WEBSTER PSEUDO-TORSION FORMULAS
OF CR MANIFOLDS

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Abstract. In this article, we obtain a formula for Webster pseudo-torsion
for the link of an isolated singularity of an $n$-dimensional complex subvariety in
$\mathbb{C}^{n+1}$ and we present an alternative proof of the Li-Luk formula for Webster
pseudo-torsion for a real hypersurface in $\mathbb{C}^{n+1}$.

1. Introduction

The complete local invariants in the pseudoconformal geometry of a nondege-
nerate $CR$ manifold $M$ are defined on an $SU(p+1,q+1)$-bundle $Y$ over $M$, which
generalizes the bundle of $Q$-frame as a real hyperquadric [1]. To reduce the struc-
ture group, Webster singles out a real nowhere vanishing one form $\theta$ on $M$ which
annihilates the $CR$ structure of $M$. A $CR$ manifold $M$ with such a choice $\theta$ is called
a pseudohermitian manifold [6]. The contact form $\theta$ is called a pseudohermitian
structure. The structure group of the Chern bundle $Y$ is reduced to $U(p,q)$. In
[6], Webster showed there is a natural connection in the bundle $T^{1,0}M$ adapted
to $\theta$. This connection can be extended to a connection to $CTM$. To solve the
equivalence problem of pseudohermitian manifold, Webster derived the structure
equations for $M$, from which the Webster Ricci curvature and Webster torsion
tensor are defined. In [3], the author derived a formula for Webster pseudo-torsion
for a real hypersurface in $\mathbb{C}^{n+1}$. In this article, we derive a formula for Webster
pseudo-torsion for the link of an isolated singularity of an $n$-dimensional complex
subvariety in $\mathbb{C}^{n+1}$ and we present an alternative proof of the Li-Luk formula for
Webster pseudo-torsion for a real hypersurface in $\mathbb{C}^{n+1}$ [3]. The main idea of the
alternative proof is to describe the $CR$ structure using all Euclidean coordinates
$z^1, z^2, \ldots, z^{n+1}$ (see [39]). This new description of $CR$ structure using all Euclidean
coordinates is originated in [4]. In other words, we dispense with distinguishing
one coordinate, say $z^{n+1}$, such that $\frac{\partial r}{\partial z^{n+1}} \neq 0$, as is required in Chern-Moser
and subsequent works. The organization of this article is as follows. In Section 2 we
review pseudohermitian geometry following Webster and Tanaka. In Section 3 we
derive a key identity for Webster pseudo-torsion computation in subsequent

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sections. In Section 4 we present the alternative proof of the Li-Luk formula for Webster pseudo-torsion for a real hypersurface in \( \mathbb{C}^{n+1} \). In Section 5 we obtain an explicit formula for Webster pseudo-torsion for the link of an isolated singularity of a \( n \)-dimensional complex subvariety in \( \mathbb{C}^{n+1} \). To the best knowledge of the author, this formula obtained in Section 5 is a new result.

2. Pseudohermitian structures

In this section, we collect the basic facts on pseudohermitian geometry. Let \( M \) be a CR manifold with structure bundle \( T^{1,0}M \) satisfying \( T^{1,0}M \cap \overline{T^{1,0}} = \{0\} \) and \( [T^{1,0}M, T^{1,0}M] \subset T^{1,0}M \). Let \( T^{0,1}M := \overline{T^{1,0}} \). Set \( HM = \text{Re} (T^{1,0}M \oplus T^{0,1}M) \). \( HM \) is a 2\( n \) dimensional subbundle of \( TM \) which carries a complex structure \( J: HM \to HM \) given by \( J(X + \overline{X}) = i(X - \overline{X}) \) for \( X \in T^{1,0}M \). Let \( E \subset TM^* \) denote the real line subbundle which annihilates \( HM \). Assuming \( M \) is orientable, \( E \) has a global nowhere vanishing section \( \theta \). A choice of such a 1-form \( \theta \) is called a pseudohermitian structure on \( M \). The Levi form of \( \theta \) is the Hermitian form \( L_\theta \) on \( TM^{1,0} \) defined by

\[
L_\theta(V, \overline{W}) = L_\theta(\overline{W}, V) = -2 \text{id} \theta(V \wedge \overline{W}).
\]

For a nondegenerate (resp. strongly pseudoconvex) \( CR \) manifold, \( L_\theta \) is a nondegenerate (resp. positive definite) Hermitian form for any choice of \( \theta \). The choice of \( \theta \) determines a unique real vector field \( \xi \) transverse to \( HM \) such that \( \theta(\xi) = 1 \), \( \xi|d\theta = 0 \). An admissible coframe on an open subset of \( M \) is a set of complex \((1,0)\)-forms \( \{\theta^1, \ldots, \theta^n\} \) form basis for \( TM^{1,0} \) and satisfies \( \theta^\alpha(\xi) = 0 \). Then we have \( d\theta = i h_{\alpha \beta} \theta^\alpha \wedge \theta^\beta \) for some hermitian matrix of functions \( h_{\alpha \beta} \). In [6], Webster showed there is a natural connection in the bundle \( T^{1,0}M \) adapted to \( \theta \). This connection can be extended to a connection to \( CTM \). Webster showed that there are uniquely determined 1-forms \( \omega^\alpha_\beta, \tau^\beta \) on \( M \) satisfying

\[
\begin{align*}
(1) & \quad d\theta = i \theta^\gamma \wedge \theta^\overline{\gamma}, \\
(2) & \quad d\theta^\alpha = \theta^\beta \wedge \omega^\alpha_\beta + \theta \wedge \tau^\alpha, \\
(3) & \quad \omega^\alpha_\beta + \omega^\beta_\alpha = 0, \quad \text{where} \quad \omega^\alpha_\beta = \overline{\omega^\beta_\alpha}, \\
(4) & \quad \tau^\alpha = A_{\alpha \gamma} \theta^\gamma, \quad \text{where} \quad \tau^\alpha = \overline{\tau^\alpha},
\end{align*}
\]

with

\[
\begin{align*}
(5) & \quad A_{\alpha \gamma} = A_{\gamma \alpha}, \\
(6) & \quad h_{\alpha \overline{\beta}} = \delta_{\alpha \beta}.
\end{align*}
\]

This connection is called Webster connection. The curvature of the Webster connection, expressed in terms of the coframe is,

\[
\Omega^\alpha_\beta := d\omega^\alpha_\beta - \omega^\gamma_\beta \wedge \omega^\alpha_\gamma - i \theta^\beta \wedge \tau^\alpha + i \tau^\beta \wedge \theta^\alpha,
\]

\[
= R_{\beta \alpha \gamma \rho} \theta^\rho \wedge \theta^\sigma + W_{\beta \alpha \gamma \rho} \theta^\rho \wedge \theta - W_{\alpha \beta \gamma \rho} \theta^\rho \wedge \theta.
\]
where

(8) \[ R_{\beta\alpha\rho\sigma} = \overline{R}_{\alpha\beta\rho\sigma} = R_{\overline{\alpha}\overline{\beta}\rho\sigma}, \]

(9) \[ R_{\beta\alpha\rho\sigma} = R_{\rho\alpha\beta\sigma}, \]

(10) \[ W_{\alpha\rho\sigma} = W_{\sigma\rho\alpha}, \]

since by (6), \( \Omega_{\alpha} = \Omega_{\beta} \). By (4), (7), we have

(11) \[
d\omega_{\alpha} - \omega_{\alpha} \wedge \omega_{\beta} = -iA_{\alpha\gamma}\theta_{\gamma} \wedge \theta^{\alpha} + R_{\beta\alpha\rho\sigma} \theta_{\rho} \wedge \theta^{\sigma} + i\overline{A}_{\alpha\gamma} \theta^{\beta} \wedge \theta^{\overline{\gamma}} + W_{\beta\alpha\rho\sigma} \theta_{\rho} \wedge \theta - W_{\overline{\alpha}\overline{\beta}\rho\sigma} \theta_{\rho} \wedge \theta.
\]

We also put \( \Omega^\alpha = d\tau^\alpha - \tau^\beta \wedge \omega_{\beta} \),

(12) \[
\Omega^\alpha : = d\tau^\alpha - \tau^\beta \wedge \omega_{\beta} = W_{\alpha\rho\sigma} \theta_{\rho} \wedge \theta^{\sigma} - A_{\alpha\gamma} \tau^{\gamma} \wedge \theta + B_{\alpha\gamma} \theta^{\sigma} \wedge \theta,
\]

where

(13) \[ B_{\alpha\sigma} = B_{\overline{\sigma}\alpha}. \]

Let \((\xi, X_\alpha, X_{\overline{\alpha}})\) be the dual frame to \((\theta, \theta^\alpha, \theta^{\overline{\alpha}})\). Define an operator \( D \) locally by

(14) \[
DX_\alpha = \omega_{\alpha}^\beta X_\beta, \quad D : \Gamma(H(M)) \rightarrow \Gamma((T^*(M) \otimes H(M)).
\]

\( D \) defines a connection on \( H(M) \), see [6, p. 32]. We can define an hermitian metric \((\cdot,\cdot)\) in the fibres of \( H(M) \) by

(15) \[
(X_\alpha, X_\beta) = \delta_\alpha^\beta.
\]

Next, we turn to a formulation of the Webster connection by N. Tanaka [5]. We have \( T^{1,0}M = \{ X - iJX \mid X \in HM \} \) and using the decomposition \( C^{TM} = T^{1,0}M \oplus T^{0,1}M \oplus \mathbb{C}\xi \), we extend \( J \) to \( C^{TM} \) with \( J\xi = 0 \). Then we have

(16) \[
J^2X = -X + \theta(X)\xi, \quad X \in TM_x.
\]

For, let \( pr : C^{TM} \rightarrow C^{HM} \) be the natural projection. Any \( Y \in C^{TM} \) can be written as \( Y = pr(Y) + \theta(Y)\xi \) Then \( J^2Y = -pr(Y) = -Y + \theta(Y)\xi \). We put

(17) \[
\Omega = -d\theta.
\]

We define a tensor field on \( M \) by

(18) \[
g(X, Y) = \Omega(JX, Y).
\]

Then \( g(X, Y) = g(Y, X), \ g(JX, JY) = g(X, Y) \) and \( g \) is positive definite on \( HM \).

Recall \( T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \).

**Theorem 2.1** (N. Tanaka [5, p. 29]). There exists a unique affine connection

\[ \nabla : \Gamma(TM) \rightarrow \Gamma(TM \otimes TM^*) \]

on \( M \) such that

(1) The contact structure \( HM \) is parallel, i.e.,

(19) \[
\nabla_X \Gamma(HM) \subset \Gamma(HM) \quad \text{for any} \quad X \in \Gamma(TM).
\]
(2) The tensor field $\xi, j, \Omega$ are all parallel, i.e., $\nabla \xi = \nabla j = \nabla \Omega = 0$.

(3) The torsion $T$ of $\nabla$ satisfies:

$$T(X, Y) = -\Omega(X, Y) \xi,$$

$$T(\xi, JY) = -JT(\xi, Y), \quad X, Y \in HM_x.$$ 

Let $X, Y \in \Gamma(CHM)$. Denote by $[X, Y]_{HM}$ the $CHM$-component of $[X, Y]$ in the decomposition:

$$CTM = CHM \oplus C \otimes (TM/HM).$$

Also denote by $[X, Y]_{1,0}$ (resp. by $[X, Y]_{0,1}$) the $TM^{1,0}$ component (resp. the $TM^{0,1}$ component) of $[X, Y]_{HM}$ in the decomposition $CHM = TM^{1,0} \oplus TM^{0,1}$. $\nabla$ can be extended to a differential operator of $\Gamma(CTM)$ to $\Gamma(CTM) \otimes CTM^*$ in a natural manner. By (19), $\nabla J = 0$ and $T^{1,0}M = \{X - iJX \mid X \in HM\}$, we have

$$\nabla_X \Gamma(TM^{1,0}) \subset \Gamma(TM^{1,0}),$$

$$\nabla_X \Gamma(TM^{0,1}) \subset \Gamma(TM^{0,1}), \quad X \in \Gamma(CTM).$$

Then we have

**Proposition 2.2** ([5] p. 31). The extension $\nabla : \Gamma(CTM) \to \Gamma(CTM \otimes CTM^*)$ is given as follows. For $X, Y \in \Gamma(TM^{1,0})$,

(20) $\nabla_X Y = [X, Y]_{1,0},$

(21) $\nabla_X Y$ is given by $\Omega(\nabla_X Y, \overline{Z}) = X\Omega(Y, \overline{Z}) - \Omega(Y, \overline{\nabla_X Z}) \forall Z \in \Gamma(TM^{1,0}),$

(22) $\nabla_\xi Y = [\xi, Y] - \frac{1}{2}J([\xi, JY] - J[\xi, Y]) = [\xi, Y]_{1,0}.$

$\nabla_X \overline{Y}, \nabla_X \overline{\overline{Y}}, \nabla_\xi \overline{Y}$ are given by conjugations, and $\nabla_X \xi, \nabla_X \overline{\xi}, \nabla_\xi \xi$ are all zero.

In the following, we shall identify $\nabla$ with Webster’s $D$. We have

$$D_{\overline{X}_\beta} X_\alpha = \omega_\gamma^\alpha (\overline{X}_\beta) X_\gamma 2d\theta^\gamma(X_\alpha, \overline{X}_\beta) X_\gamma$$

$$= -\theta^\gamma([X_\alpha, \overline{X}_\beta]) X_\gamma = [\overline{X}_\beta, X_\alpha]_{1,0} = \nabla_{\overline{X}_\beta} X_\alpha.$$ 

And we check that

$$-d\theta(D_{\overline{X}_\beta} X_\alpha, \overline{X}_\gamma) = -i\theta^\sigma \wedge \theta^\rho (\omega_\alpha^\sigma (X_\beta) X_\sigma, \overline{X}_\gamma) = -i\omega_\alpha^\gamma (X_\beta) = i\overline{\omega}_\alpha^\gamma (X_\beta)$$

$$= X_\beta ( -i\theta^\sigma \wedge \theta^\rho (X_\alpha, \overline{X}_\gamma)) + i\theta^\rho \wedge \theta^\sigma (X_\alpha, \overline{\omega}_\rho^\sigma (X_\beta) \overline{X}_\sigma)$$

$$= X_\beta ( -d\theta(X_\alpha, \overline{X}_\gamma)) - ( -d\theta(X_\alpha, \overline{\nabla_{\overline{X}_\beta} X_\gamma}) \quad \text{for all } X_\gamma.$$ 

Hence, $D_{\overline{X}_\beta} X_\alpha = \nabla_{\overline{X}_\beta} X_\alpha$. We also have

$$D_\xi X_\alpha = \omega_\alpha^\gamma (\xi) X_\gamma 2 -d\theta^\gamma(\xi, X_\alpha) X_\gamma = \theta^\gamma([\xi, X_\alpha]) X_\gamma = [\xi, X_\alpha]_{1,0} = \nabla_\xi X_\alpha.$$
Then we identify the torsion terms. We have

\[ T(X_\alpha, X_\beta) = \nabla_{X_\alpha} X_\beta - \nabla_{X_\beta} X_\alpha - [X_\alpha, X_\beta] \]

\[ = [X_\alpha, X_\beta]_{0,1} + [X_\alpha, X_\beta]_{1,0} - [X_\alpha, X_\beta] \]

\[ = -\theta(\left[ X_\alpha, X_\beta \right]) \xi \]

\[ = d\theta(X_\alpha, X_\beta) \xi \]

\[ = i\delta^\beta_\alpha \xi = -\Omega(X_\alpha, X_\beta) \xi, \]

and

\[ T(X_\alpha, X_\beta) = (\omega^\gamma_\beta(X_\alpha) - \omega^\gamma_\alpha(X_\beta) - \theta^\gamma([X_\alpha, X_\beta]))X_\gamma \]

\[ = (\omega^\gamma_\beta(X_\alpha) - \omega^\gamma_\alpha(X_\beta) + d\theta^\gamma(X_\alpha, X_\beta))X_\gamma = 0, \]

and

\[ T(\xi, X_\alpha) = \nabla_{\xi} X_\alpha - \nabla_{X_\alpha} \xi - [\xi, X_\alpha] \]

\[ = [\xi, X_\alpha]_{0,1} - [\xi, X_\alpha] \]

\[ = -\theta^{\beta}([\xi, X_\alpha])X_\beta - \theta([\xi, X_\alpha]) \xi \]

\[ = (\theta^\gamma \wedge \omega^\beta_\gamma + \theta \wedge \tau^\beta)(\xi, X_\alpha)X_\beta \]

\[ = \tau^\beta(X_\alpha)X_\beta \]

\[ = A_{\alpha\beta}X_\beta. \]

Finally, we identify the curvatures terms. We have

\[ R(Y, Z)X_\beta = \nabla_Y \nabla_Z X_\beta - \nabla_Z \nabla_Y X_\beta - \nabla_{[Y, Z]} X_\beta \]

\[ = ((Y\omega^\alpha_\beta(Z) + \omega^\alpha_\beta(Z)\omega^\gamma_\alpha(Y)) - (Z\omega^\alpha_\beta(Y) + \omega^\gamma_\alpha(Y)\omega^\alpha_\beta(Z)) \]

\[ - \omega^\alpha_\beta([Y, Z]))X_\alpha \]

\[ = (d\omega^\alpha_\beta - \omega^\gamma_\beta \wedge \omega^\alpha_\gamma)(Y, Z)X_\alpha, \]

and

\[ R(X_\rho, X_\sigma)X_\beta = (-iA_{\beta\rho}, \theta^\gamma \wedge \theta^\alpha)(X_\rho, X_\sigma)X_\alpha \]

\[ = -iA_{\beta\rho}(\delta^\alpha_\rho \delta^\alpha_\gamma - \delta^\alpha_\gamma \delta^\alpha_\rho)X_\alpha \]

\[ = -i(A_{\beta\rho}X_\sigma - A_{\beta\sigma}X_\rho), \]

\[ R(X_\rho, \overline{X}_\sigma)X_\beta = R_{\beta\alpha\bar{\rho}\bar{\sigma}}X_\alpha, \]

\[ R(\overline{X}_\rho, \overline{X}_\sigma)X_\beta = (i\overline{A}_{\alpha\gamma} \theta^{\beta \gamma} \wedge \theta^{\alpha})(\overline{X}_\rho, \overline{X}_\sigma)X_\alpha \]

\[ = i\overline{A}_{\alpha\gamma}(\delta^\alpha_\beta \delta^\alpha_\gamma - \delta^\alpha_\gamma \delta^\alpha_\beta)X_\alpha \]

\[ = i(\delta^\alpha_\beta A_{\alpha\sigma} - \delta^\alpha_\sigma A_{\alpha\beta})X_\alpha, \]

\[ R(X_\rho, \xi)X_\beta = W_{\beta\alpha\rho\sigma}X_\alpha, \]

\[ R(\overline{X}_\sigma, \xi)X_\beta = -W_{\alpha\beta\overline{\sigma}}X_\alpha. \]
3. A key identity for Webster pseudo-torsion computation

In this section, we obtain a key identity (53) for Webster pseudo-torsion computation in Section 5.

Let \( M \) be the boundary of a strongly pseudoconvex domain in \( \mathbb{C}^{n+1} \). Let \( r \) be a smooth real-valued defining function of \( M \) i.e. \( M = \{ r = 0 \} \) and \( dr \neq 0 \). Throughout this section, the range of indices are: \( 0 \leq i, j, k \cdots \leq n + 1, 0 \leq \alpha, \beta, \gamma \cdots \leq n \). Coordinates for \( \mathbb{C}^{n+1} \) will be given by \( (z_1, z_2, \ldots, z_{n+1}) \). We will use the conventions:

\[
\begin{align*}
    r_j &= \frac{\partial r}{\partial z^j}, \quad r_{j\bar{k}} = \frac{\partial^2 r}{\partial z^j \partial \bar{z}^k}.
\end{align*}
\]

The \( CR \) structure is on \( M \) is given by

\[
T^{1,0}M = \{ X = x_j \frac{\partial}{\partial z^j} : dr(X) = x_j r_j = 0 \}.
\]

We define a 2n dimensional subbundle of \( TM \) by

\[
(24) \quad CHM = T^{1,0}M \oplus T^{0,1}M \quad \text{where} \quad T^{0,1}M := \overline{T^{1,0}M},
\]

and \( HM := \text{Re} (T^{1,0}M \oplus T^{0,1}M) \). \( HM \) carries a complex structure map

\[
(25) \quad J : HM \to HM, \quad J^2 = -Id,
\]

and we denote its extension to \( CTM \) by \( J \).

(26) \quad J : CHM \to CHM, \quad J^2 = -Id \quad \text{and} \quad J|_{T^{1,0}M} = \text{multiplication by} \ i = \sqrt{-1}.

Define a one form \( \theta \) on \( \mathbb{C}^{n+1} \) by

\[
(27) \quad \theta = -i \partial r = -i r_j dz^j.
\]

On \( CTM \), \( \theta \) is a real one form annihilating \( T^{1,0}M \oplus T^{0,1}M \),

\[
(28) \quad \theta = i \partial r = i \bar{\partial} r = \frac{i}{2} (\bar{\partial} r - \partial r).
\]

For \( X, Y \in T^{1,0}M \),

\[
(29) \quad \theta([X, Y]) = 0, \quad \theta([X, \bar{Y}]) = 0, \quad \text{and} \quad \theta([X, \bar{Y}]) = -d\theta(X, \bar{Y}) = -i \bar{\partial} dr(X, \bar{Y}).
\]

For \( X, Y \in T^{1,0}M \), the Levi form is given by

\[
(30) \quad L_{\theta}(X, \bar{Y}) = \theta([JX, \bar{Y}]) = -d\theta(JX, \bar{Y}) = \partial \bar{\partial} r(X, \bar{Y}).
\]

\( M \) is said to be strongly pseudoconvex if \( L_{\theta}(X, \bar{Y}) \) is positive definite as a Hermitian form on \( T^{1,0}M \). In other words,

\[
(31) \quad \forall \ w^j \frac{\partial}{\partial z^j} \neq 0, \quad w^j r_j = 0 \implies r_{j\bar{k}} w^j w^{\bar{k}} > 0.
\]

Note that the matrix \( r_{j\bar{k}} \) is not necessary invertible though (31) is satisfied.

**Example 3.1.** The real hyperquadric in \( \mathbb{C}^2 \) given by

\[
M := \{ (z_1, z_2) \in \mathbb{C}^2 \mid r(z_1, z_2) = 0, \ r = z_1 \bar{z}_1 - \frac{z_2 - \bar{z}_2}{2i} \} \quad \text{which is s.p.c.}
\]

\( T^{1,0}M \) is spanned by \( \frac{\partial}{\partial z_1} + 2iz_1 \frac{\partial}{\partial z_2} \). We see that

\[
\begin{pmatrix}
    r_{1\bar{1}} & r_{1\bar{2}} \\
    r_{2\bar{1}} & r_{2\bar{2}}
\end{pmatrix} = \begin{pmatrix}
    1 & 0 \\
    0 & 0
\end{pmatrix} \quad \text{while} \quad \begin{pmatrix}
    1 & 2iz_1 \\
    0 & 0
\end{pmatrix} \begin{pmatrix}
    1 \\
    2iz_1
\end{pmatrix} = 1.
\]
Neither does s.p.c., \(31\) imply the positive definiteness of \(r_{jk}\), as we see from Example 3.2. \(M := \{(z_1, z_2) \in \mathbb{C}^2 \mid r(z_1, z_2) = 0, \ r = 1 + z_1\overline{z}_1 - z_2\overline{z}_2\}\) which is s.p.c. \(T^{1,0}M\) is spanned by \(z_2 \frac{\partial}{\partial z_1} + z_1 \frac{\partial}{\partial z_2}\). We see that
\[
\begin{pmatrix}
  r_1 & r_2 \\
  r_2 & r_2
\end{pmatrix} =
\begin{pmatrix}
  1 & 0 \\
  0 & -1
\end{pmatrix}
\]
while
\[
\begin{pmatrix}
  \overline{z}_2 \\
  z_1
\end{pmatrix}
\begin{pmatrix}
  1 & 0 \\
  0 & -1
\end{pmatrix}
\begin{pmatrix}
  \overline{z}_2 \\
  z_1
\end{pmatrix} = 1.
\]

Let \(\xi\) be the unique real vector field on \(M\) such that
\[
(32) \quad \theta(\xi) = 1,
\]
\[
(33) \quad \xi \mid d\theta = 0.
\]

Let
\[
(34) \quad \xi = \xi^j \frac{\partial}{\partial z^j} + \overline{\xi^j} \frac{\partial}{\partial \overline{z}^j}.
\]

We have
\[
(35) \quad \theta(\xi) = 1 \ \text{means} \ i r_{k\overline{k}} \xi^k = 1 \ \text{or} \ r_j \xi^j = i,
\]
\[
(36) \quad \xi \mid d\theta = 0 \ \text{means} \ x^j r_j = 0 \Rightarrow x^j r_{j\overline{k}} \xi^k = 0.
\]

Let \(TM = HM \oplus \mathbb{R} \xi\), we extend \(25\),
\[
(37) \quad J : TM \to TM \quad \text{by} \quad J\xi = 0.
\]

Then, \(J\) as a \((1, 0)\) tensor satisfies
\[
(38) \quad J^2 X = -X + \theta(X)\xi.
\]

for all \(X \in TM\). With \(J\) as a \((1, 0)\) tensor, we regard \(g(X, Y) := -d\theta(JX, Y) = L_g(X, Y)\) as \((0, 2)\) tensor on \(TM\). Note that , for \(X, Y \in TM\), \(\theta([JX, Y]) \neq -d\theta(JX, Y)\) since \(\theta([X, Y])\) is not a tensor, for instance, we have \(\theta([f\xi, \xi]) = \theta(\xi(f)\xi) = \xi(f)\). In the following, we write \(\langle X, Y \rangle := g(X, Y)\). Choose \(X_1, \ldots, X_n\) in \(T_p^{1,0}M\) for some point \(p\) in \(M\). Let
\[
(39) \quad X_\alpha = x^j_\alpha \frac{\partial}{\partial z^j}
\]
satisfying
\[
(40) \quad x^j_\alpha r_j = 0,
\]
\[
(41) \quad x^j_\alpha r_{j\overline{k}} \xi^k = \delta^j_\alpha.
\]

Note that we use all Euclidean coordinates \(z^1, \ldots, z^{n+1}\) in the description of the CR structure of \(M\). In this way, we dispense with distinguishing one coordinate,
say \( z^{n+1} \), such that \( \frac{\partial r}{\partial z^{n+1}} \neq 0 \), as is required in Chern-Moser and subsequent works. Our computation is therefore symmetric in all \( z^1, \ldots, z^{n+1} \). Write

\[
\begin{align*}
J(u) : &= (-1)^{n+1} \det \begin{pmatrix} u & u_T & \cdots & u_{n+1} \\ u_1 & u_{1T} & \cdots & u_{1n+1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n+1} & u_{n+1T} & \cdots & u_{n+1n+1} \end{pmatrix}, \\
F : &= \begin{pmatrix} r & r_T & \cdots & r_{n+1} \\ r_1 & r_{1T} & \cdots & r_{1n+1} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n+1} & r_{n+1T} & \cdots & r_{n+1n+1} \end{pmatrix},
\end{align*}
\]

and

\[
\langle\langle \xi, \xi \rangle\rangle := \xi^j r_{jk} \xi^k. \tag{44}
\]

Then, we have

\[
-\langle\langle \xi, \xi \rangle\rangle r_j + ir_{jk} \xi^k = 0. \tag{45}
\]

**Proof of (45).** \( (r_j dz^j)(X_\alpha) = x^j_\alpha r_j \) \( \tag{40} \) 0 and \( (r_{jk} \xi^k dz^j)(X_\alpha) = x^j_\alpha r_{jk} \xi^k = 0 \) for all \( \alpha \), implies that, since \( dr \neq 0 \), \( r_{jk} \xi^k = br_j \) for some \( b \). By contraction with \( \xi^j \), \( \langle\langle \xi, \xi \rangle\rangle = bi \). Thus, we obtain (45). Write

\[
a_{jk} := x^j_\alpha x^k_\alpha. \tag{46}
\]

Then

\[
r_j a_{jk} = 0. \tag{47}
\]

Write

\[
X_{n+1} := \xi^j \frac{\partial}{\partial z^j} \quad \text{and} \quad x^j_{n+1} = \xi^j. \tag{48}
\]

Then

\[
\begin{pmatrix}
x^1_1 & \cdots & x^{n+1}_1 \\
\vdots & \ddots & \vdots \\
x^1_{n+1} & \cdots & x^{n+1}_{n+1}
\end{pmatrix} \begin{pmatrix}
r_1 & r_T & \cdots & r_{n+1} \\
r_1 & r_{1T} & \cdots & r_{1n+1} \\
\vdots & \vdots & \ddots & \vdots \\
r_{n+1} & r_{n+1T} & \cdots & r_{n+1n+1}
\end{pmatrix} \begin{pmatrix}
x^1_1 & \cdots & x^{n+1}_1 \\
\vdots & \ddots & \vdots \\
x^1_{n+1} & \cdots & x^{n+1}_{n+1}
\end{pmatrix} = \begin{pmatrix}
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
1 & \cdots & 1
\end{pmatrix}.
\]

\[
\langle\langle \xi, \xi \rangle\rangle
\]
Write
\[
\begin{pmatrix}
  y_1^1 & \cdots & y_1^{n+1} \\
  \vdots & \ddots & \vdots \\
  y_{n+1}^1 & \cdots & y_{n+1}^{n+1}
\end{pmatrix}
= \begin{pmatrix}
  x_1^1 & \cdots & x_1^{n+1} \\
  \vdots & \ddots & \vdots \\
  x_{n+1}^1 & \cdots & x_{n+1}^{n+1}
\end{pmatrix}^{-1}
\]
(50)

Then
\[
\begin{pmatrix}
  r_{11} & \cdots & r_{1n+1} \\
  \vdots & \ddots & \vdots \\
  r_{n+11} & \cdots & r_{n+1n+1}
\end{pmatrix}
= \begin{pmatrix}
  1 \\
  \vdots \\
  1
\end{pmatrix}
\begin{pmatrix}
  \langle\langle \xi, \xi \rangle\rangle
\end{pmatrix}
\begin{pmatrix}
  x_1^1 & \cdots & x_1^{n+1} \\
  \vdots & \ddots & \vdots \\
  x_{n+1}^1 & \cdots & x_{n+1}^{n+1}
\end{pmatrix}
\]
(51)

\[
\begin{pmatrix}
  y_1^1 & \cdots & y_1^{n+1} \\
  \vdots & \ddots & \vdots \\
  y_{n+1}^1 & \cdots & y_{n+1}^{n+1}
\end{pmatrix}
= \begin{pmatrix}
  1 \\
  \vdots \\
  1
\end{pmatrix}
\begin{pmatrix}
  \langle\langle \xi, \xi \rangle\rangle
\end{pmatrix}
\begin{pmatrix}
  x_1^1 & \cdots & x_1^{n+1} \\
  \vdots & \ddots & \vdots \\
  x_{n+1}^1 & \cdots & x_{n+1}^{n+1}
\end{pmatrix}
\]
(51)

\[
\begin{pmatrix}
  1 - (1 - \langle\langle \xi, \xi \rangle\rangle)y_{1n+1}^{1}x_{n+1}^{1} & \cdots & -(1 - \langle\langle \xi, \xi \rangle\rangle)y_{1n+1}^{n+1}x_{n+1}^{1} \\
  \vdots & \ddots & \vdots \\
  1 + (1 - \langle\langle \xi, \xi \rangle\rangle)i\kappa_1^1\xi_1^1 & \cdots & 1 + (1 - \langle\langle \xi, \xi \rangle\rangle)i\kappa_{n+1}^1\xi_{n+1}^1
\end{pmatrix}
\]

i.e.
\[
r_{ik}x_i^j = \delta_i^j + (1 - \langle\langle \xi, \xi \rangle\rangle)i\kappa_1^j.
\]

By (46), (48),
\[
r_{ik}(a_{kj} + \xi_1^j, \xi_1^j) = \delta_i^j + (1 - \langle\langle \xi, \xi \rangle\rangle)i\kappa_1^j.
\]

By (45),
\[
r_{ik}a_{kj} - i\langle\langle \xi, \xi \rangle\rangle r_{ik}^j = \delta_i^j + i\kappa_1^j - i\langle\langle \xi, \xi \rangle\rangle r_{ik}^j.
\]

Hence,
\[
(52)
- i\kappa_1^j + r_{ik}a_{kj} = \delta_i^j.
\]
By (35), (45), (47), (52),

\[
\left( \begin{array}{cccc}
\bar{r} & \bar{r}^1 & \cdots & \bar{r}^n+1 \\
r_1 & r_1^1 & \cdots & r_1^n+1 \\
\vdots & \vdots & \ddots & \vdots \\
r_n+1 & r_n+1^1 & \cdots & r_n+1^n+1 \\
\end{array} \right) \left( \begin{array}{cccc}
-\langle \xi, \xi \rangle & -i\xi^1 & \cdots & -i\xi^{n+1} \\
i\xi^1 & a^1 & \cdots & a^{n+1} \\
\vdots & \vdots & \ddots & \vdots \\
i\xi^{n+1} & a^n+1 & \cdots & a^{n+1+n+1} \\
\end{array} \right) = I.
\]

4. An alternative proof of the Li-Luk formula for Webster pseudo-torsion for a real hypersurface in $\mathbb{C}^{n+1}$

This section gives an alternative proof of the Li-Luk formula for Webster pseudo-torsion (for definition, see (69)) for a strongly pseudoconvex pseudohermitian hypersurface in $\mathbb{C}^{n+1}$. For the convenience of readers and fixing notations, we recall some facts and definitions in the beginning. We will also use some definitions and results in Section 2. Let $M$ be a strongly pseudoconvex pseudohermitian hypersurface given by $M = \{ z \in \mathbb{C}^{n+1} \mid \bar{r} = 0 \}$, where $r$ is a real valued defining function for $M$ and $r$ is $C^3$ in a neighborhood of $M$. Let $TM$ be the tangent bundle on $M$ and let $HM := TM \cap iTM$, the holomorphic tangent bundle on $M$. As in the previous sections, we fix the real one form $\theta$ be a pseudohermitian structure on $M$. Let $\theta^1, \ldots, \theta^n, \theta^1, \ldots, \theta^n$ be a local admissible coframe for $M$, $1 \leq \alpha, \beta \leq n$. As before we use the convention $\theta^\alpha := \theta^\alpha$. Webster shows that there are uniquely determined 1-forms $\omega_\alpha^\beta$, $\tau^\beta$ on $M$ satisfying

\[
d\theta = i\theta^\gamma \land \theta^\gamma,
\]

\[
d\theta^\alpha = \theta^\beta \land \omega^\alpha_\beta + \theta \land \tau^\alpha,
\]

\[
\omega^\alpha_\beta + \omega^\alpha_\beta = 0,
\]

\[
\tau^\alpha = A^\alpha_{\gamma} \theta^\gamma,
\]

\[
A^\alpha_{\gamma} = A^\gamma_{\alpha}.
\]

Let $\xi, X_1, \ldots, X_n, \bar{X}_1, \ldots, \bar{X}_n$ be the dual frame satisfying

\[
\theta(\xi) = 1, \quad d\theta(\xi, \cdot) = i\theta^\gamma \land \theta^\gamma(\xi, \cdot) = 0.
\]

And we have

\[
-\text{id} \theta(X_\alpha, \bar{X}_\beta) = i\theta^\gamma \land \theta^\gamma(X_\alpha, \bar{X}_\beta) = \delta^\beta_\alpha.
\]

The Levi form $L_\theta$ on $TM^{1,0}$ is defined by $L_\theta(\cdot, \gamma) := -\text{id} \theta(\cdot, \gamma)$. Hence,

\[
L_\theta(X_\alpha, \bar{X}_\beta) = \delta^\beta_\alpha = \langle X_\alpha, \bar{X}_\beta \rangle.
\]

Covariant differentiation is given by

\[
\nabla X_\alpha = \omega^\beta_\alpha X_\beta, \quad \nabla X_\alpha = \omega^\beta_\alpha X_\beta, \quad \nabla \xi = 0.
\]

We also have

\[
\nabla_{\bar{X}_\gamma} X_\alpha = \langle \bar{X}_\gamma, X_\alpha \rangle_{TM^{1,0}},
\]
and $\nabla X_\gamma X_\alpha$ is defined by
\begin{equation}
\langle \nabla X_\gamma X_\alpha, X_\beta \rangle = X_\gamma \langle X_\alpha, X_\beta \rangle - \langle X_\alpha, \nabla X_\gamma X_\beta \rangle.
\end{equation}

We have
\begin{equation}
\nabla_\xi X_\alpha = [\xi, X_\alpha]_{TM^1.0}.
\end{equation}
The torsion tensor is defined by $T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$ for $X, Y \in \mathbb{C}TM$.

We have
\begin{align}
T(X_\alpha, \overline{Y}_\beta) &= i\delta_\beta^\alpha \xi, \\
T(X_\alpha, X_\beta) &= 0, \\
T(\xi, X_\alpha) &= A_{\alpha\beta} X_\beta.
\end{align}
The Webster pseudo-torsion is defined as [3],
\begin{equation}
\text{Tor}(z)(U, V) = i(A_{\alpha\overline{\beta}} u^\alpha v^\beta - A_{\alpha\beta} u^\alpha v^\beta),
\end{equation}
where $U = u^j \frac{\partial}{\partial z_j}, V = v^j \frac{\partial}{\partial z_j} \in H_z M$ and $z \in M$. We will use following notations.
\begin{align}
J(r) &:= \begin{vmatrix} r & r_{\overline{k}} \\ r_j & r_{\overline{k}} \\ \end{vmatrix}, \\
H(r) &:= (r_{\overline{j}k}).
\end{align}

We shall prove the following theorem.

**Theorem 4.1** ([3]). Let $M$ be a $C^4$ strongly pseudoconvex hypersurface in $\mathbb{C}^{n+1}$. Let $r$ be a defining function for $M$ which is $C^3$ in a neighborhood of $M$. Consider the pseudohermitian structure defined by $\theta = -i\partial r$ on $M$. Then for any $U = u^j \frac{\partial}{\partial z_j}, V = v^j \frac{\partial}{\partial z_j} \in H_z M$ and $z \in M$, we have
\begin{equation}
\text{Tor}(z)(U, V) = 2 \Re \left( \frac{u^l v^k}{J(r)} (N - \det H(r)) r_{\overline{l}k} \right),
\end{equation}
where
\begin{equation}
N = \sum_i (-1)^{i+j+i} r_{\overline{j}i} - r_{ij} - \frac{\partial}{\partial z_j}.
\end{equation}

We will need some preliminaries to prove this theorem. First, by (53), we have
\begin{equation}
1 = r(-\langle (\xi, \xi) \rangle) + r_1(-i\xi^1) + r_2(-i\xi^2) + \cdots + r_{n+1}(-i\xi^{n+1}).
\end{equation}
Expanding $-J(r)$ by the 1st column, we have

$$
\begin{vmatrix}
  r & r_{\overline{\kappa}} \\
  r_j & r_{\overline{j\kappa}}
\end{vmatrix} = r
\begin{vmatrix}
  r_{1\overline{T}} & \cdots & r_{1n+1} \\
  \vdots & \vdots & \vdots \\
  r_{n+1\overline{T}} & \cdots & r_{n+1n+1}
\end{vmatrix}
- r_1
\begin{vmatrix}
  r_{1\overline{T}} & \cdots & r_{1n+1} \\
  \vdots & \vdots & \vdots \\
  r_{n+1\overline{T}} & \cdots & r_{n+1n+1}
\end{vmatrix}
+ \cdots
+ (-1)^{n+1} r_{n+1}
\begin{vmatrix}
  r_{1\overline{T}} & \cdots & r_{n+1} \\
  \vdots & \vdots & \vdots \\
  r_{n+1\overline{T}} & \cdots & r_{n+1n+1}
\end{vmatrix}
$$

Hence, by (74), (75), we have

$$
-\langle\langle \xi,\xi \rangle\rangle = \begin{vmatrix}
  r & r_{\overline{k}} \\
  r_j & r_{\overline{jk}}
\end{vmatrix} = -\frac{\det H(r)}{J(r)}
$$

and

$$
-i\xi^j = \frac{(-1)^j}{-J(r)}
\begin{vmatrix}
  r & r_{\overline{k}} \\
  r_j & r_{\overline{jk}}
\end{vmatrix}
= \frac{(-1)^j}{-J(r)}
\begin{vmatrix}
  r_{1\overline{T}} & \cdots & r_{n+1} \\
  \vdots & \vdots & \vdots \\
  r_{n+1\overline{T}} & \cdots & r_{n+1n+1}
\end{vmatrix}
- r_2
\begin{vmatrix}
  r_{1\overline{T}} & r_{1\overline{2}} & \cdots & r_{1n+1} \\
  \vdots & \vdots & \vdots & \vdots \\
  r_{n+1\overline{T}} & r_{n+1\overline{2}} & \cdots & r_{n+1n+1}
\end{vmatrix}
+ (-1)^n r_{n+1}
\begin{vmatrix}
  r_{1\overline{T}} & \cdots & r_{n+1} \\
  \vdots & \vdots & \vdots \\
  r_{n+1\overline{T}} & \cdots & r_{n+1n+1}
\end{vmatrix}
$$

$$
= \sum_{k=1}^{n+1} \frac{(-1)^j (-1)^{k+1}}{-J(r)} r_{\overline{k}}
\begin{vmatrix}
  r_{1\overline{T}} & \cdots & r_{1n+1} \\
  \vdots & \vdots & \vdots \\
  r_{n+1\overline{T}} & \cdots & r_{n+1n+1}
\end{vmatrix}
$$

**Proof of Theorem 4.1.**

Step 1. We first find a relation between the torsion tensor $T$ and the Webster torsion $\text{Tor}$. Let $U = \mu^\alpha X_\alpha$, $V = \nu^\beta X_\beta \in T^{1,0}M$. We have

$$
\text{Tor}(U, V) = \text{Tor}(\mu^\alpha X_\alpha, \nu^\beta X_\beta)
= 2\text{Re}(iA_{\alpha\beta})
= 2\text{Re}(i(\overline{T}(\xi, X_\alpha), X_\beta))\overline{\mu^\alpha \nu^\beta}
= 2\text{Re}(i(\overline{T}(\xi, \mu^\alpha X_\alpha), \nu^\beta X_\beta))
= 2\text{Re}(i(\overline{T}(\xi, U), V)).
$$
Step 2. We compute
\[
T(\xi, U) = \nabla_\xi U - \nabla_U \xi - [\xi, U]
\]
\[
= [\xi, U]_{T^{1,0} M} - [\xi, U]
\]
\[
= -[\xi, U]_{T^{0,1} M}
\]
\[
= - \left[ \xi^j \frac{\partial}{\partial z^j} + \bar{\xi}^{\bar{j}} \frac{\partial}{\partial \bar{z}^j}, U \right]_{T^{0,1} M}
\]
\[
= (U \bar{\xi}^j) \frac{\partial}{\partial z^j}.
\] (79)

We check that \((U \bar{\xi}^j) \frac{\partial}{\partial z^j} \in T^{1,0} M\) as follows. Using \(U = u^i \frac{\partial}{\partial z^i}, \) we have
\[
(U \bar{\xi}^j r^k_j) = U (\bar{\xi}^j r^k_j) - \bar{\xi}^j U r^k_j = -u^k r^k_j \bar{\xi}^j = 0.
\]

Step 3. Let \(U = u^j \frac{\partial}{\partial z^j}, V = v^k \frac{\partial}{\partial z^k}\) such that \(u^j r^i_j = 0, v^k r^k = 0\). Using (78), (79), we have
\[
\text{Tor}(U, V) = 2 \text{Re} \left( i \left( \frac{(U \bar{\xi}^j) \frac{\partial}{\partial z^j}, v^k \frac{\partial}{\partial z^k} \right) \right)
\]
\[
= 2 \text{Re} \left( i \left( \frac{\partial \xi^j}{\partial z^j}, v^k \frac{\partial}{\partial z^k} \right) \right)
\]
\[
= 2 \text{Re} \left( i u^l \frac{\partial \xi^j}{\partial z^j} r^k_j v^k \right)
\]
\[
= 2 \text{Re} \left( i u^l v^k \left( \frac{\partial}{\partial z^j} (\xi^j r^k_l) - \xi^j r^k_l \right) \right)
\]
\[
= 2 \text{Re} \left( i u^l v^k \left( \frac{\partial}{\partial z^j} (a r^k_l) - i \xi^j \frac{\partial r^k_l}{\partial z^j} \right) \right)
\]
\[
= 2 \text{Re} \left( i u^l v^k \left( -\langle \xi, \xi \rangle r^k_l - i \xi^j \frac{\partial r^k_l}{\partial z^j} \right) \right).
\] (80)

Hence, we have
\[
\text{Tor}(U, V) = 2 \text{Re} \left( \frac{u^l v^k}{J(r)} \left( -|r^k_l| r^k_l + \sum_i (-1)^{j+i} r^k_i \right) - |r^k_l| - r^k_l \right)
\]
\[
= 2 \text{Re} \left( \frac{u^l v^k}{J(r)} \left( \sum_i (-1)^{j+i} r^k_i \right) - |r^k_l| - \frac{\partial}{\partial z^j} - \det H(r) \right) r^k_l
\]
\[
= 2 \text{Re} \left( \frac{u^l v^k}{J(r)} (N - \det H(r)) r^k_l \right) .
\]
In this section we derive a formula for the Webster pseudo-torsion on the link of an isolated singularity of a $n$-dimensional complex subvariety in $\mathbb{C}^{n+1}$. Let $M := \{ f = 0 \} \cap \{ r = 0 \}$ where $r$ is a defining function of the sphere of radius $\epsilon$, centered at the origin and $f$ is a holomorphic function away from the origin, we assume that $\partial f \wedge dr \neq 0$ along $M$. Then $M$ is a strongly pseudoconvex CR manifold of real hypersurface type, of dimension $2n - 1$. We will use the result in the last section to find an explicit formula for Webster torsion of $M$. The key idea is to express the components of the characteristic vector field $\xi$ in terms of the derivatives of $f$ and $r$.

Let $N := \{ z \in \mathbb{C}^{n+1} \mid f = 0 \}$ where $f(0) = 0$, $\bar{\partial} f = 0$, $\partial f \neq 0$. Let $S := \{ z \in \mathbb{C}^{n+1} \mid |z|^2 + |z|^2 + \cdots + |z|^{n+1} - \epsilon = 0 \}$ for some $\epsilon > 0$. Let $M := N \cap S$, we assume $\partial f \wedge dr \neq 0$ along $M$. The complexified tangent bundles for $S$ and $M$ are denoted by $\mathbb{C}TS$ and $\mathbb{C}TM$ respectively. Let the pseudohermitian structure of $S$ be given by $\theta = i\partial r = -i\bar{\partial}r$ on $\mathbb{C}TS$. Then, the pseudohermitian structure of $M$ is given by $\theta|_M$. We will denote $\theta|_M$ by $\theta$. Throughout this section the ranges of indices are: $1 \leq A, B, \cdots \leq n + 1$, $1 \leq j, k, \cdots \leq n$, $1 \leq \alpha, \beta, \cdots \leq n - 1$, and we will use the summation convention. Let $\xi, X_\alpha, X_\beta$ be the dual basis. We may write

\begin{align}
\xi &= \xi^A \frac{\partial}{\partial z^A} + \bar{\xi}^A \frac{\partial}{\partial z^A}, \\
X_\alpha &= x_\alpha^A \frac{\partial}{\partial z^A}.
\end{align}

We have

\begin{align}
\xi|\theta &= 1 \Rightarrow \xi^A r_A = i, \\
\xi|\partial f &= 0 \Rightarrow \xi^A f_A = 0, \\
X_\alpha|\theta &= 0, \\
X_\alpha|\partial f &= 0, \\
X_\alpha|\theta^\beta &= \delta_\alpha^\beta, \\
\xi|\theta^\beta &= 0, \\
\xi|d\theta &= 0,
\end{align}

and

\begin{align}
d\theta &= ir_A dz^A \wedge dz^B = i\bar{\delta}^A_B dz^A \wedge dz^B = idz^A \wedge dz^A.
\end{align}
Hence, we have

\begin{align*}
(91) & \quad x^A_r A = 0, \\
(92) & \quad x^A f_A = 0, \\
(93) & \quad x^A r_A B \xi_B = 0 \Rightarrow x^A \xi_A = 0.
\end{align*}

We consider (93) as a system of linear equations in unknowns $\xi_A$. The matrix $(x^A_r)$ has rank $n - 1$. So (93) has only 2 independent solutions. On the other hand the matrix $(f_1 \cdots f_{n+1} + r_1 \cdots r_{n+1})$ has rank 2. Hence, we may write

\begin{equation}
\xi_A = a f_A + b r_A,
\end{equation}

for $a, b \in \mathbb{C}$. Contracting (94) with $\xi_A$, using (83), (86) we obtain $\|\xi\|^2 = -ib$ where $\|\xi\|^2 := \xi_A \bar{\xi}_A$. Hence,

\begin{equation}
b = i\|\xi\|^2.
\end{equation}

Contracting (94) with $f_A$, we obtain $0 = a f_A f_A + b r_A f_A$. So,

\begin{equation}
a = -\frac{b r_A f_A}{f_C f_C}.
\end{equation}

By (94), (95), (96), we have

\begin{equation}
\xi_A = -i\|\xi\|^2 \frac{r_B f_B f_A}{f_C f_C} + i\|\xi\|^2 r_A.
\end{equation}

Contracting (97) with $r_A$, using (83),

\begin{equation}
i = r_A \xi_A = -i\|\xi\|^2 \left( -\frac{r_B f_B f_D r_D}{f_C f_C} + r_D r_D \right).
\end{equation}

We solve for $\|\xi\|^2$ in (98) and using (97), we obtain

\begin{equation}
\xi_A = \frac{i \left( -\frac{r_B f_B f_A}{f_C f_C} + r_A \right)}{\frac{r_B f_B f_D r_D}{f_C f_C} - r_D r_D} = \frac{i \left(-\frac{z_B f_B f_A}{f_C f_C} + z_A \right)}{\frac{z_B f_B f_D z_D}{f_C f_C} - \epsilon}.
\end{equation}

Now, we are ready to show:

**Theorem 5.1.** Let $\mathcal{N} := \{ z \in \mathbb{C}^{n+1} \mid f = 0 \}$ where $f(0) = 0, \bar{\partial} f = 0, \partial f \neq 0$. Let $S := \{ z \in \mathbb{C}^{n+1} \mid r = |z|^2 + |z|^2 + \cdots + |z|^2 - \epsilon = 0 \}$ for some $\epsilon > 0$. Let $M := \mathcal{N} \cap S$, we assume $\partial f \wedge dr \neq 0$ along $M$. Consider the pseudohermitian
structure defined by $\theta = -i\partial r$ on $M$. Then for any $U = u^A \frac{\partial}{\partial z_A}, V = v^B \frac{\partial}{\partial z_B} \in H_z M$ and $z \in M$, we have

\[(100) \quad \text{Tor}(z)(U,V) = 2 \text{Re} \left( i u^B v^A \frac{\partial \xi^A}{\partial z^B} \right) \]

where

\[
\xi^A = i \left( \frac{-z_B f_B f_A + z_A}{f_C f_C + z_A} \right) .
\]

**Proof of Theorem 5.1.**

*Step 1.* We first find a relation between the torsion tensor $T$ and the Webster torsion $\text{Tor}$. Let $U = \mu^A X^A, V = \nu^B X^B \in T^{1,0} M$. By computation similar to (78), we have

\[(101) \quad \text{Tor}(U,V) = 2 \text{Re} \left( i \langle T(\xi, U), V \rangle \right) .
\]

*Step 2.* By computation similar to (79), we have

\[(102) \quad T(\xi, U) = (\overline{U \xi^A}) \frac{\partial}{\partial z^A} .
\]

We check that $(\overline{U \xi^A}) \frac{\partial}{\partial z^A} \in T^{1,0} M$ as follows. Using $U = u^A \frac{\partial}{\partial z^A}$, we have

\[
(\overline{U \xi^A}) f_A = \overline{U} (\xi^A f_A) - \xi^A \overline{U} (f_A) = 0 .
\]

*Step 3.* Let $U = u^A \frac{\partial}{\partial z^A}, V = v^A \frac{\partial}{\partial z^A}$ such that $u^A r_A = 0, u^A f_A = 0, v^A r_A = 0, v^A f_A = 0$. Using (101), (102), we have

\[
\text{Tor}(U,V) = 2 \text{Re} \left( i \langle (U \xi^A) \frac{\partial}{\partial z^A}, v^A \frac{\partial}{\partial z^A} \rangle \right) \\
= 2 \text{Re} \left( i \langle u^B \frac{\partial \xi^C}{\partial z^B}, v^A \frac{\partial}{\partial z^A} \rangle \right) \\
= 2 \text{Re} \left( i u^B v^A \frac{\partial \xi^A}{\partial z^B} \right) .
\]

\[\Box\]

**Example 5.2.** Let $f = (z^3)^2 - z^1 z^2$. Let $M := \{ f = 0 \} \cap \{|z^1|^2 + |z^2|^2 + |z^3|^2 = 1\}$. We may see that the the codimension 3 real hypersurface $M$ is spherical as follows. Using the map $F$ given by

\[
\tilde{z}^1 = -\frac{1}{\sqrt{2}} (z^1 - iz^2) , \\
\tilde{z}^2 = \frac{1}{\sqrt{2}} (z^1 + iz^2) , \\
\tilde{z}^3 = z^3 ,
\]

\[
\tilde{z}^1 = -\frac{1}{\sqrt{2}} (z^1 - iz^2) , \\
\tilde{z}^2 = \frac{1}{\sqrt{2}} (z^1 + iz^2) , \\
\tilde{z}^3 = z^3 ,
\]

\[
\tilde{z}^1 = -\frac{1}{\sqrt{2}} (z^1 - iz^2) , \\
\tilde{z}^2 = \frac{1}{\sqrt{2}} (z^1 + iz^2) , \\
\tilde{z}^3 = z^3 ,
\]
the CR manifold $M_0$ given by
\[
\begin{aligned}
\{(z^1)^2 + (z^2)^2 + (z^3)^2 &= 0, \\
|z^1|^2 + |z^2|^2 + |z^3|^2 &= 1
\end{aligned}
\]
is mapped to
\[
\begin{aligned}
2\tilde{z}^1\tilde{z}^2 - (\tilde{z}^3)^2 &= 0, \\
|\tilde{z}^1|^2 + |\tilde{z}^2|^2 + |\tilde{z}^3|^2 &= 1.
\end{aligned}
\]
Together with the map $\phi: S^3 \to M_0$ given by
\[
(\zeta, \eta) \mapsto \left(\frac{\zeta^2 - \eta^2}{\sqrt{2}}, \frac{i(\zeta^2 + \eta^2)}{\sqrt{2}}, \frac{2\zeta\eta}{\sqrt{2}}\right) =: (z^1, z^2, z^3)
\]
where $S^3 := \{(\zeta, \eta) \in \mathbb{C}^2 : |\zeta|^2 + |\eta|^2 - 1 = 0\}$. $\phi$ is well defined, holomorphic, onto. By [2], $M_0$ is CR diffeomorphic to $S^3/G$ where $G = \{I, -I\}$, so that $M_0$ is locally biholomorphic to $S^3$. Hence, $M$ is locally biholomorphic to $S^3$. Then $z^B f_B = 0$. By (100) $\text{Tor}(z)(U, V) = 0, \forall z \in M$.

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