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A NOTE ON GENERALIZED PSEUDO-RICCI SYMMETRIC MANIFOLD

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Abstract. The object of the present paper is to introduce the notion of generalized pseudo-Ricci symmetric space with a non-trivial example. The beauty of such space is that it has the flavour of Ricci symmetric space, Ricci recurrent space, generalized Ricci recurrent space and pseudo-Ricci symmetric space. Furthermore, having found a faulty example in [14], the present paper attempts to construct a new example of pseudo-Ricci symmetric manifold.

1. Introduction

In the sense of Chaki, a non-flat n-dimensional Riemannian manifold (M^n, g), $(n > 3)$ is said to be a pseudo-Ricci symmetric manifold [9], if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the following equation

$$(\nabla_X S)(Y, U) = 2B(X)S(Y, U) + B(Y)S(X, U) + B(U)S(X, Y) \tag{1}$$

for all vector fields $X, Y, U \in \chi(M^n)$, where B is a nonzero 1-form defined by $B(X) = g(X, \varrho)\ \forall X$, where ϱ is called the associated vector field to the 1-form, $\chi(M^n)$ denotes the Lie algebra of all smooth vector fields over $C^\infty(M^n)$ on the manifold M^n and ∇ is the operator of the covariant differentiation with respect to the metric tensor g. The local expression of the above equation is

$$R_{ik,l} = 2B_l R_{ik} + B_i R_{kl} + B_k R_{il},$$

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where B_l is non-zero co-vectors and comma followed by indices denotes the covariant differentiation with respect to the metric tensor g. An n-dimensional manifold of this kind is abbreviated by $(PRS)_n$.

Keeping the tune of Dubey [11], in the present paper, we attempt to introduce the notion of *generalized pseudo-Ricci symmetric manifold* which is abbreviated by $(GPRS)_n$-manifold and defined as follows. A non-flat n-dimensional Riemannian manifold $(M^n, g)(n > 3)$ is termed as generalized pseudo-Ricci symmetric manifold if its Ricci tensor S of type $(0, 2)$ is not identically zero and fulfills the identity

$$(2)\quad \begin{aligned}(\nabla_X S)(Y,U) &= 2B(X)S(Y,U) + B(Y)S(X,U) + B(U)S(X,Y)\\ &\quad + 2C(X)]g(Y,U) + C(Y)g(X,U) + C(U)g(X,Y),\end{aligned}$$

where B and C are two non-zero 1-forms defined by $B(X) = g(X, \varrho)$ and $C(X) = g(X, \pi)$. The local expression of the above equation is

$$\begin{aligned}R_{ik,l} &= 2B_l R_{ik} + B_i R_{lk} + B_k R_{il}\\ &\quad + 2C_l g_{ik} + C_i g_{lk} + C_k g_{il},\end{aligned}$$

where B_l and C_l are two non-zero co-vectors. The beauty of such $(GPRS)_n$-space is that it has the flavour of
(1) Ricci symmetric space in the sense of Cartan (for $B = 0 = C$),
(2) Ricci recurrent space by Patterson [4] (for $B(X) \neq 0$, $B(Y) = B(U) = 0$ and $C(V) = 0$, $\forall V$),
(3) generalized Ricci recurrent space by De et al. [13] (for $B(X) \neq 0$, $C(X) \neq 0$ and $B(Y) = B(U) = 0 = C(Y) = C(U)$),
(4) pseudo-Ricci symmetric space [8] (for $B \neq 0$ and $C(V) = 0$, $\forall V$).

We structured our paper as follows: Section 2 is concerned with *generalized pseudo-Ricci symmetric manifold* and obtained some interesting results of conformally flat $(GPRS)_n$*-manifold*. In section 3, we cite an example of a manifold $(\mathbb{R}^4, g)$ which is a pseudo-Ricci symmetric for some choice of the 1-forms but fails to be a generalized pseudo-Ricci symmetric space. Finally, we observe that there exists a manifold $(\mathbb{R}^4, g)$ which is a generalized pseudo-Ricci symmetric space in some cases and a pseudo-Ricci symmetric space in some other cases depending on the choice of the 1-forms.

2. $(GPRS)_n$-*manifold*

In this section, we assume a non-flat n-dimensional Riemannian manifold (M^n, g), $(n > 3)$ to be a generalized pseudo-Ricci symmetric manifold. Next, if the 1-form B is co-directional to C, that is,

$C(X) = \phi B(X)\ \forall X$, where ϕ is a constant, then the relation (2) turns into

$$(\nabla_X Z)(Y,U) = 2B(X)Z(Y,U) + B(Y)Z(X,U) + B(U)Z(X,U),$$

where $Z(X,Y) = S(X,Y) + \phi g(X,Y)$ is a well known Z-tensor introduced in ([2], [3]). This leads to the following statement.

Theorem 1. *Every $(GPRS)_n$-manifold is a pseudo Z-symmetric manifold provided the vector fields associated to the 1-forms B and C are co-directional to each other.*

Definition 1. A non-flat Riemannian manifold $(M^n, g), (n > 3)$ is said to be a quasi-Einstein manifold [10] if its Ricci tensor S of type (0, 2) is not identically zero and satisfies the condition

$$S(X,Y) = \lambda g(X,Y) + \mu\psi(X)\psi(Y),$$

where $\lambda, \mu \in \mathbb{R}$ and ψ is a non-zero 1-form such that $g(X,U) = \psi(X)$, for all vector fields X, where U is a unit vector.

Now, contracting Y over U in (1) we obtain

$$\mathrm{dr}(X) = 2rB(X) + 2\bar{B}(X) + (2n+2)C(X),$$

where $\bar{B}(X) = S(X,\theta)$. Again, from (1), one can easily see

$$\begin{aligned}(\nabla_X S)(Y,U) - (\nabla_U S)(X,Y) &= B(X)S(Y,U) - B(U)S(X,Y) \\ &\quad + C(X)g(Y,U) - C(U)g(X,Y)\end{aligned} \tag{3}$$

after contraction, which leaves

$$\mathrm{dr}(X) = 2rB(X) - 2\bar{B}(X) + 2(n-1)C(X). \tag{4}$$

It is known ([7], p. 41) that a conformally flat (M^n, g) possesses the relation

$$(\nabla_X S)(Y,U) - (\nabla_U S)(X,Y) = \frac{[g(Y,U)dr(X) - g(X,Y)dr(U)]}{2(n-1)}. \tag{5}$$

By virtue of (3), (4), and (5) we find

$$\begin{aligned}&(n-1)[B(X)S(Y,U) - B(U)S(X,Y)] \\ &\qquad = [rB(X) - \bar{B}(X)]g(Y,U) - [rB(U) - \bar{B}(U)]g(X,Y),\end{aligned} \tag{6}$$

which yields

$$B(X)\bar{B}(U) = B(U)\bar{B}(X) \tag{7}$$

for $Y = \varrho$. Assuming that the Ricci tensor of the manifold is codazzi type (in the sense of [12]) and then using (4), we obtain, from (7), that

$$B(X)C(U) = B(U)C(X), \forall X \text{ and } U. \tag{8}$$

This motivate us to state the following.

Proposition 1. *In a conformally flat $(GPRS)_4$-manifold with codazzi type of Ricci tensor, the vector fields associated to the 1-forms B and C are co-directional.*

Again, for constant scalar curvature tensor (or codazzi type of Ricci tensor) by virtue of (4), (6), (8), we can easily get

$$S(Y,U) = -\frac{C(\varrho)}{B(\varrho)}g(Y,U) + \frac{1}{B(\varrho)}[rB(Y) + nC(Y)]B(U), \tag{9}$$

where $\frac{C(U)}{B(U)} = k$, $\forall U$. If the vector fields associated to the 1-forms B and C are co-directional, then (9) takes the following form

$$S(Y,U) = \alpha g(Y,U) + \beta B(Y)B(U).$$

This leads to the followings.

Theorem 2. *A conformally flat $(GPRS)_n$-manifold with codazzi type of Ricci tensor is a quasi-Einstein manifold.*

Corollary 1. *A conformally flat generalized pseudo-Ricci symmetric manifold with constant scalar curvature is a space of quasi constant curvature.*

3. Existence of $(\mathrm{PRS})_4$-space

In the example given in ([14], Example 3.1, p. 214-215) authors have calculated or assumed the value of the covariant derivatives corresponding to the vanishing components of the Ricci tensor R_{14}, R_{24}, and R_{34} (namely, $R_{14},_1$, $R_{24},_2$, and $R_{34},_3$) to be zero. But these values are found to be $R_{14},_1 = R_{24},_2 = R_{34},_3 = -\frac{8}{9(x^4)^{5/3}}$ which is non-zero. Consequently, their choice of the 1-forms

$$B_i(x) = \begin{cases} -\dfrac{1}{x^4} & \text{for } i = 4, \\ 0 & \text{otherwise;} \end{cases}$$

and the relations

$$R_{14},_1 = 2B_1R_{14} + B_1R_{14} + B_4R_{11}$$
$$R_{24},_2 = 2B_2R_{24} + B_2R_{24} + B_4R_{22}$$
$$R_{34},_3 = 2B_3R_{34} + B_3R_{34} + B_4R_{33}$$

do not stand as $R_{11} = R_{22} = R_{33} \neq 0$. Hence, the $(\mathbb{R}^4, g)$ underconsidered metric ([14], p. 214) cannot be a pseudo-Ricci symmetric manifold.

Example 1. Let $(\mathbb{R}^4, g)$ be a 4-dimensional Riemannian space endowed with the Riemannian metric g given by

$$ds^2 = g_{ij}dx^i dx^j = (dx^2)^2 + 2e^{x^2}[dx^1 dx^2 + dx^3 dx^4]$$

for $i, j = 1, 2, 3, 4$. The non-zero components of Riemannian curvature tensor, Ricci tensors and scalar curvature are (up to symmetry and skew-symmetry)

$$R_{2324} = \frac{1}{4}e^{x^2}, \qquad R_{22} = \frac{1}{2}, \qquad r = 0.$$

Covariant derivatives of Ricci tensors is expressed as

$$R_{22,2} = -1.$$

For the 1-form

$$B_i = \begin{cases} -1 & \text{for } i = 2, \\ 0 & \text{otherwise} \end{cases}$$

one can easily get the followings

$$\begin{aligned}
R_{12,k} &= 2B_k R_{12} + B_1 R_{k2} + B_2 R_{1k}, \\
R_{13,k} &= 2B_k R_{13} + B_1 R_{k3} + B_3 R_{1k}, \\
R_{14,k} &= 2B_k R_{14} + B_1 R_{k4} + B_4 R_{1k}, \\
R_{23,k} &= 2B_k R_{23} + B_2 R_{k3} + B_3 R_{2k}, \\
R_{24,k} &= 2B_k R_{24} + B_2 R_{k4} + B_4 R_{2k}, \\
R_{34,k} &= 2B_k R_{34} + B_3 R_{k4} + B_4 R_{3k}, \\
R_{11,k} &= 2B_k R_{11} + B_1 R_{k1} + B_1 R_{1k}, \\
R_{22,k} &= 2B_k R_{22} + B_2 R_{k2} + B_2 R_{2k}, \\
R_{33,k} &= 2B_k R_{33} + B_3 R_{k3} + B_3 R_{3k}, \\
R_{44,k} &= 2B_k R_{44} + B_4 R_{k4} + B_4 R_{4k},
\end{aligned}$$

for $k = 1, 2, 3, 4$.

Consequently, we can state the following.

Theorem 3. *There exists a manifold* $(\mathbb{R}^4, g)$ *which is pseudo-Ricci symmetric for the above mentioned choice of the 1-forms.*

It can be easily shown that the manifold $(\mathbb{R}^4, g)$ under consideration fails to be a generalized pseudo-Ricci symmetric space.

4. Existence of a $(GPRS)_4$-space

Example 2. Let $(\mathbb{R}^4, g)$ be a 4-dimensional Riemannian space endowed with the Riemannian metric g given by

$$ds^2 = g_{ij}dx^i dx^j = e^{-x^1}[(dx^1)^2 + (dx^2)^2 + 2dx^3 dx^4]$$

for $i, j = 1, 2, 3, 4$. The non-zero components of Riemannian curvature tensor, Ricci tensors and scalar curvature are (up to symmetry and skew-symmetry)

$$R_{2324} = -\frac{1}{4}e^{-x^1} = R_{3434}, \qquad R_{22} = -\frac{1}{2} = R_{34}, \qquad r = -\frac{3}{2}e^{x^1}.$$

Covariant derivatives of Ricci tensors (up to symmetry) are expressed as

$$R_{12,2} = R_{13,4} = R_{14,3} = -\frac{1}{4}, \qquad R_{22,1} = R_{34,1} = -\frac{1}{2}.$$

For the following choice of the 1-forms

$$B_i = \begin{cases} \frac{1}{4} & \text{for } i = 4, \\ 0 & \text{otherwise;} \end{cases}$$

$$C_i = \begin{cases} -\frac{e^{x^1}}{8} & \text{for } i = 4, \\ 0 & \text{otherwise} \end{cases}$$

one can verify the followings

$$R_{12,k} = (A_k+B_k)\,R_{12}+A_1R_{k2}+A_2R_{1k}+(C_k+D_k)\,g_{12}+C_1g_{k2}+C_2g_{1k},$$
$$R_{13,k} = (A_k+B_k)\,R_{13}+A_1R_{k3}+A_3R_{1k}+(C_k+D_k)\,g_{13}+C_1g_{k3}+C_3g_{1k},$$
$$R_{14,k} = (A_k+B_k)\,R_{14}+A_1R_{k4}+A_4R_{1k}+(C_k+D_k)\,g_{14}+C_1g_{k4}+C_4g_{1k},$$
$$R_{23,k} = (A_k+B_k)\,R_{23}+A_2R_{k3}+A_3R_{2k}+(C_k+D_k)\,g_{23}+C_2g_{k3}+C_3g_{2k},$$
$$R_{24,k} = (A_k+B_k)\,R_{24}+A_2R_{k4}+A_4R_{2k}+(C_k+D_k)\,g_{24}+C_2g_{k4}+C_4g_{2k},$$
$$R_{34,k} = (A_k+B_k)\,R_{34}+A_3R_{k4}+A_4R_{3k}+(C_k+D_k)\,g_{34}+C_3g_{k4}+C_4g_{3k},$$
$$R_{11,k} = (A_k+B_k)\,R_{11}+A_1R_{k1}+A_1R_{1k}+(C_k+D_k)\,g_{11}+C_1g_{k1}+C_1g_{1k},$$
$$R_{22,k} = (A_k+B_k)\,R_{22}+A_2R_{k2}+A_2R_{2k}+(C_k+D_k)\,g_{22}+C_2g_{k2}+C_2g_{2k},$$
$$R_{33,k} = (A_k+B_k)\,R_{33}+A_3R_{k3}+A_3R_{3k}+(C_k+D_k)\,g_{33}+C_3g_{k3}+C_3g_{3k},$$
$$R_{44,k} = (A_k+B_k)\,R_{44}+A_4R_{k4}+A_4R_{4k}+(C_k+D_k)\,g_{44}+C_4g_{k4}+C_4g_{4k},$$

where $k = 1, 2, 3, 4$.

This motivates us to state the following.

Theorem 4. *There exists a manifold $(\mathbb{R}^4, g)$ which is generalized pseudo-Ricci symmetric for the above choice of the 1-forms.*

However, for the 1-form

$$B_i = \begin{cases} \dfrac{1}{2} & \text{for } i = 1, \\ 0 & \text{otherwise} \end{cases}$$

one can easily obtain the followings

$$\begin{aligned}
R_{12},_k &= 2B_k R_{12} + B_1 R_{k2} + B_2 R_{1k}, \\
R_{13},_k &= 2B_k R_{13} + B_1 R_{k3} + B_3 R_{1k}, \\
R_{14},_k &= 2B_k R_{14} + B_1 R_{k4} + B_4 R_{1k}, \\
R_{23},_k &= 2B_k R_{23} + B_2 R_{k3} + B_3 R_{2k}, \\
R_{24},_k &= 2B_k R_{24} + B_2 R_{k4} + B_4 R_{2k}, \\
R_{34},_k &= 2B_k R_{34} + B_3 R_{k4} + B_4 R_{3k}, \\
R_{11},_k &= 2B_k R_{11} + B_1 R_{k1} + B_1 R_{1k}, \\
R_{22},_k &= 2B_k R_{22} + B_2 R_{k2} + B_2 R_{2k}, \\
R_{33},_k &= 2B_k R_{33} + B_3 R_{k3} + B_3 R_{3k}, \\
R_{44},_k &= 2B_k R_{44} + B_4 R_{k4} + B_4 R_{4k},
\end{aligned}$$

for $k = 1, 2, 3, 4$. In consequence of the above, one can see the following.

Theorem 5. *There exists a manifold* $(\mathbb{R}^4, g)$ *which a generalized pseudo-Ricci symmetric in some cases and a pseudo-Ricci symmetric in some other cases depending on the choice of the 1-forms.*

It is obvious that the manifold under consideration cannot be Ricci symmetric, Ricci recurrent and generalized Ricci recurrent.

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