Applied Mathematics E-Notes, 3(2003), 58-61 ©
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/
ISSN 1607-2510

An Oscillation Theorem For Higher Order Nonhomogeneous Superlinear Differential Equations *

Wan-Tong Li†, Sui Sun Cheng‡

Received 20 December 2002

Abstract

We show that subtle modifcations of the arguments in [1] can lead us to an oscillation criterion for a higher order superlinear nonhomogeneous differential equation which depends only on the behavior of the forcing function on a sequence of intervals.

In [1], Agarwal and Grace derive an oscillation theorem for the n-th order nonhomogeneous superlinear differential equation

$$y^{(n)}(t) + q(t)|y(t)|^{\beta-1}y(t) = f(t),\ \beta > 1, t \geq t_0, \tag{1}$$

where $n \geq 1$ and $q, f \in C([t_0,\infty);\mathbf{R})$. Besides the assumption $q(t) < 0$ for $t \geq t_0$, their result also requires the global behavior of the function f on $[t_0,\infty)$. By means of the following subtle modifications, we will obtain an oscillation result that only requires behaviors of q and f on a sequence of intervals.

Recall first that a solution of (1) is a function $y : [T_y,\infty) \to R$ for some $T_y \geq t_0$, which has the property $y \in C^{(n)}[T_y,\infty)$ and satisfies (1). We restrict our attention only to the nontrivial solution $y(t)$ of (1), i.e., to the solution $y(t)$ such that $\sup\{|y(t)| : t \geq T\} > 0$ for all $T \geq T_y$. A nontrivial solution of (1) is called oscillatory if it has arbitrary large zeros.

Let $D(a, b)$ be the set of all functions H in $C^{(n)}[a, b]$ such that $H(t) > 0$ for $t \in (a, b)$ and $H^{(j)}(a) = H^{(j)}(b) = 0$ for $0 \leq j \leq n-1$.

THEOREM 1. Suppose that for any $T \geq t_0$, there exist $T \leq s < \tau$ such that $q(t) < 0$ on $[s,\tau]$ and $f(t) \geq 0$ for $t \in [s,\tau]$. If there exists $H \in D(s,\tau)$ such that

$$\int_s^\tau H(t)f(t)dt > (\beta-1)\,\beta^{\beta/(1-\beta)} \int_s^\tau \left(\frac{\left|H^{(n)}(t)\right|^\beta}{H(t)}\right)^{1/(\beta-1)} |q(t)|^{1/(1-\beta)}\,dt, \tag{2}$$

then Eq.(1) cannot have an eventually positive solution.

PROOF. We will need the well known fact that if A and B are nonnegative and $\beta > 1$, then $A^\beta + (\beta-1)B^\beta \geq \beta AB^{\beta-1}$ and equality holds if and only if $A = B$. Now

*Mathematics Subject Classifications: 34C10, 34C15.

†Department of Applied Mathematics, Gansu University of Technology, and Department of Mathematics, Lanzhou University, Lanzhou, 730000, P. R. China.

‡Department of Mathematics, Tsing Hua University, Hsinchu, Taiwan 30043, R. O. China

suppose that $y(t)$ is an eventually positive solution which is positive, say $y(t) > 0$ when $t \geq T_0 \geq t_0$ for some T_0 depending on the solution $y(t)$. By assumption, we can choose $s, \tau \geq T_0$ so that $f(t) \geq 0$ on the interval $I = [s, \tau]$ with $s < \tau$. On the interval I, we multiply Eq.(1) by $H(t)$ for $t \geq t_0$ and integrate from s to τ, we obtain

$$\begin{aligned} \int_s^\tau H(t)f(t)dt &= \int_s^\tau H(t)y^{(n)}(t)dt + \int_s^\tau H(t)q(t)\left|y(t)\right|^{\beta-1} y(t)dt \\ &= \int_s^\tau H(t)y^{(n)}(t)dt - \int_s^\tau H(t)\left|q(t)\right| y^\beta(t)dt. \end{aligned} \tag{3}$$

Now, since

$$\int_s^\tau H(t)y^{(n)}(t)dt = -\int_s^\tau H'(t)y^{(n-1)}(t)dt = ... = (-1)^n \int_s^\tau H^{(n)}(t)y(t)dt,$$

thus $\int_s^\tau H(t)y^{(n)}(t)dt$ is equal to $\int_s^\tau H^{(n)}(t)y(t)dt$ if n is even and when n is odd, it is equal to $-\int_s^\tau H^{(n)}(t)y(t)dt$. Hence

$$\int_s^\tau H(t)f(t)dt = \int_s^\tau H^{(n)}(t)y(t)dt - \int_s^\tau H(t)\left|q(t)\right| y^\beta(t)dt, \text{ if } n \text{ is even,}$$

and

$$\int_s^\tau H(t)f(t)dt = -\int_s^\tau H^{(n)}(t)y(t)dt - \int_s^\tau H(t)\left|q(t)\right| y^\beta(t)dt, \text{ if } n \text{ is odd.}$$

But then

$$\int_s^\tau H(t)f(t)dt \leq \int_s^\tau \left|H^{(n)}(t)\right| y(t)dt - \int_s^\tau H(t)\left|q(t)\right| y^\beta(t)dt.$$

Set

$$A = \left[H(t)\left|q(t)\right|\right]^{1/\beta} y(t),$$

and

$$B = \left[\frac{1}{\beta}\left|H^{(n)}(t)\right| \left(H(t)\left|q(t)\right|\right)^{-1/\beta}\right]^{1/(\beta-1)},$$

then in view of the inequality mentioned above, we see that

$$\int_s^\tau H(t)f(t)dt \leq (\beta-1)\,\beta^{\beta/(1-\beta)} \int_s^\tau \left(\frac{\left|H^{(n)}(t)\right|^\beta}{H(t)}\right)^{1/(\beta-1)} \left|q(t)\right|^{1/(1-\beta)} dt,$$

which contradicts our assumption (2). The proof is complete.

EXAMPLE 1. Consider the differential equation

$$y'(t) + q\left|y(t)\right|^2 y(t) = \sin t, \tag{4}$$

where q is a negative constant to be determined. The forcing function $\sin t$ is positive on $[2k\pi, 2k\pi + \pi]$ for $k = 0, 1, 2, ...$. Let $H(t) = \sin t$. Set $s = 2k\pi$ and $\tau = (2k+1)\pi$ where k is a sufficiently large integer. Then

$$\int_s^\tau H(t)f(t)dt = \int_0^\pi \sin^2 t dt = \frac{\pi}{2} > 0,$$

and

$$\begin{aligned} & (\beta - 1)\, \beta^{\beta/(1-\beta)} \int_s^\tau \left(\frac{|H'(t)|^\beta}{H(t)} \right)^{1/(\beta-1)} |q|^{1/(1-\beta)}\, dt \\ = \quad & 2 \times 3^{-3/2}\, |q|^{-1/2} \int_0^\pi \left(\frac{|\cos t|^3}{\sin t} \right)^{1/2} dt \\ = \quad & 2 \times 3^{-3/2}\, |q|^{-1/2} \times 3.7081..., \end{aligned}$$

where we have used the fact that the singular integral

$$\int_0^{\pi/2} \left(\frac{|\cos t|^3}{\sin t} \right)^{1/2} dt$$

exists in view of

$$\lim_{x \to 0+} \frac{x^{1/2}(\cos x)^{3/2}}{(\sin x)^{1/2}} = 1,$$

and its numerical value is 1.8541...

In order that

$$\frac{\pi}{2} > 2 \times 3^{-3/2}\, |q|^{-1/2} \times 3.7081...,$$

it is sufficient that

$$|q|^{1/2} > \frac{4 \times 3^{-3/2} \times 3.7081...}{\pi} \approx 0.90861...$$

Thus, when $q < -(0.90861...)^2$, Eq. (4) cannot have an eventually positive solution.

Similarly, the differential equation

$$x'(t) + r\, |x(t)|^2\, x(t) = -\sin t \tag{5}$$

cannot have an eventually positive solution by taking $H(t) = -\sin t$ and $s = (2k+1)\pi$ and $\tau = (2k+2)\pi$, and $r < -(0.90861...)^2$.

Since an eventaully positive solution of (4) is an eventually positive solution of (5), thus when $q < -(0.90861...)^2$, every solution of (4) oscillates.

We remark that in eqaution (4), we may replace the constant q with a function $q(t)$ such that $q(t) < 0$ on each $[2k_i\pi, 2(k+1)\pi_i]$, where $\{k_i\}$ is an unbounded subsequence of $\{1, 2, 3, ...\}$.

We remark further that the results of Agarwal and Grace [1] cannot be applied to Eq.(4), since

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{t_0}^{t} (t-s)^m \sin t dt = \limsup_{t\to\infty} \frac{-1}{t^m} (t-t_0)^m \cos t_0 \neq +\infty,$$

and

$$\liminf_{t\to\infty} \frac{1}{t^m} \int_{t_0}^{t} (t-s)^m \sin t dt = \liminf_{t\to\infty} \frac{-1}{t^m} (t-t_0)^m \cos t_0 \neq -\infty.$$

Finally, we remark that the same arguments in the proof of Theorem 1 will enable us to derive the following integral type condition: Let $q \in C[a,b]$ such that $q(t) < 0$ for $a < t < b$ and let $y \in C^{(n)}[a,b]$ such that $y(t) > 0$

$$(Ly)(t) \equiv y^{(n)}(t) + q(t)y^{\beta}(t) \geq 0, \ \beta > 1,$$

for $a \leq t \leq b$. Then for any $H \in D(a,b)$, we have

$$\int_a^b H(t)(Ly)(t)dt \leq (\beta-1)\beta^{\beta/(1-\beta)} \int_a^b \left(\frac{\left|H^{(n)}(t)\right|^{\beta}}{H(t)} \right)^{1/(\beta-1)} |q(t)|^{1/(1-\beta)} dt,$$

where equality holds only if

$$H^{(n)}(t) = (-1)^{n+1}\beta q(t) y^{\beta-1}(t) H(t), \ a < t < b.$$

Acknowledgment. The first author is supported by the NNSF of China (10171040), the NSF of Gansu Province of China (ZS011-A25-007-Z), the Foundation for University Key Teacher by the Ministry of Education of China, and the Teaching and Research Award Program for Outstanding Young Teachers in Higher Education Institutions of Ministry of Education of China.

References

[1] R. P. Agarwal and S. R. Grace, Forced oscillation of n^{th}-order nonlinear differential equations, Appl. Math. Lett., 13(2000), 53-57.