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METRIZATION OF CONNECTIONS WITH REGULAR CURVATURE

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Abstract. We discuss Riemannian metrics compatible with a linear connection that has regular curvature. Combining (mostly algebraic) methods and results of [4] and [5] we give an algorithm which allows to decide effectively existence of positive definite metrics compatible with a real analytic connection with regular curvature tensor on an analytic connected and simply connected manifold, and to construct the family of compatible metrics (determined up to a scalar multiple) in the affirmative case. We also breafly touch related problems concerning geodesic mappings and projective structures.

1. Introduction

According to the fundamental theorem of (pseudo-)Riemannian geometry, given a metric g on a manifold, there is a unique symmetric connection ∇ (its Levi-Civita, or Riemann connection) which preserves the scalar product, $\nabla g = 0$. We contribute to its reciprocal. Metrization Problem, MP, for linear connections means: given a manifold M with a symmetric linear connection ∇, decide whether the connection arises from some metric tensor g as the Levi-Civita connection of the corresponding (pseudo-)Riemannian manifold (M, g). If $\nabla g = 0$ holds we say that the metric and the connection are *compatible*.

The MP problem was discussed - in various spaces (in manifolds endowed with a connection, in vector bundles), eventually under various constraint conditions - by various authors, both by mathematicians and mathematical physicists (L. P. Eisenhart and O. Veblen, S. Gołab, A. Jakubowicz, B. G. Schmidt, S. B. Edgar, O. Kowalski, L. Tamássy, M. Anastasiei, G. Thompson, K. S. Cheng and W. T. Ni, M. Cocos etc.). In [8], a possibility to use holonomy groups and holonomy algebras is pointed out, and difficulties arising in C^∞-class are discussed; in [4], among others, positive definite metrics for a symmetric connection with regular curvature are constructed in the favourable case; in [5], positive definite metrics for analytic connections on analytic manifolds are determined by means of an algorithm based on the de Rham decomposition and holonomy algebras; cf. [9] (the case of indefinite metrics, particularly Lorentzian, is different).

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We will keep the following notation. If M is a smooth n-dimensional manifold, $p\colon TM \to M$ denotes its tangent bundle, $\mathcal{X}(M)$ is the $\mathcal{F}(M)$-module of smooth vector fields on M where $\mathcal{F}(M)$ denotes the ring of smooth functions on M. Consider the vector bundles $\Lambda^2(TM)$, $\Lambda^2(T^*M)$, $\mathrm{Hom}(TM,TM)$, the vector space $\mathcal{L}(T_xM)$ of all homomorphisms $\Lambda^2(T_xM) \to \mathrm{End}(T_xM)$, and the space $\mathcal{L}(TM)$ of all smooth bundle morphisms $\Lambda^2(TM) \to \mathrm{Hom}(TM,TM)$.

If (M,g) is a (pseudo-)Riemannian manifold (i.e. g is a metric on M of arbitrary signature) then its curvature tensor[1] R of type $(1,3)$ gives rise to the $(0,4)$ curvature tensor R_g, $R_g(X,Y,Z,W) = g(R(X,Y)Z,W)$, which is usually denoted by the same symbol R. It is a well known fact that among others, $R = R_g$ satisfies $R(X,Y,Z,W) = -R(Y,X,Z,W)$, $R(X,Y,Z,W) = -R(X,Y,W,Z)$, and $R(X,Y,Z,W) = R(Z,W,X,Y)$. Moreover it can be verified that at any point $x \in M$, R induces a homomorphism[2] $\hat{R}_x\colon \Lambda^2(T_xM) \to \mathrm{End}(T_xM)$, $\sigma \mapsto \hat{R}_x(\sigma)$, such that if $\sigma = \sum_i c_i X_i \wedge Y_i \in \Lambda^2(T_xM)$ then

$$\hat{R}_x(\sigma)(Z) = \sum_i c_i R(X_i,Y_i)Z \quad \text{for any} \quad Z \in T_xM\,. \tag{1}$$

Consequently, a bundle morphism $\hat{R}\colon \Lambda^2(TM) \to \mathrm{Hom}(TM,TM)$ is induced.

Let us pay attention to some related algebraic structures with similar characteristic algebraic features or behaviour.

2. Curvature structures for inner product

Let us keep the following notation: if V is an n-dimensional real vector space, V^* denotes its dual, $\mathrm{End}(V) = \mathrm{Hom}(V,V)$ is the vector space of all endomorphisms of V. The second exterior power[3] of V, $\Lambda^2(V)$, consists of antisymmetric type $(0,2)$ tensors on V. The space $\Lambda^2(V^*)$ of antisymmetric $(0,2)$ tensors on the dual V^* will be identified with the dual of $\Lambda^2(V)$, i.e. we use the identification $(\Lambda^2(V))^* \approx \Lambda^2(V^*)$. $S^2(V^*)$ denotes the space of all symmetric bilinear forms on V. $\mathcal{L}(V)$ denotes the space of all homomorphisms $\varrho\colon \Lambda^2(V) \to \mathrm{End}(V)$.

A linear map $\varrho \in \mathcal{L}(V)$ will be called *regular* if any non-vanishing[4] decomposable bivector is mapped onto a non-zero endomorphism[5],

$$X, Y \in V\,, \quad X \wedge Y \neq 0 \implies \varrho(X \wedge Y) \neq 0\,.$$

Let $G \in S^2(V^*)$ be a fixed positive definite symmetric bilinear form on V.

Definition 1. Under a *curvature structure with respect to G* we mean a linear map $\varrho \in \mathcal{L}(V)$ such that the following two conditions hold $(X_1, X_2, Y_1, Y_2 \in V)$:

[1] In terms of the Riemannian (Levi-Civita) connection ∇ of (M,g), the curvature (Riemannian) tensor is defined by $R(X,Y)(Z) = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$ for $X,Y,Z \in \mathcal{X}(M)$.

[2] Given $X_1, X_2, Z \in T_xM$, $X'_i = a^j_i X_j$ then $X'_1 \wedge X'_2 = \det(a^j_i)\, X_1 \wedge X_2$, and we find easily that $R(X'_1, X'_2)Z = \det(a^j_i)\, R(X_1,X_2)Z$.

[3] Its elements, called bi-vectors, are of the form $\sum_{i,j} c^{ij} Z_i \wedge Z_j$, $Z_i \in V$, $c^{ij} \in \mathbb{R}$; elements of the form $X \wedge Y$ are called *decomposable*.

[4] Recall that $X \wedge Y = 0$ if and only if either $Y = 0$ or $X = kY$ for some $k \in \mathbb{R}$.

[5] Of course, we can characterize regularity by the equivalent condition that any non-zero bi-vector is mapped onto a non-zero endomorphism but the above condition is easier to check.

(i) the map $G(\varrho(X_1 \wedge X_2)-,-)\colon V^2 \to \mathbb{R}$ is antisymmetric, i.e. it satisfies

$$G(\varrho(X_1 \wedge X_2)(Y_1), Y_2) = -G(\varrho(X_1 \wedge X_2)(Y_2), Y_1)\,;$$

(ii) the pairs (X_1, X_2), (Y_1, Y_2) are interchangeable,

$$G(\varrho(Y_1 \wedge Y_2)(X_1), X_2) = G(\varrho(X_1 \wedge X_2)(Y_1), Y_2)\,.$$

All curvature structures belonging to a fixed $G \in S^2(V^*)$ form a linear subspace $\mathcal{L}(V,G) \subset \mathcal{L}(V)$. The property (i) can be equivalently written as

$$\text{(i')}\colon G(\varrho(X_1 \wedge X_2)(Y_1), Y_2) + G(Y_1, \varrho(X_1 \wedge X_2)(Y_2)) = 0\,.$$

Remark that for any $\varrho \in \mathcal{L}(V)$ and $G \in S^2(V^*)$, the assignment

$$(\varrho, G) \mapsto \varrho_G\,, \quad \varrho_G(w, X \otimes Y) = G(\varrho(w)(X), Y)\,, \quad w \in \Lambda^2(V)\,,\ X, Y \in V$$

gives rise to a map $S^2(V^*) \times \mathcal{L}(V) \to \Lambda^2(V^*) \otimes (V \otimes V)^*$. There is a canonical injection $\iota\colon \Lambda^2(V^*) \otimes \Lambda^2(V^*) \to \Lambda^2(V^*) \otimes (V \otimes V)^*$. If we denote by $\mathcal{C}(V)$ the linear subspace of all symmetric tensors from $\Lambda^2(V^*) \otimes \Lambda^2(V^*)$, we can check:

ϱ is a curvature structure w.r.t. G if and only if $\varrho_G \in \mathcal{C}(V)$.

Lemma 1. *Let $\varrho \in \mathcal{L}(V,G)$ be a regular curvature structure with respect to a positive definite symmetric bilinear form $G \in S^2(V^*)$. Then for any vectors $X \in V\backslash\{0\}$, $Y \in V$ satisfying $G(X,Y) = 0$ (i.e. forming a G-orthogonal pair) there exists a bivector $w \in \Lambda^2(V)$ such that $\varrho(w)(X) = Y$.*

Proof. For arbitrary $X \in V$, $X \neq 0$, the subset of images of the above shape forms a linear subspace $W_X = \{\varrho(w)(X) \,|\, w \in \Lambda^2(V)\}$ in V. Since G is positive definite, $G(X,X) \neq 0$ holds, and $V = W_X \oplus W_X^{\perp}$. Let us check that W_X is just the orthogonal complement of span$\{X\}$, or equivalently, span$\{X\}^{\perp} = W_X$. Due to symmetry and (i'), $G(\varrho(w)(X), X) = 0$, therefore $\varrho(w)(X) \perp X$. Assume $Y \neq 0$ with $Y \perp X$. Consider the orthogonal decomposition $Y = Y_1 + Y_2$, $Y_1 \in W_X$, $Y_2 \in {W_X}^{\perp}$. Obviously, $Y_2 \perp \varrho(w)(X)$ for any $w \in \Lambda^2(V)$. Consequently, for any $Z_1, Z_2 \in V$, $G(\varrho(X \wedge Y_2)(Z_1), Z_2) = G(\varrho(Z_1 \wedge Z_2)(X), Y_2) = 0$. Hence $\varrho(X \wedge Y_2) = 0$. Due to $X \neq 0$ and regularity, the zero morphism can arise only if $Y_2 = kX$ for certain $k \in \mathbb{R}$. But $0 = G(X, Y_1 + Y_2) = G(X, Y_2) = kG(X,X)$, that is, $k = 0$, and $Y = Y_1 \in W_X$. Hence ${W_X}^{\perp} = \text{span}\{X\}$, and $Y = \varrho(w)(X)$ for some w whenever Y and X are G-orthogonal. □

For any $\varrho \in \mathcal{L}(V)$, let us introduce a linear subspace H_ϱ in $S^2(V^*)$ by

$$(2) \quad H_\varrho = \{F \in S^2(V^*) \mid F(\varrho(X_1 \wedge X_2)(Y_1), Y_2) + F(Y_1, \varrho(X_1 \wedge X_2)(Y_2) = 0\}\,.$$

That is, endomorphisms $\varrho(w)$, $w \in \Lambda^2(V)$ are skew-adjoint relative to any symmetric form $F \in H_\varrho \subset S^2(V^*)$. Obviously, $G \in H_\varrho$ whenever ϱ is a curvature structure relative G.

Theorem 1. *Let $G \in S^2(V^*)$ be positive definite. If ϱ is a regular curvature structure w.r.t. G then the space H_ϱ is 1-dimensional, $H_\varrho = \text{span}\{G\}$.*

Proof. Let $F \in H_\varrho$. We find $k \in \mathbb{R}$ such that $F = kG$. In (V, G), choose a G-orthonormal basis $\langle e_1, \dots e_n \rangle$ of V. For any pair $X \perp Y$ (orthogonal w.r.t. G), $X \neq 0$, we get orthogonality w.r.t. F. Indeed, by Lemma 1, $Y = \varrho(w)(X)$ for some $w \in \Lambda^2(V)$. Due to symmetry and (2), $F(X, Y) = F(X, \varrho(w)(X)) = -F(\varrho(w)(X), X) = -F(Y, X) = 0$. Consequently, $F(e_i, e_j) = 0$ for $i \neq j$, $1 \leq i, j \leq n$, and, since $e_i + e_j \perp e_i - e_j$, we get $0 = F(e_i + e_j, e_i - e_j) = F(e_i, e_i) - F(e_j, e_j)$. That is, $k = F(e_i, e_i) = F(e_j, e_j)$ must be a fixed constant. Hence $F(X, Y) = \sum_{i,j} X^i Y^j F(e_i, e_j) = \sum_{i=1}^n X^i Y^i F(e_i, e_i) = kG(X, Y)$, and $F = kG$ with $k = F(e_1, e_1)$. □

3. Riemannian metrics

Let (M, ∇) be an n-dimensional manifold endowed with a linear connection, and let R be its curvature. Let us use the above algebraic results on any fibre T_xM of TM, $x \in M$. We say that $x \in M$ is a *regular point* of $\varrho \in \mathcal{L}(TM)$ if ϱ_x is regular on T_xM, and that ϱ is *regular* on M if all points of M are regular.

If $G_x \in S^2(T_x^*M)$ is a positive definite scalar product on the tangent space T_xM, $x \in M$, then $\hat{R}_x \in \mathcal{L}(T_xM)$, derived from R by the formula (1), is surely a curvature structure[6] for G_x. If g is a Riemannian metric on M we define a curvature structure with respect to g pointwise, and introduce the subspace $\mathcal{L}(M, g) \subset \mathcal{L}(M)$; the curvature tensor R of (M, g) satisfies $R \in \mathcal{L}(M, g)$. Similarly as in (2), for every $x \in M$ we introduce a subspace $H_{\hat{R}_x} =: H^0(x)$ consisting of all $G_x \in S^2(T_x^*M)$ relative to which all elements $\hat{R}_x(X_1 \wedge X_2)$ are skew-adjoint, i.e. the following holds for any $X_1, X_2, Y_1, Y_2 \in T_xM$:

$$G_x(\hat{R}_x(X_1 \wedge X_2)Y_1, Y_2) + G_x(Y_1, \hat{R}_x(X_1 \wedge X_2)Y_2 = 0\,.$$

Their collection forms the bundle

$$H^0(M) \to M, \qquad H^0(M) = \bigcup_{x \in M} H_{\hat{R}_x}\,. \tag{3}$$

As a consequence of Lemma 1 and Theorem 1 we get

Corollary 1. *Let (M, g) be a Riemannian manifold such that each point of M is regular w.r.t. the curvature tensor R. Then at each point $x \in M$, the space $H^0(x) = H_{\hat{R}_x}$ is 1-dimensional, that is, $H^0(M)$ is a line-bundle.*

On a connected manifold M with $\dim M \geq 3$, a Riemannian metric is determined by its curvature R, provided the subset of R-regular points is dense, uniquely up to a scalar multiple, [4, p. 133].

[6] As above, we can introduce $R_{G_x,x}$ by $R_{G_x,x}(\sigma, Y \otimes Z) = G_x(\hat{R}_x(\sigma)(Y), Z)$ for $Y, Z \in T_xM$, $\sigma \in \Lambda^2(T_xM)$. Then we have a map $(G_x, \hat{R}_x) \mapsto R_{G_x,x}$ of $S^2(T_x^*M) \times \mathcal{L}(T_xM)$ to a particular subspace of $\Lambda^2(T_x^*M) \otimes (T_xM \times T_xM)^*$.

4. Riemannian metrizability in regular case

Let us formulate necessary and sufficient metrizability conditions for linear connection with regular curvature tensor.

Recall that a one-form $\omega\colon M \to T^*M$ on M is *exact* (= gradient) if $\omega = df$ for a certain function f on M.

Theorem 2. *Let (M, ∇) be a manifold with torsion-free linear connection ∇, let the curvature R be regular on M, and let $H^0(M) = \bigcup_{x\in M} H_{\hat{R}_x}$ be the bundle corresponding to the curvature tensor. Then ∇ is a Riemannian connection of a positive-definite metric g if and only if the following conditions hold:*

(1) *$H^0(M)$ is the line bundle,*
(2) *the bundle $H^0(M)$ is metric,*
(3) *any Riemannian metric $\tilde{g}\colon M \to H^0(M)$ is recurrent, $\nabla\tilde{g} = \omega \otimes \tilde{g}$, and the 1-form ω is exact on M.*

Proof. To verify that the conditions are sufficient, let $\tilde{g}\colon M \to H^0(M)$ be a Riemannian metric, and let $\nabla\tilde{g} = \mathrm{d}f \otimes \tilde{g}$ for some function f. Then the tensor field $g = \exp(-f) \cdot \tilde{g}$ is parallel; $\nabla g = 0$. Therefore ∇ is the Levi-Civita connection of (M, g). The conditions are necessary according to [4]. □

Since the condition (1) means that $H^0(x)$ is one-dimensional at any point x, it is sufficient to suppose that the third condition (3) is satisfied for an arbitrary fixed metric. The second condition tells that $H^0(x)$ involves a positive definite symmetric bilinear form on each fibre T_xM, $x \in M$.

5. Real analytic case with regular curvature

In [4], [9], an algorithm is discussed which allows to answer the MP (even without regularity assumption) for positive definite metrics on an analytic, connected and simply connected manifold with an analytic linear connection. The procedure is based on the philosophy that a manifold carries a structure invariant under parallel transport if and only if this stucture is invariant at a single point under the holonomy group (which can be expressed in terms of the corresponding Lie algebra). The Lie algebra of the holonomy group is generated by the curvature endomorphisms, arising from the curvature tensor and its covariant derivatives. All compatible positive metrics can be described explicitely. If the curvature tensor is regular, the process is simplified considerably.

So let M be a connected simply connected analytic n-manifold endowed with an analytic symmetric linear connection ∇ whose curvature R is regular. Recall that in the analytic case, the holonomy group $\mathrm{Hol}(x)$ is a connected Lie subgroup of the automorphism (transformation) group $GL(T_xM)$ of the fibre, coincides with the restricted holonomy group (component of unit), $\mathrm{Hol}(x) = \mathrm{Hol}_0(x)$, and is therefore uniquely determined by its Lie algebra $\underline{hol}(x)$, i.e. its tangent space at unit. Holonomy groups in different points are isomorphic, hence we can define the abstract holonomy group of the connection, Hol^∇, [3, I], with the Lie *holonomy algebra* $\underline{hol}$. Recall that Hol_0^∇ is trivial if and only if the connection is flat.

Furthermore, in the analytic case $\mathrm{Hol}_0(x) = \mathrm{Hol}'(x)$ (the infinitesimal holonomy group), the same for Lie algebras. But for smooth connections, $\underline{hol}'(x)$ is, as a vector space, a span of endomorphisms $\nabla^k R(X, Y; Z_1, \dots, Z_k)$, $0 \le k < \infty$, $X, Y, Z_1, \dots, Z_k \in T_x M$, [3, I]. Hence the restricted holonomy group of a real analytic connection is fully determined by values of all $\nabla^k R$, $0 < k$, in a point x.

The restricted holonomy group of any Riemannian manifold (M, g) is a closed connected subgroup of the orthogonal group, and in particular it is compact, [2]; $\mathrm{Hol}(x)$ identifies with a subgroup of $O(T_x M)$, g is $\mathrm{Hol}(x)$-invariant. For connected, simply connected M, it is sufficient to find a $\mathrm{Hol}(x)$-invariant positive definite $G_x \in S^2(T_x^* M)$ in one point $x \in M$, and to induce a compatible metric via parallel transport, [8], [5], [9]. The space of all $\mathrm{Hol}(x)$-invariant forms is characterized as a subspace $H(x) \subset S^2(T_x^* M)$ consisting just of all forms G_x satisfying

$$G(AX, Y) + G(X, AY) = 0 \tag{4}$$

for all $A \in \underline{hol}(x)$, $X, Y \in T_x M$. Introduce a sequence of subalgebras in $\underline{hol}(x)$ by

$$\underline{h}^{(r)}(x) = \mathrm{span}\,\{\nabla^k R(X, Y; Z_1, \dots, Z_k) \,|\, 0 \le k \le r\}.$$

Note that $H^0(x)$ consists just of all forms with respect to which all elements $A \in \underline{h}^{(0)}(x)$ are self-adjoint (i.e. satisfy (4)); $H(x) \subset H^0(x)$ for all $x \in M$.

Lemma 2. *Let M be connected, simply connected manifold endowed with a torsion-free linear connection ∇, $x \in M$. A symmetric bilinear form G_x on $T_x M$ is* Hol*-invariant if and only if $G_x \in H(x)$,* [9, L. 3]*,* [5, p. 3].

If the manifold is connected it is sufficient to know the metric form at one point, and to enlarge it by parallel transport, hence the following holds.

Theorem 3. *Let us given (M, ∇), M connected, ∇ symmetric. If there is a (non-degenerate) symmetric bilinear form $G_x \in H(x)$ in one point $x \in M$ then there exists on M a metric of the same signature and compatible with ∇,* [9, Th. 1]*,* [8]*.*

If $\dim \underline{h}^{(r)}(x)$ attains its maximum in some nbd U_x of $x \in M$ for all r, the point is called $\mathrm{Hol}(x)$*-regular*. If this is the case, there exists $N \in \mathbb{N}$ such that $\underline{h}^{(N)}(x) = \underline{h}^{(N+1)}(x) = \dots$, and the same holds in some neighborhood $U_x \ni x$. Consequently, for all $y \in U_x$, $\underline{h}^{(N)}(y) = \underline{hol}(y)$. Hence in a local chart, we are able to decide whether the point is $\mathrm{Hol}(x)$-regular and to calculate $\underline{hol}(y)$ if the answer is affirmative; the algorithm proceeds as follows:

Step (1). Choose a local chart (U, x^i). Calculate the curvature and its covariant derivatives at a $\mathrm{Hol}(x)$-regular point x up to the lowest order N for which the sequence $\underline{h}^{(r)}(x)$, $r \in \mathbb{N}$, stabilizes.

Step (2). Calculate $H^0(x)$, $H(x)$. If $\dim H(x) = 0$ the connection is not metrizable, [5]. In the Riemannian metrizable case, $\dim H^0(x) = 1$ must be satisfied according to the above. Hence the only case favourable for Riemannian metrizability is $\dim H(x) = \dim H^0(x) = 1$.

Step (3). If $H(x) = H^0(x) = \operatorname{span}\{G\}$ for some positive definite form G take $\tilde{g} = G$ (if not ∇ is not Riemannian).

The rest of the algorithm from [5] is trivial: the only endomorphism is identical, $S = id_{T_xM}$, with $N_x = T_xM$ being the null-space of the trivial commutant $C_x = \{0\}$, and $\tilde{g}|N_x = G$. We have the only generator $S = S^{(1)}$ with T_xM as its eigenspace, hence the required decomposition of the tangent bundle is trivial, $L = N_x = T_xM$, $\tilde{g}|L = G$ is positive definite. By [5], G must be recurrent, with the corresponding 1-form exact.

Step (4). We determine a function f with $\nabla G = df \otimes G$ if possible. In case there is no such function the connection ∇ is not metrizable.

Step (5). Compatible metrics are of the form $g = c \cdot \exp(-f) \cdot G$, $c > 0$.

Let us give a pair of easy examples for demonstration.

Example 1 ([10])**.** Let us given a symmetric connection ∇ with non-zero components

$$\Gamma^1_{12} = \cot y\,, \quad \Gamma^1_{13} = \cot z\,, \quad \Gamma^2_{11} = -\sin y \cos y\,,$$
$$\Gamma^2_{23} = \cot z\,, \quad \Gamma^3_{11} = -\sin z \cos z \sin^2 y\,, \quad \Gamma^3_{22} = -\sin z \cos z$$

on the definition domain $M = \mathbb{R} \times (0,\pi) \times (0,\pi)$, with coordinates (x,y,z) (calculations are related to the standard basis $\langle e_1, e_2, e_3\rangle$ of T_pM). Non-zero components of the curvature R are

$$R^1_{212} = \sin^2 z\,, \quad R^2_{112} = -\sin^2 z \sin^2 y\,, \quad R^1_{313} = 1\,,$$
$$R^3_{113} = -\sin^2 y \sin^2 z\,, \quad R^2_{323} = 1\,, \quad R^3_{223} = -\sin^2 z\,,$$

and $\hat{R}(X \wedge Y) = R(X,Y)$ is regular, with matrix representation

$$\begin{pmatrix} 0 & R^1_{212}(X^1Y^2 - X^2Y^1) & R^1_{313}(X^1Y^3 - X^3Y^1) \\ R^2_{112}(X^1Y^2 - X^2Y^1) & 0 & R^2_{323}(X^2Y^3 - X^3Y^2) \\ R^3_{113}(X^1Y^3 - X^3Y^1) & R^3_{223}(X^2Y^3 - X^2Y^1) & 0 \end{pmatrix}.$$

We check $\nabla R = 0$. Hence

$$\underline{h}^{(0)}(x) = \underline{hol}(x) = \operatorname{span}\{R(e_1,e_2), R(e_1,e_3), R(e_2,e_3)\}\,.$$

Let us find a generator of $H(x) = H^0(x)$: so that to calculate G, it is sufficient to consider (4) with $A = R(e_i, e_j)$, $i < j$. We get $G = \operatorname{diag}(\sin^2 y \sin^2 z, \sin^2 z, 1)$, $\nabla G = 0$, $f = \text{const}$, hence ∇ is metrizable, with compatible positive metrics given up to a scalar multiple, $\{cG, c > 0\}$.

Example 2 ([10])**.** Let us take $\Gamma^1_{12} = a > 0$, $\Gamma^2_{11} = b > 0$ as the only non-zero Christoffels of the connection on $\mathbb{R}^2$ with coordinates (x,y). In any point (x,y), the curvature is regular, $R^2_{112} = -a^2$, $R^1_{212} = ab$, zero otherwise; the space $H^0(x,y)$ is generated by a positive definite form: $H^0(x,y) = \operatorname{span}\{G\}$, $G = \operatorname{diag}(b,a)$, $ab > 0$. If we calculate $\nabla R(e_1,e_2;e_1)$ we check that it does not satisfy (4), hence $H(x) = \{0\}$, and the connection is not metrizable. An alternative argumentation: the covariant derivative of $G = b\,dx \otimes dx + a\,dy \otimes dy$ cannot be writen is the form $df \otimes G$ for a function f since $\nabla G = -2ab(dx \otimes dx \otimes dx + dx \otimes dy \otimes dx + dy \otimes dx \otimes dx)$.

Recall that if Hol of (M, g) is reducible then the universal cover of M is a Riemannian product; it is never the case if R is regular on M. The irreducible Hol_0 for Riemannian manifolds are listed and discussed in [1, pp. 643–647].

6. Geodesic mappings, projective structures and metrizability

The topic can be reformulated in terms of geodesic mappings.

Recall that if M and $\bar{M}$ are smooth n-manifolds endowed with smooth linear connections ∇ and $\bar{\nabla}$, respectively, a diffeomorphism $f\colon M \to \bar{M}$ is called a *geodesic mapping* if any (canonically parametrized) geodesic of (M, ∇) is mapped onto an unparametrized geodesic (= pregeodesic) of $(\bar{M}, \bar{\nabla})$, [7] and the references therein. Due to diffeomorphism, the manifolds M and $\bar{M}$ can be in fact idetified (via suitable atlases), and we can work on a common underlying manifold $M \equiv \bar{M}$ (instead of using pull-backs). Introduce the type $(1,2)$ "difference tensor" P of the given connections, $\bar{\nabla}_X Y = \nabla_X Y + P(X,Y)$. There is a geodesic mapping of M onto $\bar{M}$ if and only if there is a 1-form ψ such that $P(X,Y) = \psi(X)Y + X\psi(Y)$; if this is the case we calculate $\psi(X) = \frac{1}{n+1}\mathrm{Tr}\,(Y \mapsto P(*,Y))$.

Two torsion-free connections ∇ and $\hat{\nabla}$ on the same manifold M are *projectively equivalent* if they have the same geodesics as unparametrized curves; the corresponding equivalence class $[\nabla]$ is called a *projective structure* on M. In these terms, a problem closely related to MP can be formulated as follows: given a projective structure $(M, [\nabla])$, we ask whether it may be represented by a metric connection or not. A more or less equivalent formulation: given a pair (M, ∇), find all possible geodesic mappings $f\colon M \to M$ of (M, ∇) onto (pseudo-)Riemannian manifolds (M, g).

Corollary 2. *Let (M, ∇) be a manifold with symmetric ∇ and regular curvature R (or, let $[\nabla]$ be a regular projective structure on M, respectively). If $\dim H^0(x) = 1$ in every $x \in M$ and there is a recurrent positive definite symmetric bilinear form $\tilde{g}\colon M \to H^0(M)$, $\nabla\tilde{g} = \omega \otimes \tilde{g}$, with $\omega = df$ exact, then there is a geodesic mapping of (M, ∇) onto Riemannian spaces (the given projective structure is representable by a metric connection corresponding to $g = e^{-f}\tilde{g}$, respectively).*

A bit more generally, we may be interested in all geodesic mappings $f\colon M \to \bar{M}$ of the given (M, ∇) onto (pseudo-)Riemannian manifolds $(\bar{M}, \bar{g})$. A system of equations (for components $\bar{g}_{ij}(x)$, components $\psi_i(x)$ of a 1-form and a certain function $\mu(x)$) that (locally) controls this question has been found by J. Mikeš, [6], [7, Th.5.3, p. 87].

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