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GENERIC ONE-STEP BRACKET-GENERATING DISTRIBUTIONS OF RANK FOUR

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Abstract. We give a uniform, explicit description of the generic types of one–step bracket–generating distributions of rank four. A manifold carrying such a structure has dimension at least five and no higher than ten. For each of the generic types, we give a brief description of the resulting class of generic distributions and of geometries equivalent to them. For dimensions different from eight and nine, these are available in the literature. The remaining two cases are dealt with in my doctoral thesis.

1. Introduction

A smooth distribution $H \subset TM$ is said to be bracket-generating if all iterated brackets among its sections generate, at each point, the whole tangent space to the manifold M. Bracket–generating distributions play a key role in non-holonomic mechanics, control theory and subriemannian geometry. The structure of generic type is encoded, at least locally, by a model nilpotent graded Lie algebra. Contact structures are remarkable generic examples. According to a classic result in differential geometry known as Pfaff theorem, any contact structure locally looks like the same canonical model, so contact structures do not admit local invariants. Moreover, the automorphism group of a contact structure is infinite-dimensional. E. Cartan showed the existence of structures with completely different behavior. In his "five variables paper" ([7]), he considered generic distributions of rank two in dimension five, which are bracket generating in two steps. To any such structure, he associated a Cartan geometry related to the exceptional Lie group G_2, thus showing the existence of local invariants for the structure and of an upper bound for the dimension of its automorphism group. Nowadays, it is clear that this is a special case of a more general phenomenon arising in parabolic geometry. Indeed, generic bracket-generating distributions underlie several parabolic geometries and in some cases, as in Cartan's example, they are actually equivalent. The main result of the article, stated in Theorem 1, is the classification of generic types of one-step bracket-generating distributions of rank four. As we will see, from the point of view of the intrinsic properties, there are several examples in rank four. We find a contact

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structure, but also distributions admitting local invariants with infinite-dimensional automorphism group, parabolic geometries and, finally, a non-parabolic geometry equipped with a canonical linear connection.

The structure of the article can be briefly described as follows. First, we define the Levi bracket $\mathcal{L}\colon \Lambda^2 H \to TM/H$ associated to a (k,n)-distribution $H \subset TM$. The bracket-generating condition in one step writes in terms of the natural action of $G = GL(k,\mathbb{R}) \times GL(n-k,\mathbb{R})$ on the set of surjective linear maps $\Lambda^2\mathbb{R}^k \to \mathbb{R}^{n-k}$. In Proposition 1, we show that $\mathcal{L}$ generates G-orbits which we assume, as a genericity condition, to be open. By linear algebra, the G-orbits are equivalent to orbits for a natural action of $GL(k,\mathbb{R})$. A key remark concerning $GL(4,\mathbb{R})$, formulated in Proposition 2, leads to a characterization of open orbits as non-degenerate restrictions for the wedge product on $\Lambda^2\mathbb{R}^4$. Then, the classification in Theorem 1 is obtained by counting the possible nondegenerate restrictions. The result in Theorem 1 can be read as a special case of the classification of rigid Carnot algebras, which is given in [1]. However, our description is more explicit and leads directly to a nice presentation of model algebras. In this picture, one can easily deduce model algebras of generic $(4,9)$ and $(4,10)$-types. The remaining generic types are described through suitable generalizations of the real Heisenberg algebra. We conclude with an overview of model structures. These are described in the literature, except for dimensions eight and nine. A detailed study of these two cases will appear in my doctoral thesis.

2. Generic one-step bracket-generating distributions

Definition 1. Let M be a smooth manifold of dimension n and $H \subset TM$ a smooth distribution of rank k. We say that H is *bracket generating in one step* (or a (k,n)-*distribution*) if $TM = H + [H,H]$.

Let $H \subset TM$ be a (k,n)–distribution. Denote by $Q = TM/H$ the quotient bundle and by $q\colon TM \to Q$ the canonical projection. Put $\mathrm{gr}_{-1}(TM) = H$ and $\mathrm{gr}_{-2}(TM) = Q$. Then $\mathrm{gr}(TM) = \mathrm{gr}_{-1}(TM) \oplus \mathrm{gr}_{-2}(TM)$ is the associated graded vector bundle. The Levi bracket $\mathcal{L}\colon \Lambda^2 H \to Q$, defined by the formula

$$\mathcal{L}(\xi,\eta) := q([\xi,\eta])\,, \quad \xi,\eta \in \Gamma(H)$$

gives, at each point $x \in M$, a surjective linear map $\mathcal{L}_x\colon \Lambda^2 H_x \to Q_x$. Extending trivially $\mathcal{L}_x$ to the remaining components, we endow $\mathrm{gr}(T_xM)$ with a nilpotent graded Lie algebra structure, in general depending on the base point x. Consider the natural action of the Lie group $G = GL(k,\mathbb{R}) \times GL(n-k,\mathbb{R})$ on the set of linear maps $\Lambda^2\mathbb{R}^k \to \mathbb{R}^{n-k}$, explicitely given by the formula

$$\begin{aligned} &G \times L(\Lambda^2\mathbb{R}^k,\mathbb{R}^{n-k}) \longrightarrow L(\Lambda^2\mathbb{R}^k,\mathbb{R}^{n-k}) \\ &(A,B)\cdot F(v,w) := B \cdot F(A^{-1}\cdot v, A^{-1}\cdot w) \end{aligned} \tag{1}$$

Observe that the subset $L_s(\Lambda^2\mathbb{R}^k,\mathbb{R}^{n-k})$ of surjective linear maps is G-invariant.

Proposition 1. *Let be k, n nonnegative integers such that $1 \le n-k \le \binom{k}{2}$. Denote by $L(\Lambda^2\mathbb{R}^k,\mathbb{R}^{n-k}) = \{\phi\colon \Lambda^2\mathbb{R}^k \to \mathbb{R}^{n-k}$ linear$\}$, by $L_s(\Lambda^2\mathbb{R}^k,\mathbb{R}^{n-k})$ the subset of* *surjective linear maps and by $G = GL(k,\mathbb{R}) \times GL(n-k,\mathbb{R})$ the product of general linear groups.*

(a) *The Levi bracket associated to a (k,n)-distribution determines, for each $x \in M$, an orbit $\mathcal{O}_x \subset L_s(\Lambda^2\mathbb{R}^k, \mathbb{R}^{n-k})$ for the natural action of G.*

(b) *The G-orbits in $L_s(\Lambda^2\mathbb{R}^k, \mathbb{R}^{n-k})$ are in one-to-one correspondence with the $GL(k,\mathbb{R})$-orbits for the natural smooth action on the Grassmannian of ℓ-planes in $\Lambda^2\mathbb{R}^k$, where $\ell = \binom{k}{2} - n + k$.*

Proof. (a) Let $H \subset TM$ be a (k,n)-distribution and $\mathcal{L}\colon \Lambda^2 H \to Q$ the associated Levi bracket. For $x \in M$, choose a pair (ϕ_x, ψ_x) of linear isomorphisms $\phi_x\colon \mathbb{R}^k \to H_x$ and $\psi_x\colon \mathbb{R}^{n-k} \to Q_x$. The set of all isomorphisms $\mathbb{R}^k \oplus \mathbb{R}^{n-k} \to \mathrm{gr}(T_xM)$ writes as

$$\{\ (\phi_x \circ A, \psi_x \circ B) \mid (A,B) \in G\ \}\,.$$

The pair (ϕ_x, ψ_x), together with $\mathcal{L}_x$, defines $T_x \in L_s(\Lambda^2\mathbb{R}^k, \mathbb{R}^{n-k})$ via

$$T_x(v,w) = ((\psi_x)^{-1} \circ \mathcal{L}_x \circ \Lambda^2\phi_x)(v,w)\,, \quad v, w \in \mathbb{R}^k\,.$$

Similarly, the pair $(\phi_x \circ A, \psi_x \circ B)$ induces a map $S_x \in L_s(\Lambda^2\mathbb{R}^k, \mathbb{R}^{n-k})$ for any $(A,B) \in G$. Now S_x is easily seen to lie in the G-orbit of T_x. Indeed, for all $v, w \in \mathbb{R}^k$:

$$\begin{aligned}(A,B)\cdot S_x(v,w) &= (A,B)\cdot((\psi_x \circ B)^{-1} \circ \mathcal{L}_x \circ \Lambda^2(\phi_x \circ A))(v,w)\\ &= (A,B)\cdot(B^{-1} \circ (\psi_x)^{-1} \circ \mathcal{L}_x \circ \Lambda^2\phi_x)(Av, Aw)\\ &= T_x(v,w)\,.\end{aligned}$$

shows that $(A,B)\cdot S_x = T_x$. We conclude that the G-orbit of T_x does not depend on the choice of isomorphism $\mathbb{R}^k \oplus \mathbb{R}^{n-k} \to \mathrm{gr}(T_xM)$ and it is therefore defined only by $\mathcal{L}_x$.

(b) Observe that $\mathrm{Ker}(T) \subset \Lambda^2\mathbb{R}^k$ is a linear subspace of dimension ℓ, hence an element in the Grassmannian $Gr(\ell, \Lambda^2\mathbb{R}^k)$, for any $T \in L_s(\Lambda^2\mathbb{R}^k, \mathbb{R}^{n-k})$. In formula (1), the factor $GL(n-k,\mathbb{R})$ acts transitively on $\mathbb{R}^{n-k}$. It thus follows that two surjective linear maps are in the same G-orbit if and only if their kernels are mapped to each other by an element of $GL(k,\mathbb{R})$. □

Consider the natural topology on $L_s(\Lambda^2\mathbb{R}^k, \mathbb{R}^{n-k})$, which is induced by the Euclidean topology on domain and target space.

Definition 2. Let be k, n nonnegative integers such that $1 \le n-k \le \binom{k}{2}$ and call (k,n) a *bidimension*. Consider the natural action of the Lie group $G = GL(k,\mathbb{R}) \times GL(n-k,\mathbb{R})$ on $L_s(\Lambda^2\mathbb{R}^k, \mathbb{R}^{n-k})$.

(1) The bidimension (k,n) is said to be *rigid* if there exist open G-orbits in $L_s(\Lambda^2\mathbb{R}^k, \mathbb{R}^{n-k})$.

(2) Let be $\mathcal{O} \subset L_s(\Lambda^2\mathbb{R}^k, \mathbb{R}^{n-k})$ an open orbit. A (k,n)-distribution $H \subset TM$ is said to be *generic of type* $\mathcal{O}$ if $\mathcal{O}_x = \mathcal{O}$ for all $x \in M$, with $\mathcal{O}_x$ denoting the orbit generated by $\mathcal{L}_x$ as in Proposition 1 (a). We say that $T \in \mathcal{O}$ is a *model bracket* for H.

An orbit $\mathcal{O} \subset L_s(\Lambda^2\mathbb{R}^k, \mathbb{R}^{n-k})$ is equivalent to an isomorphism class of nilpotent graded Lie algebras $\mathfrak{n} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}$ such that $\dim(\mathfrak{g}_{-1}) = k$ and $\dim(\mathfrak{g}_{-2}) = n - k$. For this reason, we can equivalently speak about model bracket or model algebra for a generic type $\mathcal{O}$. Observe that every open orbit $\mathcal{O}$ of bidimension (k, n) is realized as the type of some generic (k, n)-distribution. To see this, first consider a nilpotent graded Lie algebra $\mathfrak{n}$ corresponding to the orbit $\mathcal{O}$. Let N be the connected and simply connected Lie group with Lie algebra $\mathfrak{n}$. Then $\mathfrak{g}_{-1} \subset \mathfrak{n}$ generates a left-invariant distribution on N, which is generic of type $\mathcal{O}$. Therefore, a classification of generic types of (k, n)-distributions consists of a list of open orbits of bidimension (k, n). These are a special case of rigid Carnot algebras, classified in [1]. There, it is shown that the list of rigid bidimensions is given by three infinite series and several exceptional cases. Put $p = k + \binom{k}{2}$ for arbitrary integers $k \geq 2$. Each of the following bidimensions corresponds to a unique open orbit:

- *Darboux bidimensions* $(k, k + 1)$;
- *dual Darboux bidimensions* $(k, p - 1)$;
- *free bidimensions* (k, p).

Interesting in this sense is the rank-four case where, beyond Darboux, dual Darboux and free, we find three exceptional rigid bidimensions. Observe that, for every k, there is a unique orbit of free bidimension (k, p). This is the orbit of isomorphisms $\Lambda^2\mathbb{R}^k \to \mathbb{R}^{p-k}$ and coincides with $L_s(\Lambda^2\mathbb{R}^k, \mathbb{R}^{p-k})$.

2.1. **Case of rank four.** The natural action of $GL(4, \mathbb{R})$ on $\Lambda^2\mathbb{R}^4$ clearly respects the wedge product $\Lambda^2\mathbb{R}^4 \times \Lambda^2\mathbb{R}^4 \to \Lambda^4\mathbb{R}^4$, which is symmetric, up to scale. Hence, this action maps $GL(4, \mathbb{R})$ to the conformal group of the wedge product, which is isomorphic to $CO(3, 3)$. Observing that the two groups have the same dimension, one easily deduces the following

Proposition 2 ([8, p. 117])**.** *The wedge product defines a conformal class of quadratic forms of signature* $(3, 3)$ *on* $\Lambda^2\mathbb{R}^4$. *The resulting Lie group homomorphism* $GL(4, \mathbb{R}) \to CO(3, 3)$ *restricts to a two–fold covering between the connected components of the identities.*

Nonempty orbits of bidimension $(4, n)$ as in Definition 2, a priori, may exist for $5 \leq n \leq 10$. Since the free type was discussed above, we restrict to $5 \leq n \leq 9$. The orbits of bidimension $(4, n)$ are equivalent, by Proposition 1 (b), to orbits in a Grassmannian for the action of $GL(4, \mathbb{R})$. If $P \subset \Lambda^2\mathbb{R}^4$ is a linear subspace, one can restrict the wedge product to P.

Theorem 1. *The open* $GL(4, \mathbb{R})$*-orbits in* $Gr(10 - n, \Lambda^2\mathbb{R}^4)$ *for* $n = 5, \ldots, 9$ *are those consisting of subspaces for which the restriction of the wedge product is nondegenerate. Hence, there is one generic type in Darboux, dual Darboux and free bidimensions* $(4, 5)$*,* $(4, 9)$ *and* $(4, 10)$ *and there are two generic types in bidimensions* $(4, 6)$*,* $(4, 7)$ *and* $(4, 8)$*.*

Proof. Let b be a quadratic form on $\Lambda^2\mathbb{R}^4$ in the conformal class defined by the wedge product. More precisely, the wedge product determines b through a choice of isomorphism $\Lambda^4\mathbb{R}^4 \cong \mathbb{R}$. Recall that the conformal group of b is isomorphic to $CO(3,3)$. First, we show that the $O(3,3)$-orbit of a linear subspace $P \subset \Lambda^2\mathbb{R}^4$ is uniquely determined by rank and signature of the restriction $b_{|P}$. Let $P \subset \Lambda^2\mathbb{R}^4$ be a linear subspace of dimension $10-n$. Then $b_{|P}$ is a symmetric bilinear form on P of signature $[t,s]$ with rank $t+s \leq 10-n$. Clearly, rank and signature of $b_{|A\cdot P}$ are the same for every $A \in O(3,3)$, hence they are constant on the $O(3,3)$-orbit of P. If $r = t+s \leq 10-n$, the nullspace $N = P \cap P^\perp \subset P$ for $b_{|P}$ is a $(10-n-r)$-dimensional subspace and b descends to a nondegenerate pseudoscalar product of signature $[t,s]$ on W/N. Moreover, $N \subset W \subset N^\perp$ and b descends to a nondegenerate pseudoscalar product on $N^\perp/N$ of signature $[t',s'] = [t-(10-n-r), s-(10-n-r)]$. We can thus find a basis $\{w_1,\dots,w_{10-n}\}$ for P, which can be completed to a basis $\{w_1,\dots,w_6\}$ for $\Lambda^2\mathbb{R}^4 \cong \mathbb{R}^6$ such that b writes as

$$\begin{pmatrix} 0 & 0 & 0 & I_{10-n-r} \\ 0 & I_{t,s} & 0 & 0 \\ 0 & 0 & I_{t'',s''} & 0 \\ I_{10-n-r} & 0 & 0 & 0 \end{pmatrix}$$

with respect to $\{w_1,\dots,w_6\}$, with $t+t''=t'$ and $s+s''=s'$. If $P,P' \subset \Lambda^2\mathbb{R}^4$ are $(10-n)$-dimensional subspaces on which b restricts with same rank and signature, we can apply the argument above to each of them. In this way, we find bases $\{w_1,\dots,w_6\}$ and $\{w'_1,\dots,w'_6\}$ for $\mathbb{R}^n$ such that $\{w_1,\dots,w_{10-n}\}$ and $\{w'_1,\dots,w'_{10-n}\}$ are bases respectively of P and P'. There exists a unique $A \in GL(6,\mathbb{R})$ such that $Aw_i = w'_i$ for all $i = 1,\dots,6$. By construction, A maps P onto P'. Moreover, since b coincides on $\{w_1,\dots,w_6\}$ and on $\{Aw_1,\dots,Aw_6\}$, A actually lies in $O(3,3)$. Hence, P and P' lie in the same $O(3,3)$-orbit. We thus showed that rank and signature of $b_{|P}$ characterizes the $O(3,3)$-orbit of $P \subset \Lambda^2\mathbb{R}^4$. In particular, the open orbits correspond to nondegenerate restrictions. Passing to $CO(3,3)$-orbits, the same characterization holds if we identify the values $[t,s]$ and $[s,t]$ for the signature. By Proposition 2, $GL(4,\mathbb{R})$–orbits and $CO(3,3)$-orbits coincide, thus proving the first statement. In order to prove the second statement, we shall count the nondegenerate restrictions of the wedge product to a linear subspace $P \subset \Lambda^2\mathbb{R}^4$ of dimension $(10-n)$ for $n = 5,\dots,9$. Observe that $b_{|P}$ is nondegenerate if and only if the same holds for the restriction to the orthogonal complement $b_{|P^\perp}$. Therefore, it is enough to consider the following cases:

(a) orbits of lines ($n = 5,9$);

(b) orbits of two-dimensional planes ($n = 6,8$);

(c) orbits of three-dimensional planes ($n = 7$).

On the one hand, it is clear that generic Darboux and dual Darboux types are unique, since there is a unique nondegenerate restriction to a line. On the other hand, for each exceptional bidimension there are two distinct open orbits. To see this, suppose that $P \subset \Lambda^2\mathbb{R}^4$ is a linear subspace of dimension $10-n = 2,3$ such that $b_{|P}$ is nondegenerate of signature $[t,s]$, with $t+s = 10-n$ and $t \geq s$. In the case (b), hyperbolic and elliptic orbits are given by signatures $[1,1]$ and $[2,0]$, while for (c) they correspond to signatures $[2,1]$ and $[3,0]$. □

Remark 1. The characterization of $GL(4,\mathbb{R})$-orbits proves the existence of open orbits of bidimension $(4,n)$ for each admissible value of n.

Observe that the characterization of $GL(4,\mathbb{R})$-orbits gives an explicit description of model brackets. Suppose that $\mathcal{O} \subset L_s(\Lambda^2\mathbb{R}^4, \mathbb{R}^{n-4})$ is an open orbit. Then, $\mathcal{O}$ corresponds to a unique value $[t,s]$ for the signature such that $t \geq s$ and $t+s = 10-n$. If the wedge product restricts to a linear subspace $P \subset \Lambda^2\mathbb{R}^4$ of dimension $10-n$ with signature $[t,s]$, then the canonical projection $\Lambda^2\mathbb{R}^4 \to \Lambda^2\mathbb{R}^4/P$ onto the quotient is in $\mathcal{O}$. In this picture, the model algebras for the types $(4,10)$ and $(4,9)$ are immediately deduced. In dimension ten, we have the free algebra $\mathbb{R}^4 \oplus \Lambda^2\mathbb{R}^4$, which can be realized as the negative graded part of a grading on $\mathfrak{g} = \mathfrak{so}(9)$ such that the first cohomology $H^1(\mathfrak{g}_-, \mathfrak{g})$ is concentrated in negative homogeneity (see [6, p. 430]). It follows from a general result ([6, Theorem 3.1.14 p. 271]) that any generic $(4,10)$-distribution is equivalent to a parabolic geometry.

For the $(4,9)$-type, observe that the restriction of the wedge product to a line in $\Lambda^2\mathbb{R}^4$ is nondegenerate if and only if the nonzero elements in this line are nondegenerate as bilinear forms on $(\mathbb{R}^4)^*$. A model bracket is thus given by the projection $\Lambda^2\mathbb{R}^4 \to \Lambda^2_0\mathbb{R}^4$ onto the kernel of a nondegenerate skew-symmetric bilinear form on $\mathbb{R}^4$. This generalizes to a description for generic dual Darboux types of even rank. Indeed, one can consider a nondegenerate skew-symmetric bilinear form on the real vector space $\mathbb{R}^{2k}$ and analogously define a model bracket of type $(2k,n)$, where $n = 2k + \binom{2k}{2} - 1$. A detailed description of generic dual Darboux distributions of even rank will appear in my doctoral thesis. There, it will be proved that every such distribution determines a canonical linear connection. In particular, torsion and curvature of the canonical connection are local invariants for the structure, whose automorphism group is related to a conformal symplectic group and has finite dimension.

A simpler description for the remaining types is obtained through generalizations of the real Heisenberg algebra of type $(4,5)$. Generic distributions of $(4,5)$-type are contact structures in dimension five. Model algebras of hyperbolic and elliptic $(4,6)$-types are, respectively, the two-fold product of three-dimensional real Heisenberg algebras and the complex three-dimensional Heisenberg algebra. The locally flat geometries of the types hence are three-dimensional complex contact manifolds and products of two real three-dimensional contact manifolds, respectively. In particular, these have infinite-dimensional automorphism group. As shown in [4], for each type there is a tensor, whose vanishing is equivalent to local flatness, so there are local invariants. However, there exist remarkable examples of finite type. Any generic $(4,6)$-distribution endowed with an additional almost complex structure on the subbundle H, which is compatible with the Levi bracket in an appropriate sense, is in fact equivalent to a parabolic geometry (see [6, p. 443-455]). This is related to CR-structures of dimension and codimension two, see [5] and [9].

The model bracket for a generic $(4,7)$-distribution is the imaginary part of an Hermitian form on a real four-dimensional algebra A (see [6, p. 432-436] for details). The two open orbits are realized through different choices for A. The algebra of quaternions $A = \mathbb{H}$ gives the elliptic model bracket, while the hyperbolic model comes from the choice of algebra $A = M_2(\mathbb{R})$ of real square matrices of size two. An alternative nice description in terms of nondegenerate orbits in a Grassmannian involves the Hodge operator (see [8, p. 117]). Every choice of a scalar product on $\mathbb{R}^4$ gives an Hodge operator $*\colon \Lambda^2\mathbb{R}^4 \to \Lambda^2\mathbb{R}^4$. The two eigenspaces associated to $*$ are both three-planes in $\Lambda^2\mathbb{R}^4$ and we look at their $CO(3,3)$-orbits in $Gr(3,\Lambda^2\mathbb{R}^4)$. Elliptic and hyperbolic orbits come from two different choices of a scalar product on $\mathbb{R}^4$, namely of positive definite and indefinite signature $[2,2]$. These are exactly the signatures of the quadratic forms defined by the square norm on $\mathbb{H}$ and by the determinant on $M_2(\mathbb{R})$. The corresponding generic distributions of elliptic and hyperbolic types, respectively known as quaternionic and split-quaternionic contact structures, are equivalent to parabolic geometries. Quaternionic contact structures were introduced by O. Biquard in his work about conformal infinities of quaternionic-Kähler metrics (see [3] and [2]). The model for quaternionic contact structures, arising in the gauge theory of four-dimensional manifolds, is the instanton distribution on the sphere $\mathbb{S}^7 \subset \mathbb{H}^2$.

Finally, considerations concerning open $(4,8)$-orbits are deduced, by duality, from the description of $(4,6)$-types given above. Characteristic of the hyperbolic $(4,8)$-model, for instance, is a decomposition $\mathfrak{g}_{-1} = \mathfrak{g}^E_{-1} \oplus \mathfrak{g}^F_{-1}$ into two–dimensional subspaces, which both are isotropic. This means that the bracket induces an isomorphism $\mathfrak{g}_{-2} \cong \mathfrak{g}^E_{-1} \otimes \mathfrak{g}^F_{-1}$. Elliptic and hyperbolic $(4,8)$-types are obtained as the negative part of a grading on two different real forms for $\mathfrak{g} = \mathfrak{sl}(5,\mathbb{C})$. The structures in dimension eight are equivalent to parabolic geometries. Using this equivalence, one can apply tools from the general theory to deduce local invariants for the structure. Similar results concerning structures in dimension eight will appear in my doctoral thesis. By considering smooth sections, one can prove that any hyperbolic $(4,8)$-distribution writes as direct sum $H = E \oplus F$ of smooth subbundles such that $TM/H \cong E \otimes F$. This, in turn, leads to an explicit decomposition of torsion and curvature in irreducible components.

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