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INTEGRAL REPRESENTATION OF BOUNDED AND ABSOLUTELY INTEGRABLE FUNCTIONS

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Abstract. In this paper, we obtain an integral representation formula for an even function, as a consequence, we show that if the function satisfying some conditions over $(0, 1)$ then it is completely characterized by its value in the neighborhood of 1.

1. Introduction

Let $f(x)$ be an even function over $-1 \le x \le 1$ and $G(u)$ is any even bounded function and integrable over the interval $-1 \le u \le 1$. In the first theorem we will show that the function $f(x)$ can be written as an integral representation of the function $G(u)$. Then we proved that if $f(x)$ is bounded and absolutely integrable over the interval $(0, 1-\epsilon)$, and satisfy the integral representation of $f(x)$ is bounded and absolutely integrable over the interval $(0, 1)$.

Integrable functions have frequently appeared in the literature of the last few years, for example, see [1], [3] and [4]. Before proving the main result we state and proof the following theorem.

Theorem 1.1. *Suppose $f(x)$ is even function over $-1 \le x \le 1$, and $G(u)$ be an even bounded integrable function over the interval $-1 \le u \le 1$. And that $G(u)$ together with its derivatives of all orders is continuous over the interval $(-1, 1)$ and that it vanishes with all its derivatives for $u = \mp 1$. Then, for $\int_{-1}^{1} G(u)du \neq 0$, we have*

$$f(x) = \lim_{n\to\infty} \frac{1}{n! \int_{-1}^{1} G(u)du} \int_{-1}^{1} \frac{d^{n+1}}{du^{n+1}} \Big[(u-x)^n \int_{-1}^{u} G(\bar{u})d\bar{u}\Big] f(u)du$$

Proof. Taylor series for $f(x)$ is given by

$$f(x) = f(u) + (x-u)f'(u) + \cdots + \frac{(x-u)^n}{n!} f^{(n)}(u) + \cdots$$

which we shall suppose uniformly convergent in the real argument u for $-1 \le u \le 1$ and for every x such that $-1 \le x \le 1$. Since $G(u)$ is bounded and integrable over the interval

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$-1 \leq u \leq 1$. Then

$$\begin{aligned} f(x)\int_{-1}^{1} G(u)du = \int_{-1}^{1} f(u)G(u)du + \int_{-1}^{1}(x-u)f'(u)G(u)du + \cdots \\ + \frac{1}{n!}\int_{-1}^{1}(x-u)^n f^{(n)}(u)G(u)du \end{aligned} \tag{1.1}$$

with the assumption that $G(u)$ together with its derivatives of all orders is continuous over the interval $(-1,1)$, and that it vanishes with all its derivatives for $u = \mp 1$. Then, integrating by parts yields

$$\begin{aligned} &\int_{-1}^{1}(x-u)^n f^{(n)}(u)G(u)du = \int_{-1}^{1}(x-u)^n G(u)d\{f^{(n-1)}(u)\} \\ &= \Big[(x-u)^n G(u) f^{(n-1)}(u)\Big]_{-1}^{1} - \int_{-1}^{1} f^{(n-1)}(u)d\{(x-u)^n G(u)\} \\ &= -\int_{-1}^{1}\frac{d}{du}\{(x-u)^n G(u)\}f^{(n-1)}(u)du = -\int_{-1}^{1}\frac{d}{du}\{(x-u)^n G(u)\}d\{f^{(n-2)}(u)\} \\ &= -\left\{\frac{d}{du}\Big[(x-u)^n G(u)\Big]f^{(n-2)}(u)\right\}_{-1}^{1} + \int_{-1}^{1} f^{(n-2)}(u)\left\{\frac{d^2}{du^2}\Big[(x-u)^n G(u)\Big]\right\}du \\ &\vdots \\ &= \int_{-1}^{1}\frac{d^n}{du^n}\Big[(u-x)^n G(u)\Big]f(u)du \end{aligned} \tag{1.2}$$

We shall now use the fact that $\int_{-1}^{1} G(u)du \neq 0$. We consequently obtain from (1.1) and (1.2) the formula

$$\begin{aligned} f(x) = \frac{1}{\int_{-1}^{1} G(u)du}\Big\{\int_{-1}^{1} G(u)f(u)du + \int_{-1}^{1}\frac{d}{du}[(u-x)G(u)]f(u)du + \cdots \\ + \frac{1}{n!}\int_{-1}^{1}\frac{d^n}{du^n}[(u-x)^n G(u)]f(u)du + \cdots\Big\} \end{aligned} \tag{1.3}$$

Set $\int_{-1}^{u} G(\tilde{u})d\tilde{u} = F(u)$. Then $F(u)$ will then characterized by the same properties as those have determined for $G(u)$, as to the existence and continuity of its derivatives, and as to the vanishing of the function and its derivatives at the end of the interval, except that $F(1) \neq 0$. Let us consider the expression

$$\frac{1}{n!}\frac{d^{n+1}}{du^{n+1}}[(u-x)^n F(u)]$$

Clearly,

$$\begin{aligned} \frac{1}{n!}\frac{d^{n+1}}{du^{n+1}}\Big[(u-x)^n F(u)\Big] &= \frac{1}{n!}\frac{d^n}{du^n}\frac{d}{du}\Big[(u-x)^n F(u)\Big] \\ &= \frac{1}{n!}\frac{d^n}{du^n}\Big[n(u-x)^{n-1}F(u) + (u-x)^n G(u)\Big] \end{aligned}$$

$$= \frac{1}{(n-1)!}\frac{d^n}{du^n}\Big[(u-x)^{n-1}F(u)\Big] + \frac{1}{n!}\frac{d^n}{du^n}\Big[(x-u)^nG(u)\Big]$$

$$= \frac{1}{(n-2)!}\frac{d^{n-1}}{du^{n-1}}\Big[(u-x)^{n-2}F(u)\Big]$$

$$+ \frac{1}{(n-1)!}\frac{d^{n-1}}{du^{n-1}}\Big[(x-u)^{n-1}F(u)\Big] + \frac{1}{n!}\frac{d^n}{du^n}\Big[(u-x)^nG(u)\Big]$$

$$= G(u) + \frac{d}{du}\Big[(u-x)G(u)\Big] + \cdots + \frac{1}{n!}\frac{d^n}{du^n}\Big[(u-x)^nG(u)\Big]$$

Hence (1.3) becomes

$$f(x) = \lim_{n\to\infty}\frac{1}{n!F(1)}\int_{-1}^{1}\frac{d^{n+1}}{du^{n+1}}\Big[(u-x)^nF(u)\Big]f(u)du \tag{1.4}$$

But $G(x)$ and $f(x)$ are even functions, so (1.4) becomes

$$f(x) = \lim_{n\to\infty}\frac{1}{n!F(1)}\int_{0}^{1}\frac{d^{n+1}}{du^{n+1}}\left\{\Big[(u-x)^n + (u+x)^n\Big]F(u)\right\}f(u)du \tag{1.5}$$

which ends the proof of the theorem. □

2. The Main Result

We will show that if we take our $G(u)$ the function $\exp(1/(u^2-1))$, the difference between (1.5) and an expression of the Fourier type is really essential. In particular, we will show that if $f(x)$ is bounded and absolutely integrable over $(0, 1-\epsilon)$, is zero over $(1-\epsilon, 1)$, and satisfies (1.5) at every point of $(0, 1)$, then it is identically zero over $(0, 1)$. From this it will follow at once that a function satisfying (1.5) over $(0, 1)$, bounded, and absolutely integrable, is completely characterized by its value in the neighborhood of 1. It is not even necessary, however, that the function satisfy (1.5) over the whole of $(0, 1)$; it is a sufficient condition that the following limit exist

$$\lim_{n\to\infty}\frac{2}{n!F(1)}\int_0^1\frac{d^{n+1}}{du^{n+1}}\Big[u^nF(u)\Big]f(u)du$$

.

Define the auxiliary function of a complex variable by

$$\phi(\xi) = \frac{2\xi}{\int_{-1}^{1}\exp(1/(x^2-1))dx}\int_0^{1-\epsilon}\exp(1/(\xi^2u^2-1))f(u)du$$

To find the singularities of the function $\phi(\xi)$ note that $|\exp(1/(\xi^2u^2-1))| \le |1/(\xi^2u^2-1)|$. If $|\xi u+1| > \eta, |\xi u-1| > \eta$, we have $|\exp(1/(\xi^2u^2-1))| < \exp(1/\eta^2)$. Now define the region Γ in the complex plane in which ξ lies when $|\xi u+1| > \eta, |\xi u-1| > \eta$ for every u in the interval $(0, 1-\epsilon)$. In the region Γ, the function $\exp(1/(\xi^2u^2-1))f(u)$ is uniformly bounded and integrable in u, so that $\phi(\xi)$ is defined. The related function

$$\phi'(\xi) = \frac{2}{\int_{-1}^{1}\exp(\frac{1}{x^2-1})dx}\int_0^{1-\epsilon}\exp\left(\frac{1}{\xi^2u^2-1}\right)f(u)du -$$

$$-\frac{4\xi^2}{\int_{-1}^{1}\exp(\frac{1}{x^2-1})dx}\int_0^{1-\epsilon}\frac{u^2}{(\xi^2u^2-1)^2}\exp\left(\frac{1}{\xi^2u^2-1}\right)f(u)du$$

may be proved to exist by a similar argument over the same region Γ. There is no difficulty in showing directly that

$$\lim_{|\lambda|\to 0}\frac{\phi(\xi+\lambda)-\phi(\xi)}{\lambda}=\phi'(\xi)$$

whenever ξ and $\xi+\lambda$ lie in the region Γ. Hence ϕ is analytic over Γ. Now let us consider $\phi(x/(1-y))$ as a function of y, given that $|x|<1/(1-\epsilon)$. It is clearly that ϕ is analytic in a neighborhood containing the origin, as is also $1/(1-y)\phi(x/(1-y))$. Let us put $\int_0^z \phi(\bar{z})d\bar{z}=\Phi(z)$, where the path of integration lies entirely within the circle of convergence of the Taylor series about the origin for $\phi(x)$. We shall then have

$$\begin{aligned}\frac{1}{1-y}\phi\left(\frac{x}{1-y}\right)&=\frac{\partial}{\partial x}\Phi(\frac{x}{1-y})\\ &=\Big[\frac{\partial}{\partial x}\Phi(\frac{x}{1-y})\Big]_{y=0}+y\Big[\frac{\partial^2}{\partial x\partial y}\Phi(\frac{x}{1-y})\Big]_{y=0}+\cdots\\ &+\frac{y^n}{n!}\Big[\frac{\partial^{n+1}}{\partial x\partial y^n}\Phi(\frac{x}{1-y})\Big]_{y=0}+\cdots\end{aligned}\tag{2.1}$$

Now let $x/(1-y)=z$, or $z=x+yz$. Then

$$\frac{\partial z}{\partial y}=\frac{x}{(1-y)^2}=z\frac{\partial z}{\partial x}$$

Hence,

$$\frac{\partial\Phi(z)}{\partial y}=\Phi'(z)\frac{\partial z}{\partial y}=z\Phi'(z)\frac{\partial z}{\partial x}=z\frac{\partial\Phi(z)}{\partial x}$$

Again,

$$\begin{aligned}\frac{\partial^2\Phi(z)}{\partial y^2}&=\frac{\partial}{\partial y}\left(z\frac{\partial\Phi(z)}{\partial x}\right)=\frac{\partial z}{\partial y}\frac{\partial\Phi(z)}{\partial x}+z\frac{\partial^2\Phi(z)}{\partial x\partial y}\\ &=\Phi'(z)\frac{\partial z}{\partial y}\frac{\partial z}{\partial x}+z\frac{\partial^2\Phi(z)}{\partial x\partial y}=\frac{\partial\Phi(z)}{\partial y}\frac{\partial z}{\partial x}+z\frac{\partial^2\Phi(z)}{\partial x\partial y}=\frac{\partial}{\partial x}\left(z^2\frac{\partial\Phi(z)}{\partial x}\right)\end{aligned}$$

In general,

$$\frac{\partial^n\Phi(z)}{\partial y^n}=\frac{\partial^{n-1}}{\partial x^{n-1}}\left(z^n\frac{\partial\Phi(z)}{\partial x}\right)$$

Hence,

$$\Big[\frac{\partial^{n+1}\Phi(z)}{\partial x\partial y^n}\Big]_{y=0}=\Big[\frac{\partial^n}{\partial x^n}\left(z^n\frac{\partial\Phi(z)}{\partial x}\right)\Big]_{y=0}=\frac{\partial^n}{\partial x^n}\Big[x^n\phi(x)\Big]$$

Formula (2.1) thus becomes

$$\frac{1}{1-y}\phi\left(\frac{x}{1-y}\right)=\phi(x)+y\frac{d}{dx}(x\phi(x))+\cdots+\frac{y^n}{n!}\frac{d^n}{dx^n}(x^n\phi(x))+\cdots\tag{2.2}$$

It has been given here for the purpose of showing that there is actually a region for which the two sides of (2.2) are identical, provided that as in the present case the radius of convergence of the MacLaurin series for $\phi(x)$ is greater than 1.

We now say that if $\lim_{n\to\infty}\frac{1}{n!}\Big[\frac{d^n}{dx^n}\left(x^n\phi(x)\right)\Big]_{x=1}$ exists, $\phi(x)$ is identically zero. To establish this, a consideration of the singularities of ϕ is sufficient. To begin with, ϕ is an odd function, and its singularities always occur in pairs. Again, we have already seen that all the singularities of ϕ lie on the real axis, with a modulus greater than $1/(1-\epsilon)$. Now, since we may write (2.2) in the form

$$\phi\left(\frac{1}{1-y}\right) = \phi(1) + y\left\{\left[\frac{d}{dx}(x\phi(x))\right]_{x=1} - \phi(1)\right\} + \cdots + y^n\left\{\frac{1}{n!}\left[\frac{d^n}{dx^n}(x^n\phi(x))\right]_{x=1} - \frac{1}{(n-1)!}\left[\frac{d^{n-1}}{dx^{n-1}}(x^{n-1}\phi(x))\right]_{x=1}\right\} + \cdots \tag{2.3}$$

and since this power series converges for $y = 1$, it follows that ϕ has no singularities on the finite positive real axis, and hence no singularities on the real axis at all, except possibly at infinity. The singularities at infinity, for a function with only one singularity must be single-valued (see, [3]).

Let $y \to 1$ along any path for which $\arg(1/(1-y))$ lies between $-\sin^{-1}\eta$ and $\sin^{-1}\eta$. Since the power series (2.3) converges to $\lim_{n\to\infty}\frac{1}{n!}\left[\frac{d^n}{dx^n}(x^n\phi(x))\right]_{x=1}$, if this quantity exists, it follows that

$$\lim_{y\to 1}\phi\left(\frac{1}{1-y}\right) = \lim_{n\to\infty}\frac{1}{n!}\left[\frac{d^n}{dx^n}(x^n\phi(x))\right]_{x=1}$$

The $\lim_{x\to 1}\phi(1/(1-y))$ will also exist if $y \to 1$ for any path for which $\arg(1/(1-y))$ lies between $\pi - \sin^{-1}\eta$ and $\pi + \sin^{-1}\eta$, since ϕ is odd.

Now consider the function $\phi(\xi)/\xi$, this has no singularities at the origin, and is uniformly bounded whenever $\arg(\xi)$ lies outside of the angles $(-\sin^{-1}\eta, \sin^{-1}\eta)$ and $(\pi - \sin^{-1}\eta, \pi + \sin^{-1}\eta)$. All this follows from the uniformly bounded and integrable character of $\exp(1/(\xi^2u^2-1))f(u)$. On the other hand, it follows from what we have just seen that if $\xi \to \infty$ along any path within the angles $(-\sin^{-1}\eta, \sin^{-1}\eta)$ and $(\pi - \sin^{-1}\eta, \pi + \sin^{-1}\eta)$, then $Lim_{\xi\to\infty}\phi(\xi)/\xi = 0$. It follows that $\phi(\xi)/\xi$ can neither have a pole nor an essential singularity anywhere, and so reduce to a constant, which can only be zero. Hence $\phi(\xi) \equiv 0$.

Now let $f(u) = \sum_{m=0}^{\infty} a_m u^m$. Then $G(u) = \sum_{m=0}^{\infty} m a_m u^{m-1}$ and

$$\begin{aligned}\phi(\xi) &= \frac{2\xi}{F(1)}\int_0^{1-\epsilon}\left\{\sum_{m=0}^{\infty} m a_m (\xi u)^{m-1}\right\} f(u)du \\ &= \sum_{m=0}^{\infty}\frac{2\xi^m}{F(1)}\int_0^{1-\epsilon} m a_m u^{m-1} f(u)du, \; |\xi| < \frac{1}{1-\epsilon}\end{aligned}$$

and so,

$$\begin{aligned}\frac{1}{n!}\left[\frac{d^n}{dx^n}(x^n\phi(x)\right]_{x=1} &= \\ &= \left\{\sum_{m=0}^{\infty}\frac{2(m+n)(m+n-1)\cdots(m+1)m}{n!F(1)}x^m\int_0^{1-\epsilon} a_m u^{m-1} f(u)du\right\}_{x=1} \\ &= \frac{2}{n!F(1)}\int_0^{1-\epsilon}\frac{d^{n+1}}{du^{n+1}}\left[u^n F(u)\right] f(u)du.\end{aligned}$$

That is the validity of (1.5) for $x = 0$ involves the identical vanishing of $\phi(\xi)$. In other words, if (1.5) holds,

$$\int_0^{1-\epsilon}\exp\left(\frac{1}{\xi^2u^2-1}\right) f(u)du = 0, \forall\xi. \tag{2.4}$$

Let us now consider the sequence of derivatives of $\exp(1/(\xi^2u^2-1))f(u)$. and note that the derivative is of the form

$$\Big[\frac{2A_1}{x-1}+\frac{2A_2}{(x-1)^3}+\cdots+\frac{2A_{2n-1}}{(x-1)^{2n-1}}\Big]\exp\left(\frac{1}{x-1}\right)$$

where the $A's$ are positive or negative integers. If we differentiate this expression we get

$$\Big[\frac{-2A_1}{(x-1)^2}-\frac{4A_2}{(x-1)^3}-\cdots$$
$$\mp\frac{2n}{(x-1)^{2n+1}}-\frac{2A_1}{(x-1)^3}-\cdots-\frac{2A_{2n-1}}{(x-1)^{2n-1}}\mp\frac{1}{(x-1)^{2n+2}}\Big]\exp(\frac{1}{x-1})$$

which is of the same form. Hence by mathematical induction, every derivative of $\exp(1/(1-x))$ is of this form. It follows that there is an integer k such that

$$\Big[\frac{d^n}{dx^n}\exp\left(\frac{1}{x-1}\right)\Big]_{x=0}=\frac{2k+1}{e}\neq 0$$

so that

$$\Big[\frac{d^{2n}}{dx^{2n}}\exp\left(\frac{1}{x^2-1}\right)\Big]_{x=0}\neq 0 \tag{2.5}$$

as is obvious from a comparison of the Taylor series for $\exp(1/(1-x))$ and $\exp(1/(x^2-1))$. It follows from (2.4) on differentiation that

$$0=\int_0^{1-\epsilon}\Big[\frac{\partial^{2n}}{\partial\xi^{2n}}\exp\left(\frac{1}{\xi^2u^2-1}\right)\Big]_{\xi=0}f(u)du=\int_0^{1-\epsilon}u^{2n}\Big[\frac{d^{2n}}{dx^{2n}}\exp\left(\frac{1}{x^2-1}\right)\Big]_{x=0}f(u)du$$

Hence by (2.5)

$$\int_0^{1-\epsilon}u^{2n}f(u)du=0,\quad \forall n.$$

it is a direct consequence from this and the fact that the even powers of u forms a complete set over the interval $(0,1-\epsilon)$ (as follows from Weierstrass's Theorem on polynomials representation) that except for a set of points of zero measure, $f(u)=0$ over $(0,1-\epsilon)$. From this and (1.5) it follows again that $f(u)=0$ everywhere over $(0,1)$. This complete the proof of our Theorem.

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