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EVENTUAL DISCONJUGACY OF $y^{(n)} + \mu p(x) y = 0$ FOR EVERY μ

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Abstract. The work characterizes when is the equation $y^{(n)} + \mu p(x) y = 0$ eventually disconjugate for ***every*** value of μ and gives an explicit necessary and sufficient integral criterion for it. For suitable integers q, the eventually disconjugate (and disfocal) equation has 2-dimensional subspaces of solutions y such that $y^{(i)} > 0$, $i = 0, \dots, q-1$, $(-1)^{i-q} y^{(i)} > 0$, $i = q, \dots, n$. We characterize the "smallest" of such solutions and conjecture the shape of the "largest" one. Examples demonstrate that the estimates are sharp.

1. Introduction

Given the differential equation

$$y^{(n)} + \mu p(x) y = 0 \tag{1.1}$$

where $p(x)$ is a continuous, one-signed function on $[a, \infty)$. In the study of singular eigenvalue problems on infinite intervals we came to the question when is Equation (1.1) eventually disconjugate for ***every*** value of μ (i.e., disconjugate on some interval $[x_0(\mu), \infty)$). We characterize this property and discuss the asymptotic behaviour of the corresponding solutions.

Theorem 1. (a) *Equation* (1.1) *is eventually disconjugate for every value of* μ *if and only if*

$$(1.2)_\alpha \qquad \lim_{x\to\infty} \left(x^{n-\alpha-1} \int_x^\infty s^\alpha |p(s)|\, ds \right) = 0$$

for some $\alpha \le n-1$.

(b) *If* $(1.2)_\alpha$ *holds for some real* α*, then* $(1.2)_\beta$ *holds for every* β*,* $\beta < n-1$*. The convergence of the limit* $(1.2)_\alpha$ *is uniform for* $\alpha \le n-1-\varepsilon_0$*,* $\varepsilon_0 > 0$*.*

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For $\alpha < n-1$, $\beta = n-1$ the implication $(1.2)_\alpha \to (1.2)_\beta$ is in general false.

$\alpha = n-1$ plays a special role in $(1.2)_\alpha$. Indeed, for $\alpha = n-1$, $(1.2)_\alpha$ reduces into the integrability condition

$$\int^\infty s^{n-1}|p(s)|\,ds < \infty\,, \tag{1.3}$$

which is a well known necessary and sufficient condition for every solution of Equation (1.1) to be asymptotic to some polynomial. See [Ea, p. 45]. Some equations which satisfy $(1.2)_\alpha$ for all $\alpha < n-1$ but do not satisfy (1.3) are

$$y^{(n)} + \frac{\mu}{x^n \log^q x} y = 0\,, \qquad 0 < q \le 1\,, \tag{1.4}$$

since $x^{n-\alpha-1} \int_x^\infty s^\alpha\ (s^n \log^q s)^{-1}\,ds \sim \log^{-q} x/(n-\alpha-1)$ as $x \to \infty$.

Now we turn to the shape of the solutions of our equation. If Equation (1.1) is eventually disconjugate then for every solution y there exists an integer k, $(-1)^{n-k}\mu p(x) < 0$, such that

$$\begin{aligned} y^{(i)} &> 0\,, \qquad i = 0, \dots, k-1\,, \\ (-1)^{i-k} y^{(i)} &> 0\,, \qquad i = k, \dots, n-1\,, \qquad x_0 \le x < \infty\,. \end{aligned} \tag{1.5}$$

This is equivalent to the $(k, n-k)$-disfocality if Equation (1.1) on $[x_0, \infty)$. Any solution which satisfies (1.5) is bounded, of course, by

$$0 < Ax^{k-1} \le y(x) \le Bx^k\,, \qquad x_0 \le x < \infty\,.$$

Moreover, it is known that there exists a two dimensional subspace of solutions which satisfy (1.5), and a basis $\{y_{k-1}(x), y_k(x)\}$ may be chosen so that $y_{k-1}/y_k \to 0$ as $x \to \infty$. See [Ki], [E1, Chapter 8]. This is easily observed for the Euler's equation $y^{(n)} + cx^{-n}y = 0$ with small c, where $y_{k-1} = x^{r_{k-1}}$, $y_k = x^{r_k}$, respectively, with $k-1 < r_{k-1} < r_k < k$.

When (1.3) happens to hold and all solutions are asymptotically polynomials, the pair of solutions y_{k-1}, y_k are asymptotic to x^{k-1} and x^k, respectively. Here we estimate the solutions of Equation (1.1) when only $(1.2)_\alpha$ holds.

Theorem 2. *Suppose that $(1.2)_\alpha$ holds for some α and let k, $1 \le k \le n-1$ be a fixed integer such that $(-1)^{n-k}\mu p(x) < 0$. There exists a solution $u = u(x,\mu)$ of Equation (1.1) such that for every $\gamma > k-1$,*

$$0 < Ax^{k-1} \le u(x,\mu) \le Bx^\gamma\,, \qquad x \ge x_0(\gamma)\,. \tag{1.6}$$

u is of course the "small" solution which satisfies (1.5). For the "large" solution we conjecture:

Conjecture. *Let $(1.2)_\alpha$ hold for some α and let k be a fixed integer such that $(-1)^{n-k}\mu p(x) < 0$. There exists a solution $v = v(x,\mu)$ of Equation (1.1) such that for every $\delta < k$,*

$$0 < Cx^\delta \le v(x,\mu) \le Dx^k\,, \qquad x \ge x_0(\gamma)\,. \tag{1.7}$$

In spite of the similarity to Theorem 2, we don't know to prove this conjecture.

The examples of the last section demonstrate that the estimate (1.6), (1.7) cannot been improved too much.

2. Proofs

Nonoscillation for every μ has several equivalent appearances:

(a) *Equation* (1.1) *is eventually disconjugate for every value of* μ.

(b) *For **every** integer* $k, 1 \le k \le n-1$, *Equation* (1.1) *is eventually* $(k, n-k)$-*disfocal for every value of* μ.

(c) *For **some** integer* k, $1 \le k \le n-1$, *Equation* (1.1) *is eventually* $(k, n-k)$-*disfocal for every value of* μ.

Note that for convenience (b) is formulated for 'every k', but practically only the integers k such that $(-1)^{n-k}\mu p(x) < 0$ are relevant. For values of k of the opposite parity, $(k, n-k)$-disfocality is trivial.

The equivalence (a) $\leftrightarrow$ (b) is well known and (b) $\rightarrow$ (c) is self evident, so only (c) $\rightarrow$ (b) is to be proved. To show this, we utilize the following result ([E2], [E1, Chapter 7]):

If Equation (1.1) *is* $(k, n-k)$-*disfocal on an interval* I *and* $1 \le \ell \le k$, $\ell \equiv k(\text{mod} 2)$, *then the equation*

$$y^{(n)} + \mu\left(\binom{n-1}{k}\Big/\binom{n-1}{\ell}\right)p(x)y = 0 \tag{2.1}$$

is $(\ell, n-\ell)$-*disfocal on the same interval. If* $n-1 \ge \ell \ge k$, $\ell \equiv k(\text{mod} 2)$, *then the equation*

$$y^{(n)} + \mu\left(\binom{n-1}{k-1}\Big/\binom{n-1}{\ell-1}\right)p(x)y = 0 \tag{2.2}$$

is $(\ell, n-\ell)$-*disfocal there.*

By (c), Equation (1.1) is $(k, n-k)$-disfocal for any μ on some $[x_0(\mu), \infty)$. Applying the last remark for any ℓ, $\ell \equiv k(\text{mod} 2)$, equation (1.1) is also $(\ell, n-\ell)$-disfocal on some other ray $[x_0(\mu'), \infty)$, with a suitable μ' which is determined by (2.1) or by (2.2). Thus (c) implies (b).

Proof of Theorem 1. We begin with part (b) which explains the relations among the limits $(1.2)_\alpha$ for various values of α.

If $(1.2)_\alpha$ holds for some α, $(1.2)_\beta$ evidently holds for $\beta < \alpha$ since

$$x^{n-\beta-1}\int_x^\infty s^\beta |p(s)|\, ds = x^{n-\alpha-1}\int_x^\infty \left(\frac{s}{x}\right)^{\beta-\alpha} s^\alpha |p(s)|\, ds$$

and $\left(s/x\right)^{\beta-\alpha} \le 1$ for $s \ge x$, $\beta < \alpha$.

Now we go the opposite way: Given that $x^{n-\alpha-1}\int_x^\infty s^\alpha |p(s)|\, ds < \varepsilon$ for $x \ge x_0$, we calculate $(1.2)_\beta$ with $\beta = \alpha+1$, assuming that $\beta = \alpha + 1 < n-1$. For every finite b, $x^{n-\alpha-2}\int_x^b s^{\alpha+1}|p(s)|\,ds$ is integrated by parts with $f(s)=s$, $f'=1$, $g'(s)=s^\alpha|p(s)|$, $g(s)=-\int_s^\infty \tau^\alpha|p(\tau)|\,d\tau$:

$$\begin{aligned}
&x^{n-\alpha-2}\int_x^b s^{\alpha+1}|p(s)|\,ds\\
&\qquad= x^{n-\alpha-2}\Big[-s\int_s^\infty \tau^\alpha|p(\tau)|\,d\tau\Big|_{s=x}^{b}-\int_x^b 1\Big(-\int_s^\infty \tau^\alpha|p(\tau)|\,d\tau\Big)\,ds\Big]\\
&\qquad= x^{n-\alpha-2}\Big[-b\int_b^\infty \tau^\alpha|p(\tau)|\,d\tau+x\int_x^\infty \tau^\alpha|p(\tau)|\,d\tau+\int_x^b\Big(\int_s^\infty \tau^\alpha|p(\tau)|\,d\tau\Big)\,ds\Big]
\end{aligned}$$

Since $x^{n-\alpha-1}\int_x^\infty s^\alpha|p(s)|\,ds<\varepsilon$ for $x\ge x_0$, and since $n-\alpha-1>1$, we let $b\to\infty$ and get that for $x>x_0$

$$\begin{aligned}
&x^{n-\alpha-2}\int_x^\infty s^{\alpha+1}|p(s)|\,ds\\
&\qquad= x^{n-\alpha-2}\Big[x\int_x^\infty \tau^\alpha|p(\tau)|\,d\tau+\int_x^\infty\Big(\int_s^\infty \tau^\alpha|p(\tau)|\,d\tau\Big)\,ds\Big]\\
&\qquad\le x^{n-\alpha-2}\Big[x\int_x^\infty \tau^\alpha|p(\tau)|\,d\tau+\int_x^\infty \frac{\varepsilon}{s^{n-\alpha-1}}\,ds\Big]\\
&\qquad<\varepsilon+\frac{\varepsilon}{n-\alpha-2}=\varepsilon\frac{n-\beta}{n-\beta-1}\,.
\end{aligned}$$

This verifies $(1.2)_\alpha\to(1.2)_\beta$ for $\beta=\alpha+1<n-1$ and that the convergence of the limit $(1.2)_\beta$ is uniform for $\beta\le n-1-\varepsilon_0$.

By combining these two types of steps we arrive from any given α to any β, $\beta<n-1$.

Now we turn to part (a). According to Theorems 2.8, 2.9 of [KC], if

$$\limsup_{x\to\infty}\Big(x\int_x^\infty s^{n-2}|p(s)|\,ds\Big)>c_n$$

where c_n is a certain, explicitly known positive constant, then the equation $y^{(n)}+p(x)y=0$ has an oscillatory solution. If Equation (1.1) has no oscillatory solutions for any μ, it is necessary that

$$\lim_{x\to\infty} x\int_x^\infty s^{n-2}|p(s)|\,ds=0\,.$$

By virtue of the proved above this implies that $(1.2)_\alpha$ holds for every $\alpha<n-1$ and the necessity part is proved.

To prove the sufficiency of $(1.2)_\alpha$, recall Lemma 1.6, Lemma 1.18 and Lemma 1.19 of [KC]: *If $y^{(n)}+P(x)y=0$ is $(q,n-q)$-disfocal on $[x_0,\infty)$ and $\int_x^\infty s^{q-1}|P(s)|\,ds\ge\int_x^\infty s^{q-1}|p(s)|\,ds$, then also equation $y^{(n)}+p(x)y=0$ is $(q,n-q)$-disfocal there.*

Euler's equation $y^{(n)}+cx^{-n}y=0$ is eventually $(q,n-q)$-disfocal for well known values of c. So, with $P(x)=cx^{-n}$, we get that

$$\int_x^\infty s^{q-1}|p(s)|\,ds \le \frac{|c|}{n-q}x^{-n+q} \tag{2.3}$$

is a sufficient condition for the eventual $(q,n-q)$-disfocality of $y^{(n)}+p(x)y=0$.

Now, if $(1.2)_\alpha$ holds for some α, then it is clear that $(1.2)_\beta$ also holds for $\beta=q-1<n-1$, i.e., $x^{n-q}\int_x^\infty s^{q-1}|p(s)|\,ds\to 0$. Consequently (2.3) is satisfied for large values of x and the sufficiency part of Theorem 1 is completed. □

Proof of Theorem 2. This theorem considers the asymptotic behavior of the solutions of Equation (1.1) when $(1.2)_\alpha$ holds (but not necessarily (1.3)).

The required solution of (1.1) will be obtained as a solution of the integral equation

$$\begin{aligned} y(x) = (x-x_1)^{k-1} &+ (-1)^{n-k-1}\int_{x_1}^x \frac{(x-\tau)^{k-1}}{(k-1)!} \\ &\times\Big(\int_\tau^\infty \frac{(s-\tau)^{n-k-1}}{(n-k-1)!}\mu p(s)y(s)\,ds\Big)\,d\tau\,. \end{aligned} \tag{2.4}$$

A straightforward differentiation of (2.4) shows that its solution satisfies Equation (1.1). Moreover, if y is a positive solution of (2.4) then

$$y^{(k)}(x) = (-1)^{n-k-1}\int_x^\infty \frac{(s-x)^{n-k-1}}{(n-k-1)!}\,\mu p(s)\,y(s)\,ds > 0\,, \tag{2.5}$$

due to $(-1)^{n-k}\,\mu p(x)<0$. Further differentiations of (2.5) verify that $(-1)^{i-k}$ $y^{(i)}>0$ for $i=k,\dots,n-1$ and integrations of (2.5) on $[x_1,x]$ show that $y^{(i)}>0$, $i=0,\dots,k-1$. Hence inequalities (1.5) are satisfied.

Take a number γ, $k-1<\gamma<k$, and let x_0 be a fixed point and

$$M=m(x_0)=\max_{[x_0,\infty)}\frac{(x-x_0)^{k-1}}{x^\gamma}\,.$$

We choose in (2.4) $x_1\ge x_0>0$ such that

$$x^{k-\gamma}\int_{x_1}^\infty s^{(n-1)-(k-\gamma)}|p(s)|\,ds<\varepsilon/|\mu|M\,.$$

The solution of (2.4) on $[x_1,\infty)$ is obtained as the limit of the iterations

$$y_0(x)=2Mx^\gamma,\qquad y_i(x)=T[y_{i-1}]\,,$$

where $T[y]$ denotes the right hand side of (2.4). First,

$$\begin{aligned}\Big|\mu(-1)^{n-k-1}\int_\tau^\infty \frac{(s-\tau)^{n-k-1}}{(n-k-1)!}p(s)y_0(s)\,ds\Big| &\le 2|\mu|M\int_\tau^\infty s^{n-k-1+\gamma}|p(s)|\,ds \\ &< 2\varepsilon\tau^{\gamma-k}\end{aligned}$$

for $\tau > x_1$. Therefore

$$\begin{aligned} y_1(x) = T[y_0] &\le (x-x_1)^{k-1} + \int_{x_1}^{x} \frac{(x-\tau)^{k-1}}{(k-1)!} 2\varepsilon\tau^{\gamma-k}\,d\tau \\ &= (x-x_1)^{k-1} + \int_{x_1}^{x} \cdots \int 2\varepsilon\tau^{\gamma-k}\,d\tau \qquad (k \text{ integrations }) \\ &= (x-x_1)^{k-1} + 2\varepsilon \frac{(x-x_1)^{\gamma}}{(\gamma-k+1)\cdots\gamma} \\ &\le \Big(M + \frac{2\varepsilon}{(\gamma-k+1)\cdots\gamma}\Big)x^{\gamma} \end{aligned}$$

for $x \ge x_1$, since $m(x_1) \le m(x_0) = M$. Finally, we determine ε to be sufficiently small (perhaps by an additional increase of x_1) so that the last bound is not bigger than $2Mx^{\gamma}$. This completes the estimate

$$(x-x_1)^{k-1} \le y_1(x) \le 2Mx^{\gamma} = y_0(x)\,, \qquad x \ge x_1\,.$$

Since T is a positive operator, the next iteration yields $y_2 = T[y_1] \le T[y_0] = y_1$ and finally

$$(x-x_1)^{k-1} \le \cdots \le y_2 \le y_1 \le y_0 = 2Mx^{\gamma}\,, \qquad x \ge x_1\,.$$

A standard argument shows that this sequence converges to some solution $u = u(x,\mu,\gamma)$ of (1.1) such that $0 < Ax^{k-1} \le u(x,\mu,\gamma) \le Bx^{\gamma}$ for the fixed γ which we took.

However, we claim more, namely, that there exists a solution $u(x,\mu)$ which satisfies the inequality $u(x,\mu) \le Bx^{\gamma}$ for ***every*** $\gamma > k-1$ on some suitable $[x_0(\gamma),\infty)$.

For every two solutions $y_1(x), y_2(x)$ of (1.1), $\lim_{x\to\infty} y_1(x)/y_2(x)$ exists, finite or infinite. Otherwise, if $L = \liminf y_1(x)/y_2(x) \ne \limsup y_1(x)/y_2(x) = M$ then for every $L < c < M$, $y_1 - cy_2$ would be an oscillatory solution. In particular $\lim_{x\to\infty} u(x,\mu,\gamma_1)/u(x,\mu,\gamma_2)$ exists for every γ_1,γ_2. The solutions in the set $\{u(x,\mu,\gamma)\,|\,k-1 < \gamma < k\}$ may have at most two different orders of magnitude, i.e., there cannot exist three solutions such that $u(x,\mu,\gamma_1)/u(x,\mu,\gamma_2) \to \infty$, $u(x,\mu,\gamma_2)/u(x,\mu,\gamma_3) \to \infty$. Otherwise each solution in the 3-dimensional subspace that they span would satisfy inequalities (1.5) (up to $\pm$ sign), which is known to be impossible. Thus $\{u(x,\mu,\gamma)\,|\,k-1 < \gamma < k\}$ consists of at most two subsets Γ_1,Γ_2 such that $u(x,\mu,\gamma_1)/u(x,\mu,\gamma_2) \to L$ as $x \to \infty$, $0 < |L| < \infty$, when $u(x,\mu,\gamma_1), u(x,\mu,\gamma_2) \in \Gamma_1$ and when $u(x,\mu,\gamma_1), u(x,\mu,\gamma_2) \in \Gamma_2$ but $u(x,\mu,\gamma_1)/u(x,\mu,\gamma_2) \to 0$ when $u(x,\mu,\gamma_1) \in \Gamma_1, u(x,\mu,\gamma_2) \in \Gamma_2$ (if $\Gamma_2 \ne \emptyset$). Now we choose $u(x,\mu) = u(x,\mu,\gamma_1)$ as an arbitrary solution in Γ_1 and it satisfies (1.6) for every $\gamma > k-1$ on some suitable $[x_0(\gamma),\infty)$. □

Unfortunately we cannot say anything about the other, "large", solution of (1.1) which satisfies inequalities (1.5). While in the proof above the solution was obtained as a fixed point of a contractive map, a similar technique is not available for the "large" solution. This happens probably because for the "large" solution the integral $\int^{\infty} s^{n-k-1}|p(s)|y(s)\,ds$ is too close to $\int^{\infty} s^{n-1}|p(s)|\,ds$ which does not necessarily exist.

3. Examples

The following examples demonstrate the asymptotic forms of the solutions of equations which satisfy $(1.2)_\alpha$ for $\alpha < n-1$ and are not asymptotic to polynomials.

Example 1. Equation (1.4) with $n = 2$,

$$y'' + \frac{\mu}{x^2 \log x} y = 0\,, \qquad x > 1\,, \tag{3.1}$$

is transformed by $t = \log x$ into $tv'' - tv' + \mu v = 0$, which has a solution

$$v(t,\mu) = \sum_{k=0}^{\infty} \frac{(1-\mu)(2-\mu)\cdots(k-\mu)}{k!\,(k+1)!} t^{k+1}\,.$$

For a fixed ***noninteger*** μ, $(1-\mu)(2-\mu)\cdots(k-\mu)/k! \sim ck^{-\mu}$ as $k \to \infty$ by the Stirling formula, so up to a multiplicative constant

$$v(t,\mu) \sim \sum_{k=1}^{\infty} \frac{t^{k+1}}{k^{\mu}\,(k+1)!} \sim \frac{e^t}{t^{\mu}}$$

(see [PSz, Part IV, 70]) and

$$y(x,\mu) = \sum_{k=0}^{\infty} \frac{(1-\mu)(2-\mu)\cdots(k-\mu)}{k!\,(k+1)!} (\log x)^{k+1} \sim \frac{x}{(\log x)^{\mu}}$$

as $x \to \infty$. Another solution of (3.1) is

$$z(x,\mu) = y \int_x^{\infty} y^{-2}\,dx = x(\log x)^{-\mu} \int_x^{\infty} x^{-2} (\log x)^{2\mu}\,dx \sim (\log x)^{\mu}$$

(verify by l'Hopital rule). $y(x,\mu), z(x,\mu)$ are the "large" and "small" solutions of (3.1), respectively.

It is remarkable that for an integer valued μ, say $\mu = m$, the roles interchange and $y(x,m) = \sum_{k=0}^{m-1} \frac{(1-m)(2-m)\cdots(k-m)}{k!\,(k+1)!} (\log x)^{k+1} \sim c(\log x)^m$ becomes the "small" solution while $z(x,\mu) = y\int y^{-2}\,dx = (\log x)^m \int (\log x)^{-2m}\,dx \sim x/(\log x)^m$ is the "large" solution.

Example 2. $u = x^{k-1}(\log x)^q$ satisfies inequalities (1.5) and it is a solution of

$$u^{(n)} + (-1)^{n-k-1} \frac{q\,(k-1)!\,(n-k)!\,(1+o(1))}{x^n \log x} u = 0\,, \qquad 1 < x < \infty\,, \tag{3.2}$$

where $o(1)$ is a certain polynomial of $(\log x)^{-1}$.

$v = x^k/(\log x)^q$ satisfies (1.5) as well and it is a solution of another equation,

$$v^{(n)} + (-1)^{n-k-1} \frac{q\,k!\,(n-k-1)!\,(1+o(1))}{x^n \log x} v = 0\,, \qquad 1 < x < \infty\,. \tag{3.3}$$

(3.2) and (3.3) are verified by direct calculation. Examples 1 and 2 show that the estimates (1.6), (1.7) are not far away from reality.

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