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SECOND VARIATIONAL DERIVATIVE OF LOCAL VARIATIONAL PROBLEMS AND CONSERVATION LAWS

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Abstract. We consider cohomology defined by a system of local Lagrangian and investigate under which conditions the variational Lie derivative of associated local currents is a system of *conserved* currents. The answer to such a question involves Jacobi equations for the local system. Furthermore, we recall that it was shown by Krupka *et al.* that the invariance of a closed Helmholtz form of a dynamical form is equivalent with local variationality of the Lie derivative of the dynamical form; we remark that the corresponding local system of Euler–Lagrange forms is variationally equivalent to a global one.

1. Introduction

Geometric definitions of conserved quantities in field theories have been proposed within formulations based on symmetries of field equations rather than of the Lagrangian (see *e.g.* [18]). In particular, a definition of variation of conserved currents for gauge-natural theories [6] has been proposed in [7] where the field equations content and the symmetry information have been used, in particular, relatively to the existence of preferred lifts of infinitesimal principal automorphisms. Considering the first variation of a 'variational' Lagrangian (depending on gauge-natural lifts and Euler-Lagrange equations); then performing repeated integrations by parts with respect to covariant derivatives (with respect to fixed connections) of the components of the infinitesimal symmetry — due to the particular structure of gauge-natural lifts — a potential was obtained by which to define a concept of *variations of conserved quantities*. Independently, and without the fixing of a connection *a priori*, the gauge-natural second variational derivative of gauge-natural invariant Lagrangian has been exploited in order to construct Noether covariant conserved current [16, 15], by showing the relationship of the Bergmann–Bianchi morphism with the variational derivative of what we called the deformed Lagrangian (coinciding with the concept of 'variational' Lagrangian mentioned above). In all the mentioned approaches generalized symmetries (*i.e.* symmetries of field equations) - and in many of them some version of Noether Theorem (II) - play a fundamental role. In this view that we shall now investigate

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how much can we say about existence and globality of conserved quantities without resorting to the structure of a gauge-natural lift. The important fact is that we consider symmetries of equations of motions supposed to be Euler–Lagrangian equations of some (local) Lagrangian; thus we will be faced with inverse problems (in general, at any degree of variational forms) in the calculus of variations.

In fact, we recall that the geometrical formulations of the calculus of variations on fibered manifolds include a large class of theories for which the Euler–Lagrange operator is a morphism of an exact sequence [2, 13, 19, 20, 21]. The module in degree $(n+1)$, contains dynamical forms; a given equation is globally an Euler-Lagrange equation if its dynamical form is the differential of a Lagrangian and this is equivalent to the dynamical form being closed in the complex, *i.e.* Helmholtz conditions hold true, and its cohomolgy class being trivial. Dynamical forms which are only *locally variational*, *i.e.* which are closed in the complex and define a non trivial cohomology class, admit a system of local Lagrangians, one for each open set in a suitable covering, which satisfy certain relations among them. We shall consider global projectable vector field on a jet fiber manifold which are symmetries of dynamical forms, in particular of locally variational dynamical forms and corresponding formulations of Noether theorem (II) in order to determine obstruction to globality of associated conserved quantities. In this perspective it is clear the relevant role played by the *variational Lie derivative*, a differential operator acting on equivalence classes of variational forms in the variational sequence defined in [10], by which Noether Theorems can be formulated.

It is of great interest for the implications in the calculus of variations to investigate the role of the variational Lie derivative operator for the cohomology. We relate the cohomology class defined by a system of local Lagrangian with the cohomology class defined by the local variational problem given by the system of their local variational Lie derivative. We see that variational forms defining nontrivial cohomology class are transformed in variational forms with trivial cohomology class. This is of relevance since iterated variational derivatives define higher order variations [9, 11]. Thus variations of currents can be recognized in this approach. We consider cohomology defined by a system of local Lagrangian and investigate under which conditions the variational Lie derivative of associated local currents is a system of *conserved* currents. The answer to such a question involves Jacobi equations for the local system.

Symmetries of the Helmholtz form are also taken into account and as a consequence of the above mentioned property of the variational Lie derivative with respect to cohomology we show that not only the invariance of a closed Helmholtz form ζ_{η_i}, *i.e.* $\mathcal{L}_{\Xi}\zeta_{\eta_i} = 0$ is equivalent with local variationality of the Lie derivative $\mathcal{L}_{\Xi}\eta_i$, *i.e.* $\zeta_{\mathcal{L}_{\Xi}\eta_i} = 0$ meaning that the dynamical form $\mathcal{L}_{\Xi}\eta_i$ is locally the Euler-Lagrange form of a Lagrangian, as shown in [14], but also that the local system of Euler-Lagrange forms $\mathcal{L}_{\Xi}\eta_i$ is variationally equivalent to a well defined global Euler-Lagrange form (in order to simplify notation, here we denote by Ξ a suitable jet prolongation of a projectable vector field).

2. Local variational problems and cohomology

We shall consider the variational sequence [13] defined on a fibered manifold $\pi\colon \boldsymbol{Y} \to \boldsymbol{X}$, with $\dim \boldsymbol{X} = n$ and $\dim \boldsymbol{Y} = n+m$. For $r \geq 0$ we have the r–jet space $J_r\boldsymbol{Y}$ of jet prolongations of sections of the fibered manifold π. We have also the natural fiberings $\pi^r_s\colon J_r\boldsymbol{Y} \to J_s\boldsymbol{Y}$, $r \geq s$, and $\pi^r\colon J_r\boldsymbol{Y} \to \boldsymbol{X}$; among these the fiberings π^r_{r-1} are *affine bundles* which induce the natural fibered splitting

$$J_r\boldsymbol{Y} \times_{J_{r-1}\boldsymbol{Y}} T^*J_{r-1}\boldsymbol{Y} \simeq J_r\boldsymbol{Y} \times_{J_{r-1}\boldsymbol{Y}} (T^*\boldsymbol{X} \oplus V^*J_{r-1}\boldsymbol{Y}),$$

which, in turn, induces also a decomposition of the exterior differential on $\boldsymbol{Y}$ in the *horizontal* and *vertical differential*, $(\pi^{r+1}_r)^* \circ d = d_H + d_V$. By $(j_r\Xi, \xi)$ we denote the jet prolongation of a *projectable vector field* (Ξ, ξ) on $\boldsymbol{Y}$, and by $j_r\Xi_H$ and $j_r\Xi_V$ the horizontal and the vertical part of $j_r\Xi$, respectively.

We have the *sheaf splitting* $\mathcal{H}^p_{(s+1,s)} = \bigoplus_{t=0}^{p} \mathcal{C}^{p-t}_{(s+1,s)} \wedge \mathcal{H}^t_{s+1}$ where if Λ^p_s is the standard sheaf of p–forms on $J_s\boldsymbol{Y}$, the sheaves of *horizontal forms* $\mathcal{H}^p_{(s,q)}$ and $\mathcal{H}^p_s$ $(q \leq s)$ the sections of which are local *fibered morphisms* over π^s_q and π^s of the type $\alpha\colon J_s\boldsymbol{Y} \to \overset{p}{\wedge} T^*J_q\boldsymbol{Y}$ and $\beta\colon J_s\boldsymbol{Y} \to \overset{p}{\wedge} T^*\boldsymbol{X}$, respectively and the subsheaves $\mathcal{C}^p_{(s,q)} \subset \mathcal{H}^p_{(s,q)}$ of *contact forms* the sections of which are local morphisms $\alpha \in \mathcal{H}^p_{(s,q)}$ with values into $\overset{p}{\wedge}(J_q\boldsymbol{Y} \times_{J_{q-1}\boldsymbol{Y}} V^*J_{q-1}\boldsymbol{Y})$, *i.e.* forms which do not have a variational role. In fact, let us denote by $d\ker h$ the sheaf generated by the corresponding presheaf and set then $\Theta^*_r \equiv \ker h + d\ker h$. The quotient sequence

$$0 \longrightarrow I\!R_{\boldsymbol{Y}} \longrightarrow \cdots \overset{\mathcal{E}_{n-1}}{\longrightarrow} \Lambda^n_r/\Theta^n_r \overset{\mathcal{E}_n}{\longrightarrow} \Lambda^{n+1}_r/\Theta^{n+1}_r \overset{\mathcal{E}_{n+1}}{\longrightarrow} \Lambda^{n+2}_r/\Theta^{n+2}_r \overset{\mathcal{E}_{n+2}}{\longrightarrow} \cdots \overset{d}{\longrightarrow} 0$$

defines the *r-th order variational sequence* associated with the fibered manifold $\boldsymbol{Y} \to \boldsymbol{X}$. It turns out that it is an exact resolution of the constant sheaf $I\!R_{\boldsymbol{Y}}$ over $\boldsymbol{Y}$.

The quotient sheaves (the sections of which are classes of forms modulo contact forms) in the variational sequence can be represented as sheaves $\mathcal{V}^k_r$ of k-forms on jet spaces of higher order. In particular, currents are classes $\nu \in (\mathcal{V}^{n-1}_r)_{\boldsymbol{Y}}$; Lagrangians are classes $\lambda \in (\mathcal{V}^n_r)_{\boldsymbol{Y}}$, while $\mathcal{E}_n(\lambda)$ is called a Euler–Lagrange form (being $\mathcal{E}_n$ the Euler-Lagrange morphism); dynamical forms are classes $\eta \in (\mathcal{V}^{n+1}_r)_{\boldsymbol{Y}}$ and $\mathcal{E}_{n+1}(\eta) := \tilde{H}_{d\eta}$ is a Helmohltz form (being $\mathcal{E}_{n+1}$ the corresponding Helmholtz morphism).

The cohomology groups of the corresponding complex of global sections

$$0 \longrightarrow I\!R_{\boldsymbol{Y}} \longrightarrow \cdots \overset{\mathcal{E}_{n-1}}{\longrightarrow} (\Lambda^n_r/\Theta^n_r)_{\boldsymbol{Y}} \overset{\mathcal{E}_n}{\longrightarrow} (\Lambda^{n+1}_r/\Theta^{n+1}_r)_{\boldsymbol{Y}} \overset{\mathcal{E}_{n+1}}{\longrightarrow} (\Lambda^{n+2}_r/\Theta^{n+2}_r)_{\boldsymbol{Y}} \overset{\mathcal{E}_{n+2}}{\longrightarrow} \cdots \overset{d}{\longrightarrow} 0$$

will be denoted by $H^*_{\mathrm{VS}}(\boldsymbol{Y})$.

Since the variational sequence is a soft resolution of the constant sheaf $I\!R_{\boldsymbol{Y}}$ over $\boldsymbol{Y}$, the cohomology of the complex of global sections is naturally isomorphic to both the Čech cohomology of $\boldsymbol{Y}$ with coefficients in the constant sheaf $I\!R$ and the de Rham cohomology $H^k_{\mathrm{dR}}\boldsymbol{Y}$ [13].

Let $\boldsymbol{K}^p_r := \mathrm{Ker}\ \mathcal{E}_p$. We have the short exact sequence of sheaves

$$0 \longrightarrow \boldsymbol{K}^p_r \overset{i}{\longrightarrow} \mathcal{V}^p_r \overset{\mathcal{E}_p}{\longrightarrow} \mathcal{E}_p(\mathcal{V}^p_r) \longrightarrow 0\,.$$

In particular $\mathcal{E}_n(\mathcal{V}^n_r)$ is the sheave of Euler–Lagrange morphisms: for a global section $\eta \in (\mathcal{V}^{n+1}_r)_{\boldsymbol{Y}}$ we have $\eta \in (\mathcal{E}_n(\mathcal{V}^n_r))_{\boldsymbol{Y}}$ if and only if $\mathcal{E}_{n+1}(\eta) = 0$, which are the Helmholtz conditions of local variationality. A global inverse problem is to find necessary and sufficient conditions for such a locally variational η to be globally variational.

The above exact sequence gives rise to the long exact sequence in Čech cohomology

$$0 \longrightarrow (\boldsymbol{K}^p_r)_{\boldsymbol{Y}} \longrightarrow (\mathcal{V}^p_r)_{\boldsymbol{Y}} \longrightarrow (\mathcal{E}_p(\mathcal{V}^p_r))_{\boldsymbol{Y}} \xrightarrow{\delta_p} H^1(\boldsymbol{Y}, \boldsymbol{K}^p_r) \longrightarrow 0\,.$$

Hence, every $\eta \in (\mathcal{E}_n(\mathcal{V}^n_r))_{\boldsymbol{Y}}$ (*i.e.* locally variational) defines a cohomology class $\delta(\eta) \equiv \delta_n(\eta) \in H^1(\boldsymbol{Y}, \boldsymbol{K}^n_r) \simeq H^{n+1}_{VS}(\boldsymbol{Y}) \simeq H^{n+1}_{dR}(\boldsymbol{Y}) \simeq H^{n+1}(\boldsymbol{Y}, \mathbb{R})$. Furthermore, every $\mu \in (d_H(\mathcal{V}^{n-1}_r))_{\boldsymbol{Y}}$ (*i.e.* variationally trivial) defines a cohomology class $\delta'(\mu) \equiv \delta_{n-1}(\mu) \in H^1(\boldsymbol{Y}, \boldsymbol{K}^{n-1}_r) \simeq H^n_{VS}(\boldsymbol{Y}) \simeq H^n_{dR}(\boldsymbol{Y}) \simeq H^n(\boldsymbol{Y}, \mathbb{R})$.

The above gives rise to a well known diagram of cochain complexes:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^0(Y, K^p_r) & \xrightarrow{i} & C^0(Y, \mathcal{V}^p_r) & \xrightarrow{\mathcal{E}_p} & C^0(Y, \mathcal{E}_p(\mathcal{V}^p_r)) & \longrightarrow & 0 \\
 & & \downarrow \mathfrak{d} & & \downarrow \mathfrak{d} & & \downarrow \mathfrak{d} & & \\
0 & \longrightarrow & C^1(Y, K^p_r) & \xrightarrow{i} & C^1(Y, \mathcal{V}^p_r) & \xrightarrow{\mathcal{E}_p} & C^1(Y, \mathcal{E}_p(\mathcal{V}^p_r)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

whereby we recognize the *connecting homomorphism* $\delta_p = i^{-1} \circ \mathfrak{d} \circ \mathcal{E}_p^{-1}$ as a mapping of cohomologies (as usual for any countable open covering $\mathfrak{U} = \{U_i\}_{i \in I}$, $I \subset \mathbb{Z}$, $C^q(\mathfrak{U}, \mathcal{S})$ is the set of q–cochains with coefficients in a sheaf $\mathcal{S}$ and $\mathfrak{d} \colon C^q(\mathfrak{U}, \mathcal{S}) \to C^{q+1}(\mathfrak{U}, \mathcal{S})$ is the *coboundary operator*).

Note that η is globally variational if and only if $\delta(\eta) = 0$; if instead $\delta(\eta) \neq 0$ then $\eta = \mathcal{E}_n(\lambda)$ can be solved only locally, *i.e.* for any countable good covering of $\boldsymbol{Y}$ there exists a local Lagrangian λ_i over each subset $\boldsymbol{U}_i \subset \boldsymbol{Y}$ such that $\eta_i = \mathcal{E}_n(\lambda_i)$. A system of local sections λ_i of $(\mathcal{V}^n_r)_{U_i}$ such that $\mathcal{E}_n((\lambda_i - \lambda_j)|_{U_i \cap U_j}) = 0$, is what we call a *local variational problem*; two local variational problems are *equivalent* if and only if they give rise to the same Euler–Lagrange form: $(\{\boldsymbol{U}_i\}_{i \in \mathbb{Z}}, \lambda_i)$ is a *presentation* of the local variational problem [8]; every cohomology class in $H^{n+1}_{dR}(\boldsymbol{Y}) \simeq H^{n+1}(\boldsymbol{Y}, \mathbb{R})$ gives rise to local variational problems. In general, infinitesimal symmetries of different presentations can be different.

For any countable open covering of $\boldsymbol{Y}$, $\lambda = \{\lambda_i\}_{i \in I}$ is then a 0-cochain of Lagrangians in Čech cohomology with values in the sheaf $\mathcal{V}^n_r$, *i.e.* $\lambda \in C^0(\mathfrak{U}, \mathcal{V}^n_r)$. By an abuse of notation we shall denote by η_λ the 0–cochain formed by the restrictions $\eta_i = \mathcal{E}_n(\lambda_i)$ (and so will do at any degree of forms). Of course, $\mathfrak{d}\lambda \equiv \{\lambda_{ij}\} \equiv (\lambda_i - \lambda_j)|_{U_i \cap U_j} = 0$ if and only if λ is globally defined on $\boldsymbol{Y}$; analogously, if $\eta \in C^0(\mathfrak{U}, \mathcal{V}^{n+1}_r)$, then $\mathfrak{d}\eta = 0$ if and only if η is global. Note that $\mathfrak{d}\lambda = 0$ implies $\mathfrak{d}\eta_\lambda = 0$, while by $\mathbb{R}$–linearity we only have $\mathfrak{d}\eta_\lambda = \eta_{\mathfrak{d}\lambda} = 0$ *i.e.* $\mathfrak{d}\lambda \in C^1(\mathfrak{U}, \boldsymbol{K}^n_r)$ [3].

2.1. **Local variational problems equivalent to global ones.** As well known Noether Theorems relate symmetries of a variational problem to conserved quantities; in order to make those theorems effective in the case of local systems, in [8] we tackled the question what the most natural choice for *symmetries of the local variational problem* might be. We use the concept of a *variational Lie derivative* operator $\mathcal{L}_{j_r\Xi}$, defined for any projectable vector field (Ξ, ξ), which was inspired by the fact that the standard Lie derivative of forms with respect to a projectable vector field preserves the contact structure induced by the affine bundles π^r_{r-1} (with $r \geq 1$) [12]. The variational Lie derivative is a local differential operator by which symmetries of Lagrangian and dynamical forms (as well as higher degree classes of forms in the variational sequence) and corresponding Noether theorems, can be characterized [10]. We notice that the variational Lie derivative sends a diagram of cochain complexes into a diagram of cochain complexes and thus defines an operator which acts on cohomology classes.

Let η_λ be the Euler–Lagrange morphism of a local variational problem. The first Noether theorem implies that $0 = \Xi_V \rfloor \eta_\lambda + d_H(\epsilon_i(\lambda_i, \Xi) - \beta(\lambda_i, \Xi))$, where $\epsilon_i := \epsilon_i(\lambda_i, \Xi) = j_r\Xi_V \rfloor p_{d_V\lambda_i} + \xi \rfloor \lambda_i$ is the usual *canonical* Noether current; the current $\epsilon(\lambda_i, \Xi) - \beta(\lambda_i, \Xi)$ is a *local* object and it is conserved along the solutions of Euler-Lagrange equations (*local conservation law*). Local conserved currents can be derived by using Lepagian equivalent of local systems of Lagrangians [4]. Note that the Noether current $\epsilon(\lambda_i, \Xi)$ is conserved if and only if Ξ is also a symmetry of λ_i.

Since a local variational problem has a global Euler–Lagrange morphism defining a topological invariant, there is a precise relation between local conservation laws [8]. In fact, let η_λ be the Euler–Lagrange morphism of a local variational problem and λ_i the system of local Lagrangians of an arbitrary given presentation. The local currents satisfy $d_H(\epsilon(\lambda_i, \Xi) - \beta(\lambda_i, \Xi) - \epsilon(\lambda_j, \Xi) + \beta(\lambda_j, \Xi)) = 0$. Note that, since $0 = \mathcal{L}_{j_r\Xi}\eta = \mathcal{E}_n(\Xi_V \rfloor \eta_\lambda)$, the horizontal n-form $\Xi_V \rfloor \eta_\lambda$ defines a cohomology class and we have that the local currents are the restrictions of a global conserved current if and only if the cohomlogy class $\delta'(\Xi_V \rfloor \eta_\lambda) \in H^n_{dR}(\boldsymbol{Y})$ vanishes. It is noteworthy that even the cohomology class $\delta'(\Xi_V \rfloor \mathcal{E}_n(\lambda))$ may be non trivial (see *e.g.* [8]).

In order to study the obstruction to the existence of a variationally global equivalent to a local variational problem it is of fundamental importance to study how the variational Lie derivative affects cohomology classes. It is notewhorty that, independently from the fact that Ξ be a dynamical form symmetry or not, *the variational Lie derivative trivializes cohomology classes* [17]. In fact, by linearity and resorting to the naturality of the variational Lie derivative we have

$$\eta_{\mathcal{L}_\Xi\lambda_i} = \mathcal{E}_n(\Xi_V \rfloor \eta_\lambda) + \mathcal{E}_n(d_H\epsilon_i) = \mathcal{E}_n(\Xi_V \rfloor \eta_\lambda) = \mathcal{L}_\Xi\eta_\lambda\,.$$

The result that $\mathcal{L}_\Xi\eta_\lambda = \mathcal{E}_n(\Xi_V \rfloor \eta_\lambda)$ is very important for the cohomology since it implies that $\delta(\mathcal{L}_\Xi\eta_\lambda) = \delta(\eta_{\mathcal{L}_\Xi\lambda_i}) = 0$ although $\delta(\eta_\lambda) \neq 0$. From this we see that the variational Lie derivative enables us to transform non trivial cohomology classes to trivial cohomology classes associated with the variational Lie derivative of local presentations. On the other hand, since $\eta_{\mathcal{L}_\Xi\lambda_i} = \mathcal{E}_n(\Xi_V \rfloor \eta_\lambda)$ we see that *Euler-Lagrange equations of the local problem defined by $\mathcal{L}_\Xi\lambda_i$ are equal to*

Euler-Lagrange equations of the global problem defined by $\Xi_V \rfloor \eta_\lambda$. Thus we have the following.

Proposition 1. *The local problem defined by the local presentation* $\mathcal{L}_\Xi \lambda_i$ *is variationally equivalent to a global one.*

It is noteworthy for the sequel that this result holds true at any degree k in the variational sequence; specifically for local potentials of variationally trivial Lagrangians. Let μ be a variationally trivial Lagrangian, *i.e.* such that $\mathcal{E}_n(\mu) = 0$, this means that we have a 0-cocycle of currents ν_i such that $\mu = d_H \nu_i$ and $\mathfrak{d}\mu_\nu = 0$ but we suppose $\delta'(\mu_\nu) \neq 0$. We can consider the Lie derivative $\mathcal{L}_\Xi \nu_i$ and the corresponding $\mu_{\mathcal{L}_\Xi \nu_i}$. By using the expression for the Lie derivative of an $(n-1)$ form according to [10] and, again, resorting to the naturality of the variational Lie derivative, we have

$$\mu_{\mathcal{L}_\Xi \nu_i} = d_H(\Xi_H \rfloor d_H(\nu_i) + \Xi_V \rfloor d_V \nu_i) = \mathcal{L}_\Xi \mu_\nu = d_H(\Xi_H \rfloor \mu_\nu + \Xi_V \rfloor p_{d_V \mu_\nu}) ,$$

so that $\delta'(\mathcal{L}_\Xi \mu_\nu) = \delta'(\mu_{\mathcal{L}_\Xi \nu_i}) = 0$, although $\delta'(\mu_\nu) \neq 0$.

In particular, from the above equation we deduce the following.

Corollary 1. *The local problem defined by* $\mathcal{L}_\Xi \nu_i$ *is variationally equivalent to the global problem defined by* $\Xi_H \rfloor \mu_\nu + \Xi_V \rfloor p_{d_V \mu_\nu}$.

We recall that $\mu_\nu = d_H \nu_i$ is assumed to satisfy $\mathfrak{d}\mu_\nu = 0$, *i.e.* it is a global object. We also notice that $\Xi_V \rfloor d_V \nu_i = \Xi_V \rfloor p_{d_V \mu_\nu} + d_H \phi_i$, with ϕ_i an $(n-2)$ cocycle and it is noteworthy that $\Xi_V \rfloor d_V \nu_i - d_H \phi_i$ is a global object.

Example 1. Let us assume Ξ be a symmetry of dynamical forms: we have $\mathcal{L}_\Xi \eta_\lambda = 0$ then, in particular, $\delta(\mathcal{L}_\Xi \eta_\lambda) \equiv 0$; furthermore, under the same assumption, we have $\mathcal{E}_n(\Xi_V \rfloor \eta_\lambda) = 0$ then there exists a 0-cocycle ν_i as above, defined by $\mu_\nu = \Xi_V \rfloor \eta_\lambda := d_H(\nu_i)$. In this case, divergence expressions of the local problem defined by $\mathcal{L}_\Xi \nu_i$ coincide with divergence expressions for the global current $\Xi_H \rfloor \Xi_V \rfloor \eta_\lambda + \Xi_V \rfloor p_{d_V(\Xi_v \rfloor \eta_\lambda)}$.

As a consequence of the above, we can associate a system of global currents with a system local currents by taking the Lie derivatives of the local system, for which $\delta'(\mathcal{L}_\Xi \mu_\nu) = \delta'(\mathcal{L}_\Xi(\Xi_V \rfloor \eta_\lambda)) = \delta'(\mathcal{L}_\Xi(d_H(\nu_i)) = \delta'(d_H(\mathcal{L}_\Xi(\nu_i)) = 0$ holds true.

A natural question is now if there exists a way to find under which conditions such a variational Lie derivative of local currents is a system of *conserved* currents. The answer to such a question involves Jacobi equations for the local system λ_i. In fact, let us consider the second variational derivative of a presentation of a local problem $\bar{\delta}^2 := \mathcal{L}_\Xi \mathcal{L}_\Xi \lambda_i$; since we are supposing Ξ being a *symmetry of dynamical forms*, we have

$$\mathcal{L}_\Xi \mathcal{L}_\Xi \lambda_i = \mathcal{L}_\Xi(\Xi_V \rfloor \eta_\lambda) + \mathcal{L}_\Xi\big(d_H(\epsilon_i)\big) = \mathcal{L}_\Xi\big(d_H(\nu_i)\big) + \mathcal{L}_\Xi\big(d_H(\epsilon_i)\big) = d_H \mathcal{L}_\Xi(\nu_i + \epsilon_i) .$$

As a consequence we can state the following important result.

Proposition 2. *Let* Ξ *be a symmetry of the Euler-Lagrange form* η_λ *and* $d_H \mathfrak{d}(\nu_i + \epsilon_i) = 0$. *If the second variational derivative is vanishing, then we have the conservation law* $d_H \mathcal{L}_\Xi(\nu_i + \epsilon_i) = 0$, *where* $\mathcal{L}_\Xi(\nu_i + \epsilon_i)$ *is a local representative of a global conserved current.*

The global current is given by

$$\Xi_H \rfloor \mu_{\nu+\epsilon} + \Xi_V \rfloor p_{d_V \mu_{\nu+\epsilon}} \equiv \Xi_H \rfloor d_H(\nu_i + \epsilon_i) + \Xi_V \rfloor p_{d_V(d_H(\nu_i+\epsilon_i))} \,.$$

Notice that the condition $\mathcal{L}_\Xi \mathcal{L}_\Xi \lambda_i = 0$ means that the symmetry Ξ is required to be a symmetry of dynamical forms *and* also a symmetry of the local variational problem $\mathcal{L}_\Xi \lambda_i$. We also stress that since Ξ is only a symmetry of dynamical forms and not a symmetry of the Lagrangian, the current $\nu_i + \epsilon_i$ *is not* a conserved current and it is such that $d_H(\nu_i + \epsilon_i)$ is locally equal to $d_H \beta_i$.

Example 2. We deduce the existence of a global conserved current associated with the Chern-Simons Lagrangian λ_{CS}. Let $\nu_i + \epsilon_i$ be a 0-cocycle of 'strong' Noether currents for the Chern-Simons Lagrangian. Since given a symmetry Ξ of the Chern-Simons equations, for the variational Lie derivative of a local Lagrangian, we have $\mathcal{L}_\Xi \lambda_{CS_i} = d_H \beta_i$ and it is easy to verify that for the Chern-Simons Lagrangian we have $\mathfrak{d} \mathcal{L}_\Xi \lambda_{CS_i} = \mathcal{L}_\Xi \mathfrak{d} \lambda_{CS_i} = \mathcal{L}_\Xi d_H \gamma_i$, we have that $\mathfrak{d} d_H \beta_i = \mathcal{L}_\Xi d_H \gamma_i$. Thus $\mathfrak{d} d_H(\nu_i + \epsilon_i) = \mathcal{L}_\Xi d_H \gamma_i$. We see that the conditions of the above propositions are verified when $\mathcal{L}_\Xi d_H \gamma_i = 0$. The explicit expression of a global conserved current associated with the Chern-Simons Lagrangian is then given by the direct application of the above Proposition:

$$\Xi_H \rfloor \mathcal{L}_\Xi \lambda_{CS_i} + \Xi_V \rfloor p_{d_V \mathcal{L}_\Xi \lambda_{CS_i}} \,.$$

and we can see that it is a global conserved current variationally equivalent to the variational derivative of the 'strong' Noether currents $\nu_i + \epsilon_i$. Generators of such a current are in the kernel of the second variational derivative and are symmetries of the variationally trivial Lagrangian $d_H \gamma_i$. This expression can be compared with results given in [1].

Remark 1. As it is well known, one of the results of the variational sequence theory, related to the inverse problem of the calculus of variations, states that a dynamical form η is locally variational if and only if its Helmholtz form vanishes. Invariance properties of classes in the variational sequence suggested to Krupka *et al.* the idea that there should exist a close correspondence between the notions of variationality of a differential form and invariance of its exterior derivative.

Let us take into account symmetries of the Helmholtz form. Since the variational Lie derivative trivializes cohomology classes, we show that not only the invariance of a closed Helmholtz form ζ_{η_i}, *i.e.* $\mathcal{L}_\Xi \zeta_{\eta_i}$ is equivalent with local variationality of the Lie derivative $\mathcal{L}_\Xi \eta_i$, *i.e.* $\zeta_{\mathcal{L}_\Xi \eta_i} = 0$ meaning that the dynamical form $\mathcal{L}_\Xi \eta_i$ is locally the Euler–Lagrange form of a Lagrangian, as shown in [14], but also that the system of local Euler–Lagrange forms $\mathcal{L}_\Xi \eta_i$ is variationally equivalent to a global Euler-Lagrange form (*i.e.* they have the same Helmholtz form).

In fact, it appears noteworty the application of our results above to the $(n+2)$ degree closed variational classes in the Krupka's sequence. Locally variational $(n+1)$ dynamical forms are dragged by the variational Lie derivative to dynamical forms always globally variational. Analogously to what seen before, suppose $\mathcal{E}_{n+2}(\zeta) = 0$ (higher degree Helmholtz conditions) which implies that there exists a local system of Euler–Lagrange forms η_i.

Let Ξ be a symmetry of ζ_{η_i}, *i.e.* $\mathcal{L}_\Xi \zeta_{\eta_i} = 0$; then we have $\mathcal{E}_{n+1}(\mathcal{L}_\Xi(\eta_i)) = 0$. Analogously to what stated above for Euler–Lagrange equations and divergence equations, one can realize that Helmholtz conditions of the local problem $\mathcal{L}_\Xi \eta_i$ are Helmholtz conditions for the global problem defined by $\Xi_V \rfloor \zeta_{\eta_i}$. □

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