i x \exp\left(\frac{d}{2\delta}x + c_i t + \gamma_i\right)} \\ & +2\delta \frac{\left[\Sigma_{i=1}^n \left(\Omega_i \sinh \Omega_i x + \frac{d}{2\delta}\cosh \Omega_i\right) \exp\left(\frac{d}{2\delta}x + c_i t + \gamma_i\right)\right]^2}{\left[\gamma_0 + \Sigma_{i=1}^n \cosh \Omega_i x \exp\left(\frac{d}{2\delta}x + c_i t + \gamma_i\right)\right]^2}, \end{aligned}$$

$$u(x,y) = 2\delta \frac{\Sigma_{i=1}^n \left(\Omega_i \cosh \Omega_i + \frac{d}{2\delta}\sinh \Omega_i\right) \exp\left(\frac{d}{2\delta}x + c_i t + \gamma_i\right)}{\gamma_0 + \Sigma_{i=1}^n \sinh \Omega_i x \exp\left(\frac{d}{2\delta}x + c_i t + \gamma_i\right)} + d,$$

$$\begin{aligned} H(x,y) \quad = \quad & -2\delta \frac{\Sigma_{i=1}^n \left[\left(\frac{1}{2}\frac{d^2}{\delta^2} + \frac{c_i}{d}\right)\sinh \Omega_i x + \frac{d}{\delta}\Omega_i \cosh \Omega_i\right] \exp\left(\frac{d}{2\delta}x + c_i t + \gamma_i\right)}{\gamma_0 + \Sigma_{i=1}^n \sinh \Omega_i x \exp\left(\frac{d}{2\delta}x + c_i t + \gamma_i\right)} \\ & +2\delta \frac{\left[\Sigma_{i=1}^n (\Omega_i \cosh \Omega_i x + \frac{d}{2\delta}\sinh \Omega_i) \exp\left(\frac{d}{2\delta}x + c_i t + \gamma_i\right)\right]^2}{\left[\gamma_0 + \Sigma_{i=1}^n \sinh \Omega_i x \exp\left(\frac{d}{2\delta}x + c_i t + \gamma_i\right)\right]^2}, \end{aligned}$$

where

$$\Omega_i = \sqrt{\left(\frac{d}{2\delta}\right)^2 + \frac{c_i}{d}}.$$

Case 5. Since (22) is a linear equation, it is easy to show that the following combined form of the above six solutions $\phi_i,\ i = 1, 2, \cdots, 6$, is also a solution of equation (22), namely,

$$\phi(x,t) = c_0 + \sum_{i=1}^{5} c_i \phi_i(x,t)$$

where $c_1, \cdots, c_6$ are arbitrary constants. By using the Bäcklund transformation (15) and (16), we get solutions of equations (4) and (5) of the form

$$u(x,t) = -2\delta \frac{\Sigma_{i=1}^6 c_i \phi_i x}{c_0 + \Sigma_{i=1}^6 c_i \phi_i(x,t)} + d,$$

and

$$H(x,t) = -2\delta \frac{\Sigma_{i=1}^5 c_i \phi_{ixx}}{c_0 + \Sigma_{i=1}^5 c_i \phi_i(x,t)} + 2\delta \frac{(\Sigma_{i=1}^5 c_i \phi_{ix})^2}{(c_0 + \Sigma_{i=1}^5 c_i \phi_i(x,t))^2},$$

where $\phi_i (i = 1, 2, \cdots, 6)$ satisfy (23), (31)-(35).

3 Concluding Remarks

In summary, based on the improved homogenous balance method, by using Bächlund transformation, serval types of exact solutions for Kupershmidt equation are obtained. These solutions contain previously found solitary wave solutions and new rational function solution and periodic solution, etc.

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