Adv. Geom. **4** (2004), 41–59 **Advances in Geometry**

Linear systems with multiple base points in $\mathbb{P}^2$

Brian Harbourne and Joaquim Roé

(Communicated by R. Miranda)

Abstract. Conjectures for the Hilbert function $h(n;m)$ and minimal free resolution of the mth symbolic power $I(n;m)$ of the ideal of n general points of $\mathbb{P}^2$ are verified for a broad range of values of m and n where both m and n can be large, including (in the case of the Hilbert function) for infinitely many m for each square $n > 9$ and (in the case of resolutions) for infinitely many m for each even square $n > 9$. All previous results require either that n be small or be a square of a special form, or that m be small compared to n. Our results are based on a new approach for bounding the least degree among curves passing through n general points of $\mathbb{P}^2$ with given minimum multiplicities at each point and for bounding the regularity of the linear system of all such curves. For simplicity, we work over the complex numbers.

1 Introduction

Consider the ideal $I(n;m) \subset R = \mathbb{C}[\mathbb{P}^2]$ generated by all forms having multiplicity at least m at n given general points of $\mathbb{P}^2$. This is a graded ideal, and thus we can consider the Hilbert function $h(n;m)$ whose value at each nonnegative integer t is the dimension $h(n;m)(t) = \dim I(n;m)_t$ of the homogeneous component $I(n;m)_t$ of $I(n;m)$ of degree t. It is well known that $h(n;m)(t) \geqslant \max\big(0, \binom{t+2}{2} - n\binom{m+1}{2}\big)$, with equality for t sufficiently large. Denote by $\alpha(n;m)$ the least degree t such that $h(n;m)(t) > 0$ and by $\tau(n;m)$ the least degree t such that $h(n;m)(t) = \binom{t+2}{2} - n\binom{m+1}{2}$; we refer to $\tau(n;m)$ as the *regularity* of $I(n;m)$.

For $n \leqslant 9$, the Hilbert function [29] and minimal free resolution [17] of $I(n;m)$ are known. For $n > 9$, there are in general only conjectures:

Conjecture 1.1. *Let* $n \geqslant 10$ *and* $m \geqslant 0$; *then*:

(a) $\alpha(n;m) \geqslant m\sqrt{n}$;

(b) $h(n;m)(t) = \max\big(0, \binom{t+2}{2} - n\binom{m+1}{2}\big)$ *for each integer* $t \geqslant 0$; *and*

(c) *the minimal free resolution of* $I(n;m)$ *is an exact sequence*

$$0 \to R[-\alpha-2]^d \oplus R[-\alpha-1]^c \to R[-\alpha-1]^b \oplus R[-\alpha]^a \to I(n;m) \to 0,$$

where $\alpha = \alpha(n;m)$, $a = h(n;m)(\alpha)$, $b = \max(h(n;m)(\alpha+1) - 3h(n;m)(\alpha), 0)$, $c =$