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On the size of maximal caps in $Q^+(5, q)$

Klaus Metsch

Dedicated to Professor Adriano Barlotti on the occasion of his 80th birthday

Abstract. Let q be an odd prime power. A cap of the hyperbolic quadric $Q^+(5, q)$ is a set of points of $Q^+(5, q)$ that does not contain three collinear points. It is called maximal, if it is not contained in a larger cap. It is easy to see that a cap has size at most $q^3 + q^2 + q + 1$ with equality if and only if it meets every plane of $Q^+(5, q)$ in a conic. Caps of this size do exist. The largest known maximal cap with less than $q^3 + q^2 + q + 1$ points has size $q^3 + q^2 + 1$. In this paper it is shown that for large odd q all caps with more than $q^3 + q^2 + 2$ points are contained in a cap of size $q^3 + q^2 + q + 1$. Also, some structural information is given on a hypothetical maximal cap of size $q^3 + q^2 + 2$. This result is an analogue of an extension result in the case that q is even. In contrast to the even case, the odd case heavily relies on algebraic arguments.

1 Introduction

For background on plane arcs, conics in planes, and quadrics of projective spaces we refer to the book of J. Hirschfeld [4] on Galois geometries.

Consider the hyperbolic quadric $Q^+(5, q)$ of $PG(5, q)$. A *cap* of $Q^+(5, q)$ is a set of points of $Q^+(5, q)$ that does not contain three collinear points. A cap C is called *maximal*, if it is not contained in larger cap, that is, if every point of $Q^+(5, q) \setminus C$ lies on a line that meets C in two points.

If q is odd, then a cap of $Q^+(5, q)$ has at most $q^3 + q^2 + q + 1$ points. This can be seen by a simple counting argument using that an arc of the plane $PG(2, q)$, q odd, has at most $q + 1$ points. Caps of size $q^3 + q^2 + q + 1$ have the property that they meet every plane of $Q^+(5, q)$ in a conic. Glynn [3] constructed caps of size $q^3 + q^2 + q + 1$ as the intersection of $Q^+(5, q)$ with another quadric of $PG(5, q)$. L. Storme [5] described the caps of size $q^3 + q^2 + q + 1$ for large odd q. More precisely, he shows for $q > 3138$ that such a cap is the intersection of $Q^+(5, q)$ with another quadric.

Now suppose that q is even. Then arcs in planes $PG(2, q)$ can have $q + 2$ points, so the same counting argument shows that a cap of $Q^+(5, q)$ has at most $(q^2 + 1)(q + 2)$ points. For $q = 2$, one can easily construct a cap of size $(q + 2)(q^2 + 1) = 20$ as the image under the Klein-correspondence of the set of the nonabsolute lines of a symplectic polarity in $PG(3, q)$. However, for $q > 2$, q even, the largest known caps have

$q+1$ points less; it is also known that every cap with more than $(q^2+1)(q+2)-q-1$ points can be extended to a cap with $(q^2+1)(q+2)$ points. This extension result has been proved by Ebert, Szönyi and Metsch [2]. So, for $q>2$ even the open question is whether or not there exists a cap with $(q^2+1)(q+2)$ points.

It is the purpose of this paper to prove an extension result for caps in the case q odd, which is in a way similar to the one in the even case. It was my hope to determine the largest size of a cap that is not contained in a cap of size q^3+q^2+q+1. There exists an example with q^3+q^2+1 points, which is due to L. Storme. The following theorem, which will be proved in this paper, misses this example only by one.

Theorem 1.1. *Every cap of $Q^+(5,q)$, q odd, $q>4363$, with more than q^3+q^2+2 points is contained in a cap of size q^3+q^2+q+1 of $Q^+(5,q)$.*

The methods developed to prove this theorem give a lot of information on the structure of a hypothetical maximal cap with q^3+q^2+2 points. The information is collected in the next result.

Theorem 1.2. *Suppose that there exists a maximal cap C of $Q^+(5,q)$, q odd, with q^3+q^2+2 points. If $q>4363$, then one of the following two cases occurs.*

(a) *There exists a line h of $Q^+(5,q)$ with $|h\cap C|=2$. The two planes of $Q^+(5,q)$ on h meet C only in the two points of $h\cap C$. Every other plane of $Q^+(5,q)$ on one of the $q-1$ points P of $h\backslash C$ meets C in a q-arc that can be extended to a conic by adjoining P. All other planes of $Q^+(5,q)$ meet C in a conic.*

(b) *There exists a point $P\in C$ with the following properties. A plane π of $Q^+(5,q)$ that contains P meets C in two points. The other planes of $Q^+(5,q)$ meet C in a conic.*

The proof of the second theorem shows even more, which might be used to show that C is almost (that is except for some points) the intersection of $Q^+(5,q)$ with another quadric Q. Some local results in the present paper in combination with the techniques developed by L. Storme in [5] might show this. I did not try to do so, since I was neither able to show that such an intersection of quadrics leads to a maximal cap with q^3+q^2+2 points, nor could I show that this is not possible.

We remark that both theorems can be translated via the Klein-correspondence to results in $\mathrm{PG}(3,q)$. We also like to mention that the proof of the above mentioned extension result in the even case uses purely combinatorial arguments, whereas the proofs of these two theorems rely on algebraic arguments. The arguments in Section 3 are inspired by the method L. Storme used in [5].

2 The example

The example discussed below is due to L. Storme. Consider in $\mathrm{PG}(3,q)$ the two ovoids $\mathcal{O}$ and E defined by the following quadratic forms:

$$f_{\mathcal{O}} = x_0^2 + bx_0x_1 + x_1^2 + x_2x_3,$$

$$f_E = x_0^2 + bx_0x_1 + x_1^2 + x_2x_3 + x_3^2.$$

These are two ovoids from a pencil of ovoids of type 1(i) given in Table 2 of [1]. The two quadrics share exactly one point Q, namely the one generated by $(0,0,1,0)$. They also have the same tangent plane in Q. While these two ovoids have all their tangents in common when q is even, they share only the $q+1$ tangents in Q when q is odd. In any case, there exists exactly one plane that is a tangent plane to both ovoids, and this is the tangent plane on Q. Using these properties, it is easy to see that every point P with $P \notin \mathcal{O}$ lies on a line l that is a secant of $\mathcal{O}$ but not a secant of E.

Now we go to the hyperbolic quadric $Q^+(5,q)$ defined by a quadratic form in $\mathrm{PG}(5,q)$. Choose a solid Γ such that $E := \Gamma \cap Q^+(5,q)$ is an ovoid of Γ. Let $\mathcal{O}$ be an ovoid such that E and $\mathcal{O}$ are related as above. Then $|\mathcal{O} \cap E| = 1$ and every point of $\Gamma \backslash \mathcal{O}$ lies on a line that meets $\mathcal{O}$ in two points and in E in at most one point.

Let τ be the polarity associated to $Q^+(5,q)$. Then $h := \Gamma^\tau$ is a line that is skew to Γ and has no point in $Q^+(5,q)$. For points $P \in E$, the plane $\langle P,h\rangle$ meets $Q^+(5,q)$ only in P, while for points $P \in \Gamma\backslash E$, the plane $\langle P,h\rangle$ meets $Q^+(5,q)$ in a conic. Let Q be the point of $\mathcal{O} \cap E$. Let C be the set consisting of the point Q and of the points of the conics $Q^+(5,q) \cap \pi$ for the planes $\pi = \langle P,h\rangle$ with $P \in \mathcal{O}\backslash\{Q\}$. Then $|C| = 1 + q^2(q+1)$. As $\mathcal{O}$ is an ovoid, no three of the planes $\langle h,P\rangle$ with $P \in \mathcal{O}$ are contained together in a 3-space. This implies that C is a cap of $Q^+(5,q)$.

Now we show that C is a maximal cap of $Q^+(5,q)$. To see this, consider a point X of $Q^+(5,q)\backslash C$. We show that X lies on a line that meets C in two points. The plane $\tau := \langle h,X\rangle$ meets Γ in a point R that is not in $\mathcal{O}$. Then R lies on a line l of Γ that meets $\mathcal{O}$ in two points P_1 and P_2, and that meets E in at most one point. If σ is the polarity of Γ associated with the ovoid E, then $l^\sigma = l^\tau \cap \Gamma$, as $E = \Gamma \cap Q^+(5,q)$. As l is not a secant of E, the line $l' := l^\sigma$ is a tangent or a secant of E. As $l' \subseteq l^\tau$ and $h = \Gamma^\tau \subseteq (l')^\tau$, the solid $S := \langle l,h\rangle$ is contained in $(l')^\tau$. Thus $(l')^\tau = S$. As $P_1, P_2 \in l$, the conics $C_1 := \langle P_1,h\rangle \cap Q^+(5,q)$ and $C_2 := \langle P_2,h\rangle \cap Q^+(5,q)$ are contained in S. Note that these two conics are disjoint. From $X \in \langle R,h\rangle$ and $R \in l$ we have $X \in \langle l,h\rangle = S$.

First consider the case that l' is a tangent of E. Then $S = (l')^\tau$ meets $Q^+(5,q)$ in a cone with a point vertex over a $Q(2,q)$. As the conics C_1 and C_2 are contained in S and are disjoint, it follows that every point of the cone $S \cap Q^+(5,q)$ lies on a line that meets C_1 and C_2 in distinct points. This applies to X.

Now consider the case when l' is a secant of E. Then $S = (l')^\tau$ meets $Q^+(5,q)$ in a hyperbolic quadric $H = Q^+(3,q)$. As C_1 and C_2 are disjoint conics of this hyperbolic quadric H, every ruling line of H meets these conics in different points. Hence every point of $S \cap Q^+(5,q) = H$ lies on a line that meets C in two points. Again this applies to X.

Remark. If q is even, then almost all of the above argument applies. There is a difference, when the plane $\langle h,X\rangle$ meets Γ in a point of E. Then $R = X$ and X is a point of E. In this case, every line of Γ on R that is not a secant of E is a tangent of E and hence a tangent of $\mathcal{O}$. Thus, it is not possible to find a line l on R that is a secant of $\mathcal{O}$ but not of E. In fact, the q^2 points of $E\backslash\mathcal{O}$ can be adjoined to C, and the set $C \cup E$ is a partial ovoid with q^2 extra points: $|C \cup E| = q^2 + 1 + q^2(q+1)$. It was shown in [2] that this cap is maximal. In fact, this is the largest cap of $Q^+(5, q)$ that is known in the case when $q > 2$ is even, cf. the introduction.

3 Results on conics

An *arc* of the plane $\mathrm{PG}(2, q)$ is a set of points such that every line contains at most two of the points in the set. A *secant* of an arc is a line that meets it in two points. A *tangent* of an arc is a line that meets it in exactly one point.

A conic of $\mathrm{PG}(2, q)$ can be defined by a quadratic form (see [4] for a precise definition). When q is odd, then every point P that is not in the conic C of $\mathrm{PG}(2, q)$ either lies on two tangents and $\frac{1}{2}(q-1)$ secants and is called an *external point* of C, or lies on no tangent and $\frac{1}{2}(q+1)$ secants and is called an *internal point* of C. The following two famous theorems are due to Segre. A proof may be found in Hirschfeld [4].

Result 3.1. *An arc of* $\mathrm{PG}(2, q)$, *q odd, has at most $q+1$ points with equality if and only if it is a conic.*

Result 3.2. *Every arc of* $\mathrm{PG}(2, q)$, *q odd, with more than $q - \frac{1}{4}\sqrt{q} + \frac{7}{4}$ points is contained in a conic.*

Lemma 3.3. (a) *Two different conics of* $\mathrm{PG}(2, q)$ *share at most four points.*

(b) *Let C be a conic and B an arc of* $\mathrm{PG}(2, q)$. *If $|B \cap C| > \frac{1}{2}(q+3)$, then $B \subseteq C$.*

Proof. For (a) see [4]. For (b) suppose that B contains a point P that is not in C. Then P lies on $\frac{1}{2}(q-1)$ or $\frac{1}{2}(q+1)$ secants of C. As $P \in B$, we find on each of these secants a point of C that is not in B. Thus $|B \cap C| \leqslant |C| - \frac{1}{2}(q-1) = \frac{1}{2}(q+3)$. □

Consider a conic C of $\mathrm{PG}(2, q)$, q odd. Embed $\mathrm{PG}(2, q)$ in $\mathrm{PG}(2, q^2)$. Then C lies in a unique conic $\overline{C}$ of $\mathrm{PG}(2, q^2)$. This conic may defined in $\mathrm{PG}(2, q^2)$ by the quadratic form that defines C in $\mathrm{PG}(2, q)$. Consider a point R of $\mathrm{PG}(2, q)$ that is not in C. Then R might be an external or internal point of C. However in $\mathrm{PG}(2, q^2)$, the point R is an external point of $\overline{C}$ and thus lies on two tangents t_1 and t_2 of $\overline{C}$. These lines belong to $\mathrm{PG}(2, q)$ if and only if R is an external point of C. In the case when R in an internal point of C, these lines do not belong to $\mathrm{PG}(2, q)$; we call them the *tangents of C to R in a quadratic extension of* $\mathrm{PG}(2, q)$.

The next result can be deduced from a paper of L. Storme [5].

Lemma 3.4. *Let C and C' be two conics of* $\mathrm{PG}(2, q)$, $q > 16$, *and let R be a point not contained in $C \cup C'$. Then the number s of lines on R that are secant to both conics satisfies one of the following*:

(i) $s = 0$, $s = \frac{(q-1)}{2}$ *or* $s = \frac{(q+1)}{2}$,

(ii) $\frac{q - 3\sqrt{q}}{4} \leqslant s \leqslant \frac{q + 3\sqrt{q}}{4}$.

Moreover, the first case can occur only if R is an external point of both conics and if *both conics have the same tangents through R, or if R is internal point of both conics and if the conics have the same tangents in a quadratic extension of* $\mathrm{PG}(2,q)$.

We will need this result in the following situation. Consider in $\mathrm{PG}(4,q)$ a cone with a point vertex R over a $Q^+(3,q)$. We denote this quadric by $RQ^+(3,q)$. It contains $2(q+1)$ planes that can be called *Greek* and *Latin* planes, such that there are $q+1$ *Greek* planes and $q+1$ *Latin* planes, such that different planes of the same type meet in R, while planes of different types meet in a line on R. Consider two Greek planes π and π', a conic C of π with $R \notin C$, and a conic C' of π' with $R \notin C'$.

Consider a projective collineation κ of $\mathrm{PG}(4,q)$ with $\pi^\kappa = \pi'$ and $\tau^\kappa = \tau$ for all Latin planes τ. Then for every Latin plane τ the line $\tau \cap \pi$ is mapped to the line $\tau \cap \pi'$.

Extend $\mathrm{PG}(4,q)$ to $\mathrm{PG}(4,q^2)$. The form that defines $RQ^+(3,q)$ defines in $\mathrm{PG}(4,q^2)$ a quadric $RQ^+(3,q^2)$ with $RQ^+(3,q^2) \cap \mathrm{PG}(4,q) = RQ^+(3,q)$. This quadric has q^2+1 Latin and Greek planes. The projective collineation κ extends uniquely to a projective collineation $\bar{\kappa}$ of $\mathrm{PG}(4,q^2)$. Also $\bar{\kappa}$ fixes each of the q^2+1 Latin planes of $RQ^+(3,q^2)$.

In $\mathrm{PG}(4,q^2)$, the plane π spans a plane $\bar{\pi}$, and the conic C is contained in a unique conic $\overline{C}$ of $\bar{\pi}$. Though R may be an internal or external point of C, it is always an external point of $\overline{C}$. Thus, it lies on two tangents of $\overline{C}$. These tangents lie in unique Latin planes of $RQ^+(3,q^2)$. We call these the two *Latin tangent planes* of the conic C. Denote by $E(\pi)$ the set consisting of those of the $q+1$ the Latin planes of $RQ^+(3,q)$ for which $\tau \cap \pi$ is a secant of C. Then $|E(\pi)| = \frac{1}{2}(q-1)$ or $|E(\pi)| = \frac{1}{2}(q+1)$. Use the same notation and teminology for E'.

Using the above isomorphism κ from π onto π', Lemma 3.4 gives the following.

Lemma 3.5. *In the above situation, if the two Latin tangents planes of C coincide with those of C', then $E(\pi) = E(\pi')$ or $E(\pi) \cap E(\pi') = \emptyset$. If they do not coincide, then* $\frac{q-3\sqrt{q}}{4} \leqslant |E(\pi) \cap E(\pi')| \leqslant \frac{q+3\sqrt{q}}{4}$.

Lemma 3.6. *Consider in* $\mathrm{PG}(4,q)$ *a quadric that is a cone with a point vertex R over a quadric $Q^+(3,q)$; call its planes Greek and Latin planes as above.*

Let Q be a second quadric of $\mathrm{PG}(4,q)$ *with $R \notin Q$. Suppose that $\pi_1, \pi_2, \ldots, \pi_s$, $s \geqslant 3$, are Greek planes such the sets $C_i := \pi_i \cap Q$ are conics with the same Latin tangent planes.*

(a) *Suppose that R is an external point of each conic $\pi_i \cap Q$. Then there exist two skew lines that both meet each conic C_i.*

(b) *Suppose that R is an internal point of each conic $\pi_i \cap C$. Then there does not exist a Greek plane π on R for which $\pi \cap Q$ is a conic such that R is an external point of $\pi \cap C$.*

Proof. (a) In this part, we do not need to go to the quadratic extension. Let T be the subspace of $\mathrm{PG}(4,q)$ that is perpendicular to R with respect to the quadric Q. As $R \notin Q$, the subspace T has dimension three and $R \notin T$. Let τ_1 and τ_2 be the common Latin tangent planes of all conics C_i. Then each line $\tau_1 \cap \pi_i$ meets Q in a unique point A_i. As RA_i is a tangent, A_i lies in T. Thus all points A_i lie on the line $\tau_1 \cap T$. Similar, we find points B_i on the line $\tau_2 \cap T$.

(b) Now we have to go to the quadratic extension. The form defining Q defines in $\mathrm{PG}(4,q^2)$ a quadric $\bar{Q}$ with $\bar{Q} \cap \mathrm{PG}(4,q) = Q$. Consider the 3-space $\bar{T}$ of $\mathrm{PG}(4,q^2)$ that is perpendicular to R with respect to the quadric $\bar{Q}$. As $R \notin \bar{Q}$, then $\bar{T}$ has dimension three and $R \notin \bar{T}$. The Greek planes π_i span Greek planes $\bar{\pi}_i$ of $\bar{Q}$. As $C_i = \pi_i \cap Q$ is a conic, $\bar{C}_i := \bar{\pi}_i \cap \bar{Q}$ is a conic of $\bar{\pi}_i$.

Let τ_1 and τ_2 be the two common Latin tangent planes of the conics $\bar{C}_i$. As in Part (a), the lines $h_1 := \tau_1 \cap \bar{T}$ and $h_2 := \tau_2 \cap \bar{T}$ contain at least s points of $\bar{Q}$. As $s \geqslant 3$, it follows that h_1 and h_2 are contained in $\bar{Q}$.

Let π be any Greek plane of $RQ^+(3,q)$ for which $C := \pi \cap Q$ is a conic. Extend π and C as above to $\bar{\pi}$ and the conic $\bar{C} := \bar{\pi} \cap \bar{Q}$. Then $\bar{\pi} \cap \bar{T}$ contains the point $A := h_1 \cap \bar{\pi}$ of $\bar{T}$. Hence RA is a tangent of $\bar{C}$. As the Latin plane $\tau_1 = \langle h_1, R \rangle$ does not belong to $\mathrm{PG}(4,q)$ but only to $\mathrm{PG}(4,q^2)$, it follows that R is an internal point of the conic $C = \pi \cap Q$. $\square$

4 A local result

In the next section we shall study large caps in the hyperbolic quadric $Q^+(5,q)$. We shall see there that many tangent hyperplanes of $Q^+(5,q)$ have the property that they meet the cap in a lot of points. It is the purpose of this section to derive local information on the intersection of the cap with such a tangent hyperplane.

Throughout this section, we consider in $\mathrm{PG}(4,q)$ a degenerate quadric that is a cone with a point vertex R over a $Q^+(3,q)$. We denote this quadric by $RQ^+(3,q)$. As in the last section we call the $2(q+1)$ planes contained in $RQ^+(3,q)$ Greek or Latin planes. Notice that every 3-space of $\mathrm{PG}(4,q)$ not containing R meets Q in a hyperbolic quadric $Q^+(3,q)$. A *cap* of $RQ^+(3,q)$ is a set of points of $RQ^+(3,q)$ that does not contain three collinear points. We shall prove the following proposition.

Proposition 4.1. *Suppose that C is a cap of $RQ^+(3,q)$, q odd, with at least q^2+q+2 points and $R \notin C$. Suppose that some Greek or Latin plane meets C in a conic and that R is an external point of this conic. If $q > 4363$, then there exists a quadric Q of $\mathrm{PG}(4,q)$ with $R \notin Q$ such that one of the following cases occurs.*

(a) *$C \subseteq Q$. Also, every Greek and Latin plane on R meets Q in a conic.*

(b) *$C \subseteq Q$. There is one Greek or Latin plane (w.l.o.g. a Greek plane) π_0 such that $|\pi_0 \cap C| = 2$ and such that $h := \pi_0 \cap Q$ is a line. The other q Greek planes meet C in a conic. The two Latin planes that contain a point of $h \cap C$ meet C in a conic. For the $q-1$ Latin planes that meet h in point P with $P \notin C$, the set $\pi \cap C$ is a q-arc and the set $(\pi \cap C) \cup \{P\}$ is a conic. Every Greek and Latin plane other than π_0 meets Q in a conic.*

(c) *We have $|C| \leqslant q^2+q+3$. One Latin plane τ_0 and one Greek plane π_0 meet Q in exactly one point. These two planes meet C in at most three points. Every other Greek and Latin plane σ has the properties that $|\sigma \cap C| \in \{q, q+1\}$, that $\sigma \cap C \subseteq \sigma \cap Q$, and that $\sigma \cap Q$ is a conic.*

(d) *We have* $|C| \leqslant q^2 + q + 12$. *Also* R *lies on a Greek plane* π *that satisfies* $\frac{1}{2}(q-5) \leqslant |\pi \cap C| \leqslant \frac{1}{2}(q+7)$ *and for which the arc* $\pi \cap C$ *is not contained in a conic.*

The proof of this proposition is divided into several lemmas. Throughout this section we suppose that the hypotheses of the proposition are satisfied.

For every Greek or Latin plane π, the set $\pi \cap C$ is an arc of π. Hence, by Result 3.1, we have $|\pi \cap C| \leqslant q+1$ with equality if and only if $\pi \cap C$ is a conic. We put $d_\pi := q + 1 - |\pi \cap C|$ and call d_π the *deficiency* of π. Define

$$e := \frac{1}{4}(\sqrt{q} - 3) \quad \text{and} \quad D := 4\sqrt{q} + 13.$$

Lemma 4.2. *The sum of the deficiencies of the Greek planes is at most* $q-1$. *More than* $q+1-D$ *Greek planes meet* C *in more than* $q+1-e$ *points. For these Greek planes* π, *the arc* $\pi \cap C$ *is contained in a unique conic of* π, *which will be denoted by* C_π. *The same statement is true for the Latin planes, and also the same notation is used for the Latin planes.*

Proof. If π_i, $i = 0, \ldots, q$, are the Greek planes, then $|C| = \sum |\pi_i \cap C| = (q+1)^2 - \sum d_{\pi_i}$, since $R \notin C$. As $|C| \geqslant q^2 + q + 2$, this gives $\sum d_{\pi_i} \leqslant q - 1$. Since $(q-1)/e < D$ (use $q > 4363$), it follows that less than D Greek planes meet C in $q+1-e$ or less points. If the Greek plane π meets C in more than $q+1-e$ points, then Lemmas 3.2 and 3.3 show that $\pi \cap C$ is contained in a unique conic. □

If π is a Greek plane, then we denote by $S(\pi)$ the set consisting of all Latin planes τ for which $\tau \cap \pi$ is a secant of C, that is for which the line $\tau \cap \pi$ meets C in two points. If $|\pi \cap C| > q + 1 - e$, so that $\pi \cap C$ is contained in the conic C_π, then we denote by $E(\pi)$ the set consisting of all Latin planes τ for which the line $\tau \cap \pi$ meets C_π in two points. Clearly $S(\pi) \subseteq E(\pi)$. For Latin planes τ, the same notation is used; of course, $S(\tau)$ and $E(\tau)$ (if defined) are sets of Greek planes.

Lemma 4.3. *Let* M *be the set of all Greek planes* (*or all Latin planes*) π *that satisfy* $|\pi \cap C| > q + 1 - e$ *and* $R \notin C_\pi$. *Let* $\pi, \pi', \pi_i \in M$.

(a) *We have* $|S(\pi)| > |E(\pi)| - e$.

(b) *Either* $E(\pi) = E(\pi')$, *or* $E(\pi) \cap E(\pi') = \varnothing$, *or* $(q - 3\sqrt{q})/4 \leqslant |E(\pi) \cap E(\pi')| \leqslant (q + 3\sqrt{q})/4$.

(c) *If* $E(\pi_1) \cap E(\pi_2) = \varnothing$ *and* $E(\pi_1) \cap E(\pi_3) = \varnothing$, *then* $E(\pi_2) = E(\pi_3)$.

Proof. Part (a) follows from $|\pi \cap C| > q + 1 - e$ and $|C_\pi| = q + 1$. Part (b) follows from Lemma 3.5. Part (c) follows from (b) using that $|E(\pi_i)| \in \left\{\frac{1}{2}(q-1), \frac{1}{2}(q+1)\right\}$ for $i = 1, 2, 3$. □

Lemma 4.4. *Let Q be a quadric of* $\mathrm{PG}(4,q)$. *Suppose that π is a Greek or Latin plane with $|\pi \cap C| > q+1-e$. If Q contains five points of $\pi \cap C$, then $\pi \cap C \subseteq C_\pi \subseteq Q$. If in addition $R \notin Q$, then $C_\pi = \pi \cap Q$.*

Proof. As $\pi \cap Q$ contains five points of an arc, $\pi \cap Q = \pi$ or $\pi \cap Q$ is a conic of π. In the first case we are done. Suppose then that $\pi \cap Q$ is a conic. As this conic shares five points with the conic C_π, we have $\pi \cap Q = C_\pi$ by Result 3.3 (a). □

Lemma 4.5. *Let Q be a quadric of* $\mathrm{PG}(4,q)$ *with $R \notin Q$. Let g (or l) be the number of Greek (or Latin) planes π that satisfy $\pi \cap C \subseteq Q$ and $|\pi \cap C| > q+1-e$. If $l \geqslant 6$, then $g, l \geqslant \frac{1}{2}(q-7) - D$.*

Proof. Let S be the set consisting of the points of C that lie in the l Latin planes τ that satisfy $\tau \cap C \subseteq Q$ and $|\tau \cap C| > q+1-e$. As the sum of the deficiencies of the Latin planes is at most $q-1$, we have $|S| \geqslant l(q+1) - q + 1$. Clearly $S \subseteq Q$.

Consider a Greek plane π with $|\pi \cap C| > q+1-e$. If the set $\pi \cap C$ contains at least five points of S, then $\pi \cap C \subseteq Q$ by Lemma 3.3. As there are at most D Greek planes that meet C in $q+1-e$ or less points, it follows that at most $g+D$ Greek planes have more than four points in S; these planes have of course at most $2l$ points in S, since they meet each of the l Latin planes that define S in a line. The remaining $q+1-g-D$ Greek planes have at most four points in S. As every point of S lies on a unique Greek plane, it follows that

$$(q+1-g-D)\cdot 4 + (g+D)2l \geqslant |S| \geqslant (q+1)l - q + 1.$$

This gives

$$g - 2 \geqslant \frac{1}{2}(q-3) - D - \frac{3q+1}{2(l-2)}.$$

As $l \geqslant 6$ and $q > 4363$, this implies that $g \geqslant 6$ and we obtain in the same way

$$l - 2 \geqslant \frac{1}{2}(q-3) - D - \frac{3q+1}{2(g-2)}.$$

Combine both bounds to

$$g - 2 \geqslant \frac{1}{2}(q-3-2D) - (3q+1)\left(q-3-2D-\frac{3q+1}{g-2}\right)^{-1}.$$

This is equivalent to

$$(q-3-2D)(g-2)^2 \geqslant \frac{1}{2}(q-3-2D)^2(g-2) - \frac{1}{2}(q-3-2D)(3q+1).$$

Divide this by $q-3-2D$ and check that the resulting inequality can be written in the form

$$0 \geqslant (g-6)(q-7-2D-2g)+q-45-8D.$$

As $q > 45+8D$ (this follows from $q > 4363$) and $g \geqslant 6$, this implies that $2g > q-7-2D$. The same bound holds for l. $\square$

Lemma 4.6. *There exists a (possibly degenerate) quadric Q of* $\mathrm{PG}(4,q)$ *with $R \notin Q$ and the following properties. At least $\frac{1}{2}(q-7)-D$ Greek planes π and at least $\frac{1}{2}(q-7)-D$ Latin planes π satisfy $\pi \cap C \subseteq Q$ and $|\pi \cap C| > q+1-e$.*

Proof. (1) Consider the set G consisting of the Greek planes that meet C in more than $q+1-e$ points. Then $|G| \geqslant q+1-D \geqslant 15$ by Lemma 4.2.

(2) In this part of the proof we show that there exist three planes $\pi_1, \pi_2, \pi_3 \in G$ such that $|S(\pi_1) \cap S(\pi_2) \cap S(\pi_3)| \geqslant 6+D$.

As the sum of the deficiencies of the Latin planes is at most $q-1$, there exists a Latin plane τ for which $\tau \cap C$ is a conic. As $R \notin C$, then R lies on at least $\frac{1}{2}(q-1)$ secant lines of this conic. Each of these secant lines lies in a Greek plane, and at most D of the Greek planes obtained in this way do not lie in G. As $\frac{1}{2}(q-1)-D \geqslant 5$, we then find five Greek planes $\pi_1, \dots, \pi_5 \in G$ such that $\pi_i \cap \tau$ is a secant line. As R lies on the secant line $\tau \cap \pi_i$ of $\pi_i \cap C$, then R is not a point of the conic C_{π_i}. Also $\tau \in S(\pi_i) \subseteq E(\pi_i)$. Then, by Lemma 4.3 (b), any two of the sets $E(\pi_i)$ share at least $\frac{1}{4}(q-3\sqrt{q})$ Latin planes.

Put $T_i := E(\pi_1) \cap E(\pi_i)$ for $i = 2, \dots, 5$. Then $|T_i| \geqslant \frac{1}{4}(q-3\sqrt{q})$. Also $T_2 \cup T_3 \cup T_4 \cup T_5 \subseteq E(\pi_1)$ and $|E(\pi_1)| \leqslant \frac{1}{2}(q+1)$. Thus, if u is the largest intersection any two of the sets T_2, T_3, T_4, T_5 have, then $\frac{1}{2}(q+1) \geqslant q-3\sqrt{q}-6u$. Using $q > 4363$, this implies that $u \geqslant \frac{1}{12}(q-1-6\sqrt{q}) \geqslant 3e+6+D$. We may assume that $|T_2 \cap T_3| \geqslant 3e+6+D$. As $T_2 \cap T_3 = E(\pi_1) \cap E(\pi_2) \cap E(\pi_3)$, then Lemma 4.3 (a) proves (2).

(3) Let π_1, π_2, π_3 be as in (2). As $|\tau \cap C| \leqslant q+1-e$ for at most D Latin planes τ, we find six Latin planes $\tau_1, \dots, \tau_6$ in $S(\pi_1) \cap S(\pi_2) \cap S(\pi_3)$ that have each more than $q+1-e$ points in C. Then each of the 18 lines $\pi_i \cap \tau_j$ is a line that meets C in two points.

Choose five points from $\pi_1 \cap C$, five points from $\pi_2 \cap C$, and from each of the three planes τ_i, $i = 1, 2, 3$, one point of $\tau_i \cap C$ that does not lie in $\pi_1 \cup \pi_2$. Since quadrics in $\mathrm{PG}(4,q)$ are determined by 14 points, these 13 points lie on at least a pencil of quadrics of $\mathrm{PG}(4,q)$. Hence these 13 points are contained in a quadric Q with $Q \neq \mathrm{PG}(4,q)$ and $Q \neq RQ^+(3,q)$.

Lemma 4.4 shows that $\pi_1 \cap C \subseteq C_{\pi_1} \subseteq Q$ and that $\pi_2 \cap C \subseteq C_{\pi_2} \subseteq Q$. As τ_i, $i = 1, 2, 3$, shares two points with $\pi_1 \cap C$, two points with $\pi_2 \cap C$, and one more point with Q, the same lemma shows that $\tau_i \cap C \subseteq C_{\tau_i} \subseteq Q$. As $\pi_3 \cap C$ shares two points with each of the arcs $\tau_1 \cap C$, $\tau_2 \cap C$ and $\tau_3 \cap C$, the same lemma shows $\pi_3 \cap C \subseteq C_{\pi_3} \subseteq Q$. Finally, for τ_i, $i = 4, 5, 6$, we now also see that six points of $\tau_i \cap C$ are contained in Q. Hence $\tau_i \cap C \subseteq C_{\tau_i} \subseteq Q$, $i = 4, 5, 6$.

Above we have seen that R does not belong to the conics C_{π_1}, C_{π_2} and C_{π_3}. Assume that $R \in Q$. Then Q meets π_i, $i = 1,2,3$, in a conic and at least one more point. This implies that the planes π_1, π_2, π_3 are contained in Q. Then every Latin plane contains three lines of Q. Hence Q contains all Latin planes, that is Q contains $RQ^+(3,q)$. But this implies that $Q = RQ^+(3,q)$ or that $Q = \mathrm{PG}(4,q)$. This contradicts our choice of Q. Hence $R \notin Q$.

As Q contains the six arcs $\tau_i \cap C$, $1 \leqslant i \leqslant 6$, Lemma 4.5 completes the proof. □

From now on, we denote by Q the quadric constructed in Lemma 4.6.

Lemma 4.7. *Let G be the set consisting of the Greek planes π that satisfy $\pi \cap C \subseteq Q$ and $|\pi \cap C| > q+1-e$. Let π' be a Greek plane with $|\pi' \cap C| > q+1-e$.*

(a) *The point R is not in the conic $C_{\pi'}$.*

(b) *If $\pi' \notin G$, then more than $\frac{2}{3}|G|$ planes π of G satisfy $E(\pi) \cap E(\pi') = \varnothing$.*

Proof. (a) Assume that $R \in C_{\pi'}$. As $C_{\pi'}$ is a conic containing the arc $\pi' \cap C$ and as $R \notin C$, then R lies on $|\pi' \cap C| > q+1-e$ lines of π' that meet C. Thus, all but at most e Latin planes have the property that they meet $\pi' \cap C$. Then Lemma 4.6 shows that we can find five Latin planes τ that meet $\pi' \cap C$ and satisfy $\tau \cap C \subseteq Q$. Thus Q shares five points with $\pi' \cap C$. Then $C_{\pi'} \subseteq Q$ (Lemma 4.4). But $R \notin Q$, a contradiction.

(b) Let L be the set consisting of the Latin planes that meet C in more than $q+1-e$ points. We first show indirectly that for any three different planes $\pi_1, \pi_2, \pi_3 \in G$, the set $S(\pi') \cap S(\pi_1) \cap S(\pi_2) \cap S(\pi_3)$ contains at most two Latin planes of L. Assume that this is not true, so that this set contains three Latin planes τ of L. These planes τ would have two points of C in common with each plane π_i. As $\pi_i \cap C \subseteq Q$ for $i = 1,2,3$, then $\tau \cap C$ has six points in Q. Then $\tau \cap C \subseteq Q$ (Lemma 4.4). As there are three such planes τ, also six points of $\pi' \cap C$ are in Q and therefore $\pi' \cap C \subseteq Q$, a contradiction.

Consider the 3-subsets T of $S(\pi') \cap L$. We have just shown that every set T is contained in at most two sets $S(\pi)$ with $\pi \in G$. As R is an internal or external point of $C_{\pi'}$, then $|S(\pi')| \leqslant \frac{1}{2}(q+1)$. Thus, if s is the number of planes of $S(\pi')$ that do not lie in L, then there are at most $\binom{\frac{1}{2}(q+1)-s}{3}$ sets T. From Lemma 4.2 we have $s \leqslant D$.

If $\pi \in G$ with $E(\pi) \cap E(\pi') \neq \varnothing$, then $|E(\pi) \cap E(\pi')| \geqslant \frac{1}{4}(q - 3\sqrt{q})$ and therefore $|S(\pi) \cap S(\pi')| \geqslant c := \frac{1}{4}(q - 3\sqrt{q}) - 2e$ (Lemma 4.3). Therefore $S(\pi)$ contains at least $c - \binom{s}{3}$ sets T.

Count in two ways the pairs (π, T) of Greek planes $\pi \in G$ and 3-subsets T of $S(\pi') \cap L$ that satisfy $T \subseteq S(\pi)$. If α is the number of planes $\pi \in G$ that satisfy $E(\pi) \cap E(\pi') \neq 0$, then $\alpha\binom{c-s}{3} \leqslant 2\binom{\frac{1}{2}(q+1)-s}{3}$. As $s \leqslant D$ and $c \leqslant \frac{1}{2}(q+1)$, this implies $\alpha\binom{c-D}{3} \leqslant 2\binom{\frac{1}{2}(q+1)-D}{3}$. Using $q > 4363$, it follows that $\alpha < 34$. As $|G| \geqslant \frac{1}{2}(q-7) - D$, this gives $|G| - \alpha \geqslant \frac{2}{3}|G|$. □

Lemma 4.8. *Suppose that there exists a Greek plane that meets C in a conic and such that R is an external point of this conic. Then every Greek plane π_0 with $|\pi_0 \cap C| > q+1-e$ satisfies $\pi_0 \cap C \subseteq Q$.*

Proof. Let G be the set consisting of the Greek planes π that satisfy $|\pi \cap C| > q+1-e$ and $\pi \cap C \subseteq Q$, and let G' be the set consisting of the Greek planes π that satisfy $|\pi \cap C| > q+1-e$ but not $\pi \cap C \subseteq Q$. From Lemma 4.2 we have $|G| + |G'| \geqslant q+1-D$, and Lemma 4.6 we have $|G| \geqslant \frac{1}{2}(q-7-2D)$. From the previous lemma we also know that $R \notin \pi \cap C$ for all $\pi \in G \cup G'$. Lemma 4.4 shows that $C_\pi = \pi \cap Q$ for all $\pi \in G$. We have to show that $G' = \emptyset$.

Assume this is not true and let $\pi_0 \in G'$. By the previous lemma, there exists a subset S of G with $|S| > \frac{2}{3}|G|$ such that $E(\pi_0) \cap E(\pi) = \emptyset$ for all $\pi \in S$. Lemma 4.3 implies that all sets $E(\pi)$ with $\pi \in S$ are the same. If $\pi' \in G'$, then the preceding lemma gives $E(\pi') \cap E(\pi) = \emptyset$ for at least one plane $\pi \in S$, and then Lemma 4.3 implies that $E(\pi') = E(\pi_0)$. Thus any two planes π of $S \cup G'$ give the same or disjoint sets $E(\pi)$. Therefore R is an external point of all conics C_π, or it is an internal point of all conics C_π, $\pi \in S \cup G'$, cf. Lemma 3.5.

First consider the case when R is an internal point of all conics C_π with $\pi \in S \cup G'$. In this case, we consider three planes $\pi_1, \pi_2, \pi_3 \in S$. Then R is an internal point of the conics $C_{\pi_i} = \pi_i \cap Q$. By hypotheses, R lies on a Greek plane σ for which $\sigma \cap C$ is a conic and R is an external point of this conic. As R is an internal point of all conics C_π with $\pi \in G'$, then $\sigma \in G$. Thus $C_\sigma = \sigma \cap Q$. Then Lemma 3.6 (b) gives a contradiction.

Now consider the case when R is an external point of all conics C_π with $\pi \in S \cup G'$. Then Lemma 3.6 (a) shows that we find two skew lines l_1 and l_2 such that l_i, $i = 1,2$, meets all conics C_π with $\pi \in S$. As l_1 as well as l_2 can contain at most two points of the cap C, we have thus found at least $w := 2(|S| - 2)$ points that belong to one of the conics C_π with $\pi \in S$ but that do not lie in C. Thus, the sum of the deficiencies of the planes $\pi \in S$ is at least w. If w' denotes the sum of the deficiencies of the planes in G', then Lemma 4.2 gives $w + w' \leqslant q-1$. We show next that this implies $w' \leqslant |G'| - 3$.

Assume on the contrary that $w' \geqslant |G'| - 2$. Using $w \geqslant 2|S| - 4$ and $|S| \geqslant 2|G|/3$ and $|G| + |G'| \geqslant q+1-D$, it follows that

$$w + w' \geqslant \frac{4}{3}|G| - 4 + |G'| - 2 \geqslant q - 5 - D + \frac{|G|}{3}.$$

As $w + w' \leqslant q-1$, we conclude that $|G| \leqslant 3D + 12$. Combining this with $|G| \geqslant \frac{1}{2}(q-7-2D)$, it follows that $q \leqslant 8D + 31$. But $q > 8D + 45$ (this was already verified in the proof of Lemma 4.5), contradiction.

Then $w' \leqslant |G'| - 3$. Hence G' contains three planes π_1, π_2, π_3 of deficiency zero. Then $S(\pi_i) = E(\pi_i)$, and as we have seen above, $E(\pi_i) = E(\pi_0)$ for $i = 1,2,3$. As in Part (3) of the proof of Lemma 4.6, we see that there exists a quadric Q' with $Q' \neq \mathrm{PG}(4,q)$ and $Q' \neq RQ^+(3,q)$ such that the conics $\pi_i \cap C = C_{\pi_i} \subseteq Q'$ are contained in Q'. As R does not lie in C and thus R does not belong to the conics C_{π_i}, we see as in Part (3) of the proof of Lemma 4.6 that $R \notin Q'$, so that $\pi_i \cap Q' = C_{\pi_i}$. Lemma 3.6 (a) shows that there exist lines l_1', l_2' such that l_i' meets the conics C_{π_1}, C_{π_2}, and C_{π_3}. As $C_{\pi_i} = \pi_i \cap C$, it follows that C contains three collinear points. Contradiction. □

Lemma 4.9. *For every Greek and Latin plane π with more than $\frac{1}{2}(q+11)$ points in C, the set $\pi \cap Q$ is a conic and $\pi \cap C \subseteq Q$. There are at least $q-1$ such Greek and at least $q-1$ such Latin planes.*

Proof. Let g be the number of Greek planes that do not satisfy $\pi \cap C \subseteq Q$, and let l be the number of Latin planes τ that do not satisfy $\tau \cap C \subseteq Q$. By hypothesis of Proposition 4.1 we may assume that R lies on a Greek plane that meets C in a conic and such that R is an external point of this conic. Then the preceding lemma shows that all Greek planes π on R with $|\pi \cap C| > q+1-e$ satisfy $\pi \cap C \subseteq Q$. Then Lemma 4.2 gives $g \leqslant D = 4\sqrt{q}+13$.

Consider a Latin plane τ. Of the $q+1$ lines of τ on R, only g lie in Greek planes π that do not satisfy $\pi \cap C \subseteq Q$, and only these lines can contain points of C that do not lie in Q. Hence, all but at most $2g$ points of $\tau \cap C$ lie in Q. Suppose that $|\tau \cap C| > \frac{1}{2}(q+3)+2g$. Then $\tau \cap Q$ contains more than $\frac{1}{2}(q+3)$ points of the arc $\tau \cap C$. As $R \notin Q$, it follows that $\tau \cap Q$ is a conic. Lemma 3.3 implies that $\tau \cap C \subseteq Q$.

Thus, every Latin plane τ that meets C in more than $(q+3)/2+2g$ points satisfies $\tau \cap C \subseteq Q$. As the sum of the deficiencies of the Latin planes is at most $q-1$ (Lemma 4.2), it follows that at most two Latin planes do not have this property. Hence $l \leqslant 2$. The same argument shows then $g \leqslant 2$. □

Lemma 4.10. *At most eight points of C do not lie in Q. In every Greek and Latin plane lie at most four points that are in C but not in Q.*

Proof. Let π be a Greek plane. Then π is the union of the lines $\pi \cap \tau$ for the Latin planes τ. By Lemma 4.9, at most two Latin planes can contain points of $C \backslash Q$. Hence, at most two of the lines $\pi \cap \tau$ contain points of $C \backslash Q$; clearly on every such line there are at most two points of C. Hence, π has at most four points in $C \backslash Q$. Also, by Lemma 4.9, at most two Greek planes can contain points of $C \backslash Q$, so $|C \backslash Q| \leqslant 8$. □

Lemma 4.11. *Every Latin or Greek plane π meets Q is a point, a line, or a conic.*

Proof. As Q is a quadric with $R \notin Q$, the set $\pi \cap Q$ is a point, a line, a conic, or the union of two lines. We just have to exclude the last possibility. We may assume that π is a Greek plane.

Assume that $\pi \cap Q$ is a union of two lines h_1 and h_2. Then at most four points of $\pi \cap Q$ can be in the cap C. By Lemma 4.9, there exists $q-1$ Latin planes τ for which $\tau \cap C \subseteq \tau \cap Q$ and for which $\tau \cap Q$ is a conic. For at most four of these, the line $\tau \cap \pi$ contains a point of C, and exactly one of these contains the point $h_1 \cap h_2$. Thus, at least $q-6$ of these planes τ have the properties that the line $\tau \cap \pi$ meets $\tau \cap Q = h_1 \cup h_2$ in two points and that these two points are not in C. As $\tau \cap Q$ is a conic, it follows that $|\tau \cap C| \leqslant |\tau \cap Q| - 2 = q-1$. Thus the sum of the deficiencies of these $q-6$ planes τ is at least $2(q-6)$. This contradicts Lemma 4.2. □

Lemma 4.12. *If some Greek or Latin plane meets Q in a line, then conclusion* (b) *of Proposition* 4.1 *holds.*

Proof. We may assume that a Greek plane π meets Q in a line h. Consider the Latin planes τ with the properties that $\tau \cap C \subseteq \tau \cap Q$ and that $\tau \cap Q$ is a conic. All these conics contain a point of h, but only two points of h can be in the cap C. So, at most two of these planes τ have deficiency zero. Then, by Lemma 4.9, at least $q-3$ Latin planes have positive deficiency. Then Lemma 4.2 shows that every Latin plane has deficiency at most three, so Lemma 4.9 implies that all Latin planes τ on R have the property that $\tau \cap C \subseteq \tau \cap Q$ and that $\tau \cap Q$ is a conic. Thus $C \subseteq Q$.

It follows that h meets C in two points, that the two Latin planes on the points of $h \cap C$ have deficiency zero, and that the $q-1$ remaining Latin planes τ on R have deficiency one. Also, for the latter planes τ, the unique point of the conic $\tau \cap Q$ that is not in the q-arc $\tau \cap C$ is the point $\tau \cap h$.

As $C \subseteq Q$, we have $|\pi \cap C| = |h \cap C| = 2$. Hence, π has deficiency $q-1$ and consequently every other Greek plane has have deficiency zero, which implies that it meets C in a conic. □

Lemma 4.13. *If some Greek or Latin plane meets Q in one point, then conclusion* (c) *of Proposition* 4.1 *holds.*

Proof. We may assume that a Greek plane π_0 meets Q in only one point. From Lemma 4.10 we obtain $|\pi_0 \cap C| \leqslant 5$, that is π_0 has deficiency at least $q-4$. If π is any other Greek plane, then π has deficiency at most $d-(q-4) \leqslant 3$, so Lemma 4.9 shows that $\pi \cap C \subseteq Q$ and that $\pi \cap Q$ is a conic. Thus, exactly $q(q+1)+1$ points of Q lie in the Greek planes. Then also exactly $q(q+1)+1$ points of Q lie in the Latin planes.

Then Lemma 4.11 implies that there is a Latin plane τ_0 that meets Q in one point. As for the Greek planes, it follows for the other q Latin planes τ that the set $\tau \cap Q$ is a conic that contains the arc $\tau \cap C$.

All points of C that do not lie in Q lie in π_0 and τ_0 and thus on the line $\tau_0 \cap \pi_0$. As there can be at most two points of C on this line, if follows that π_0 and τ_0 have at most three points in C. The q other Greek planes must thus together contain at least $|C|-3 \geqslant q^2+q-1$ points of C. Thus, all these meet C in q or $q+1$ points (in fact at most one of these meets C in q points). The same holds for the Latin planes. This is the situation described in (c) of Proposition 4.1. □

Lemma 4.14. *If every Greek and Latin plane meets Q in a conic, then one of the conclusions* (a) *and* (d) *of Proposition* 4.1 *hold.*

Proof. If every Greek or every Latin plane π satisfies $\pi \cap C \subseteq Q$, then $C \subseteq Q$ and conclusion (a) of Proposition 4.1 holds. We may thus assume that there exists a Latin plane τ_0 and a Greek plane π_0 such that $\tau_0 \cap C$ is not contained in the conic $\tau_0 \cap Q$, and that $\pi_0 \cap C$ is not contained in the conic $\pi_0 \cap Q$.

Let V be the variety in which the quadrics $RQ^+(3,q)$ and Q meet. As every Greek plane meets Q in a conic, then $|V| = (q+1)^2$. Let n be the number of points of C that are not in V. We have $n \leqslant 8$ from Lemma 4.10. Also $|V \cap C| = |C| - n \geqslant q^2+q+2-n$. Hence, at most $q-1+n$ points of V do not lie in C, that is $|V \setminus C| \leqslant q-1+n$.

If σ is a Greek or a Latin plane with $|\sigma \cap C| > \frac{1}{2}(q+3)+n$, then the conic $\sigma \cap Q = \sigma \cap V$ shares more than $\frac{1}{2}(q+3)$ points with the arc $\sigma \cap C$. In this case, Result 3.3 shows that $\sigma \cap C \subseteq \sigma \cap Q$.

Now consider a Greek or a Latin plane σ for which $\sigma \cap C$ is not contained in the conic $\sigma \cap Q$. Then $|\sigma \cap C| \leqslant \frac{1}{2}(q+3)+n$ and hence $\sigma \cap C$ shares at most $\frac{1}{2}(q+1)+n$ points with Q. In this case, at least $\frac{1}{2}(q+1)-n$ points of the conic $\sigma \cap Q = \sigma \cap V$ do not lie in C. As two different planes σ share at most two points of V (and this of course only when one is a Greek and the other a Latin plane), and as $|V \backslash C| \leqslant q-1+n \leqslant q+7$, it follows that there can be at most two such planes.

Hence τ_0 and π_0 are the only planes σ, for which $\sigma \cap C$ is not contained in V. Then $\tau_0 \cap \pi_0$ is the only line of $RQ^+(3,q)$ that contains points of C that are not in V. As this line meets C in at most two points, this gives $n \leqslant 2$. Then $|V \backslash C| \leqslant q+1$.

As τ_0 contains at least $\frac{1}{2}(q+1)-n$ points of $V \backslash C$, then τ_0 contains at least $\frac{1}{2}(q+1)-n-2 \geqslant \frac{1}{2}(q+1)-4$ points of $V \backslash C$ that do not lie in π_0. As $|V \backslash C| \leqslant q+1$, it follows that π_0 contains at most $\frac{1}{2}(q+1)+4$ points of $V \backslash C$. Since $|\pi_0 \cap V| = q+1$, it follows that $|\pi_0 \cap C \cap V| \geqslant \frac{1}{2}(q+1)-4$. Thus $|\pi_0 \cap C| \geqslant \frac{1}{2}(q+1)-3$. Above we have seen that $|\pi_0 \cap C| \leqslant \frac{1}{2}(q+3)+n \leqslant \frac{1}{2}(q+7)$. Of course, the same bounds hold for $|\tau_0 \cap C|$.

Recall that τ_0 and similarly π_0 contain each at least $\frac{1}{2}(q+1)-4$ points of $V \backslash C$. As $\pi_0 \cap \tau_0$ can have up to two points in Q, it follows that $|V \backslash C| \geqslant (q+1)-4-4-2 = q-9$. Then $|V \cap C| \leqslant |V|-(q-9) = q^2+q+10$. As $|C \backslash V| = n \leqslant 2$, this gives $|C| \leqslant q^2+q+12$.

Finally, as $\pi_0 \cap C$ is not contained in the conic $\pi_0 \cap Q$, it is not possible that $\pi_0 \cap C$ is contained in any conic, since otherwise we would have two different conics that share at least $|\pi_0 \cap V \cap C| \geqslant \frac{1}{2}(q+1)-4 > 4$ points (see Part (a) of Lemma 3.3). □

5 Proofs of the theorems

In this section, we consider $Q^+(5,q)$ embedded in PG$(5,q)$, and a cap C of $Q^+(5,q)$. For every Greek or Latin plane π, the set $\pi \cap C$ is an arc of π. Hence, by Result 3.1, we have $|\pi \cap C| \leqslant q+1$ with equality if and only if $\pi \cap C$ is a conic. We put $d_\pi := q+1-|\pi \cap C|$ and call d_π the *deficiency* of π. It is easy to see (see Part (a) of the following lemma) that $|C| \leqslant q^3+q^2+q+1$ with equality iff and only if every Greek or Latin plane meets C in a conic. In this section, we suppose that C is a maximal cap satisfying $q^3+q^2+2 \leqslant |C| \leqslant q^3+q^2+q$, that is

$$|C| = q^3+q^2+q+1-d \quad \text{with } 1 \leqslant d \leqslant q-1.$$

We also assume that $q > 4363$. We shall see that this implies that $d = q-1$ and that one of the conclusions of Theorem 1.2 is satisfied.

Lemma 5.1. (a) *The sum of the deficiencies of the Greek planes is $d(q+1)$. The sum of the deficiencies of the Latin planes is $d(q+1)$.*

(b) *If π is a Greek plane, then the sum of the deficiencies of the q^2+q+1 Latin* *planes that meet* π *in a line is* $qd_\pi + d$. *An equivalent statement holds for the Latin planes* π.

(c) *Every Greek and Latin plane has deficiency at most* d.

(d) *If* R *is a point of* $Q^+(5,q)$, *then the sum of the deficiencies of the Latin planes on* R *is equal to the sum of the deficiencies of the Greek planes on* R.

Proof. (a) Each point of C lies in $q+1$ Greek planes. Thus, if F is the set of Greek planes, then $\sum_{\pi\in F}|\pi\cap C| = |C|(q+1)$. Using $|C| = q^3+q^2+q+1-d$ and $|F| = q^3+q^2+q+1$, this proves (a) for the Greek planes. The argument for the Latin planes is the same.

(b) Let τ_i, $i = 1,\dots,q^2+q+1$, be the Latin planes that meet π in a line. Each point of $C\cap\pi$ lies in $q+1$ planes τ_i, and each point of $C\backslash\pi$ lies on one plane τ_i. Thus

$$\sum(q+1-d_{\tau_i}) = (|C| - |C\cap\pi|) + |C\cap\pi|(q+1).$$

As $|C\cap\pi| = q+1-d_\pi$, this proves the (b).

(c) This follows immediately from (a) and (b).

(d) Let $\pi_0,\dots,\pi_q$ be the Greek planes on R. If $R\notin C$, then $|R^\perp\cap C| = \sum|\pi_i\cap C|$. If $R\in C$, then $|R^\perp\cap C| = 1+\sum(|\pi_i\cap C|-1)$. As the same holds for the Latin planes on R, this proves (d). □

Let $\perp$ be the polarity of $PG(5,q)$ related to $Q^+(5,q)$. If $R\in Q^+(5,q)$, then $R^\perp$ is the tangent hyperplane of R. It is a subspace of dimension four that contains R, and that meets $Q^+(5,q)$ in a cone $RQ^+(3,q)$ with vertex R over a $Q^+(3,q)$. The set $R^\perp\cap C$ is of course a cap of this cone. If $R\notin C$ and if some Greek (or Latin) plane π on R meets C is a conic, then Lemma 5.1 (b) shows that the sum of the deficiencies of the $q+1$ Latin planes on R is at most $d\leqslant q-1$, that is that $|R^\perp\cap C|\geqslant q^3+q^2+2$. If in addition R is an external point of the conic $\pi\cap C$, then Proposition 4.1 can be applied to the quadric $RQ^+(3,q)$ and its cap $R^\perp\cap C$. Then there exists a quadric of $R^\perp$ satisfying one of the conclusion (a)–(d) of Proposition 4.1. We always denote this quadric by Q_R. We first show that conclusion (d) does not occur.

Lemma 5.2. *Conclusion* (d) *of the Proposition* 4.1 *does not occur.*

Proof. Assume that R is a point of $Q^+(5,q)\backslash C$ that satisfies the hypotheses and conclusion (d) of Proposition 4.1. Then there exists a Greek plane π_0 on R for which the arc $\pi_0\cap C$ cannot be completed to a conic and such that $\frac{1}{2}(q-5)\leqslant|\pi_0\cap C|\leqslant\frac{1}{2}(q+7)$. Thus, if δ is the deficiency of π_0, then $\frac{1}{2}(q-5)\leqslant\delta\leqslant\frac{1}{2}(q+7)$.

We shall obtain a contradiction in the following way. We show that there exists a point $R'\in\pi_0\backslash C$ to which Proposition 4.1 can also be applied and such that $|R'^\perp\cap C| > q^2+q+12$. Thus one of (a), (b), (c), (d) of 4.1 must hold. But $|R'^\perp\cap C| > q^2+q+12$ excludes (d) and the fact that the arc $\pi_0\cap C$ cannot be extended to a conic excludes (a), (b) and (c). This is the desired contradiction.

Consider a Latin plane τ on R that has deficiency zero. Then $\tau\cap C = \tau\cap Q_R$ and this set is a conic. At least $\frac{1}{2}(q-1)$ points of the line $g := \tau \cap \pi_0$ are external points of this conic. Let R_i, $i = 1, \dots, \frac{1}{2}(q-1)$, be points of g that are external points of the conic $\tau \cap Q_R$. The point R_i lies on π_0 and on q further Greek planes; let δ_i be the sum of the deficiencies of these q Greek planes. Then the sum of the deficiencies of all Greek planes that contain a point R_i is $\delta + \sum \delta_i$. As all these planes meet τ in a point and thus even in a line, we obtain $\delta + \sum \delta_i \leqslant d \leqslant q-1$ from Lemma 5.1 (b). As there are $\frac{1}{2}(q-1)$ different points R_i and as $\delta > 0$, then $\delta_i = 1$ for some i. We may assume that $\delta_1 \leqslant 1$.

Put $R' := R_1$. The sum of the deficiencies of the Greek planes on R' is δ or $\delta + 1$. Counting the points of C in the $q+1$ Greek planes on R', we obtain $|R'^{\perp} \cap C| \geqslant (q+1)^2 - 1 - \delta$. Using the upper bound for δ, we obtain $|R'^{\perp} \cap C| > q^2 + q + 12$, as desired. Notice that Proposition 4.1 can be applied to R', since R' is an external point of the conic $\tau \cap C$. □

Lemma 5.1 shows that at least one Greek and one Latin plane has positive deficiency. From now on, we denote by d_0 the smallest positive deficiency of all Greek and Latin planes.

Lemma 5.3. *Let π be a Greek or Latin plane of deficiency d_0. Then there exists a plane τ of $Q^+(5,q)$ of deficiency zero such that $\pi \cap \tau$ is a line that meets C in a unique point.*

Proof. We may assume that π is a Greek plane. As $\pi \cap C$ is an arc with $q+1-d_0$ points, there exist $(q+1-d_0)(d_0+1)$ lines in π that meet C in a unique point. Each of these lines lies in a unique Latin plane. As d_0 is the smallest positive deficiency, Lemma 5.1 (b) implies that at most $(qd_0+d)/d_0$ of these Latin planes have positive deficiency. It suffices therefore to verify that

$$(q+1-d_0)(d_0+1) > (qd_0+d)/d_0.$$

This follows from $1 \leqslant d_0 \leqslant d \leqslant q-1$. □

Lemma 5.4. (a) *If $d_0 < d$, then $d_0 = 1$.*

(b) *Suppose that $d_0 = 1$ and that π is a Greek or Latin plane of deficiency one. Then the q-arc $\pi \cap C$ can be extended to a conic by adjoining a point P. All Greek and Latin planes on P have positive deficiency, and P is the only point of π with this property.*

Proof. (a) Let π be any plane of deficiency d_0. We may assume that π is a Greek plane. From Lemma 5.3 we see that there exists a Latin plane τ of deficiency zero such that $\pi \cap \tau$ is a line that meets C in a unique point. By Lemma 5.1 (b), the sum of the deficiencies of all Greek planes that meet τ in a line is d. The plane π_0 contributes d_0 and the remaining ones contribute $d - d_0$. As d_0 is the smallest positive deficiency, it follows that $d - d_0 = 0$ or $d - d_0 \geqslant d_0$, that is $d_0 = d$ or $d_0 \leqslant d/2$.

Suppose that $d_0 < d$. Then $d_0 \leqslant d/2 \leqslant (q-1)/2$. Let R be a point of the line $\pi \cap \tau$ that is not the point of C on this line. As $\tau \cap C$ is a conic and R is an external point of this conic, we can apply Proposition 4.1 to R. Then (a), (b) or (c) of 4.1 is satisfied, since (d) is excluded by Lemma 5.2. As $|\pi \cap C| = q + 1 - d_0$ and $2d_0 \leqslant q - 1$, Proposition 4.1 shows that $\pi \cap Q_R$ is a conic containing the arc $\pi \cap C$.

Exactly $(q + 1 - d_0)d_0$ lines of π_0 meet C in one and Q_R in two points. Assume that one of these lines has the property that the Latin plane τ_0 on this line has deficiency zero. Then let R' be the point on the line $\pi \cap \tau_0$ that lies in Q_R but not in C. Then $R' \notin C$ and R' is an external point of the conic $\tau_0 \cap C$. We can thus apply Proposition 4.1 and obtain a quadric $Q_{R'}$ of $R'^{\perp}$ with $R' \notin Q_{R'}$. As π has deficiency $d_0 \leqslant d/2$, the same proposition shows that $\pi \cap Q_{R'}$ is a conic containing $\pi \cap C$. Then $\pi \cap Q_R$ and $\pi \cap Q_{R'}$ are conics containing the arc $\pi \cap C$. As $|\pi \cap C| = q + 1 - d_0 \geqslant 5$, then both conics share five points and hence they are equal. However, by the choice of R', the point R' lies in $\pi \cap Q_R$, and as $R' \notin Q_{R'}$, the point R' does not lie in $\pi \cap Q_{R'}$, contradiction.

Hence, the $(q + 1 - d_0)d_0$ lines that meet $\pi \cap C$ in one and $\pi \cap Q_R$ in two points have the property that the Latin plane on it has positive deficiency and thus deficiency at least d_0. The sum of the deficiencies of the Latin planes that meet π in a line is thus at least $(q + 1 - d_0)d_0^2$. Then Lemma 5.1 (b) gives $(q + 1 - d_0)d_0^2 \leqslant d_0 q + d$. As $2d_0 \leqslant d \leqslant q - 1$, it follows that $d_0 = 1$.

(b) We may assume that π is a Greek plane. As the arc $\pi \cap C$ has q points, it can uniquely be extended to a conic by adjoining a point P. The arguments of the proof of Part (a) show that each of the q lines of π on P that meet $\pi \cap C$ has the property that the Latin plane on it has positive deficiency. Thus P lies on at least q Latin planes of positive deficiency. As $d < q$, Lemma 5.1 (b) implies that every Greek and Latin plane on P has positive deficiency. As the sum of the deficiencies of the Latin planes that meet π in a line is $q + d$ (Lemma 5.1), at most $q + d \leqslant 2q - 1$ Latin planes of positive deficiency meet π in a line. This implies that P is unique. □

Lemma 5.5. *Suppose that there exists a point $R \in Q^+(5, q) \setminus C$ for which conclusion* (b) *of Proposition* 4.1 *holds. Then $|C| = q^3 + q^2 + 2$ and C satisfies conclusion* (a) *of Theorem* 1.2, *which is the following*:

There exists a line h of $Q^+(5, q)$ with $|h \cap C| = 2$. The two planes of $Q^+(5, q)$ on h meet C only in the two points of $h \cap C$. Every other plane of $Q^+(5, q)$ on one of the $q - 1$ points P of $h \setminus C$ meets C in a q-arc that can be extended to a conic by adjoining P. All other planes of $Q^+(5, q)$ meet C in a conic.

Proof. From (b) of Proposition 4.1 we find a plane π_0 on R that meets C in exactly two points P_1 and P_2 in C, such that the line $h := P_1P_2$ has the following properties. The point R does not lie on h (since $R \notin Q$ and $h \subseteq Q$ in (b) of Proposition 4.1). The $q - 1$ Latin planes through R and one of the $q - 1$ points P of $h \setminus \{P_1, P_2\}$ meet C in a q-arc, which can be completed to a conic by adjoining P.

For these $q - 1$ points P, Lemma 5.4 shows that all Greek and Latin planes on P have positive deficiency. This gives apart from π_0 another $(q - 1)q$ Greek planes of positive deficiency. As π_0 has deficiency $q - 1$, and as the sum of the deficiencies of all Greek planes is $d(q + 1) \leqslant (q - 1)(q + 1)$ (Lemma 5.1 (a)), it follows that $d = q - 1$, that these $(q - 1)q$ Greek planes have deficiency one, and that every other Greek plane except π_0 has deficiency zero.

Consider $P \in h$ with $P \neq P_1, P_2$. The sum of the deficiencies of the $q+1$ Greek planes on P is $2q-1$. By Lemma 5.1 (b), the same is true for the Latin planes on P. Thus, if τ_0 is the Latin plane on h, then the sum of the remaining q Latin planes on P is $2q-1-d_{\tau_0}$. As there are $q-1$ choices for P, we obtain $d_{\tau_0} + (q-1)(2q-1-d_{\tau_0}) \leqslant d(q+1)$ from Lemma 5.1 (a). This gives $d_{\tau_0} = q-1$, and as for the Greek planes, we see that the other $(q-1)q$ Latin planes on the points P have deficiency one.

Thus, π_0 and τ_0 have deficiency $q-1$, the Greek and Latin planes other than π_0 and τ_0 on a point P of $h \backslash \{P_1, P_2\}$ have deficiency one, and all other Greek and Latin planes have deficiency zero.

Consider any Greek or Latin plane σ of deficiency one. Then σ meets h in a point P with $P \notin C$. We have seen that all Greek and Latin planes on P have positive deficiency. Then Lemma 5.4 (b) shows that P extends the q-arc $\sigma \cap C$ to a conic. □

Lemma 5.6. *If* $d_0 = 1$, *then* $|C| = q^3 + q^2 + 2$ *and* C *satisfies conclusion* (a) *of Theorem* 1.2.

Proof. Suppose $d_0 = 1$ and consider a plane π that meets C in q points. We may assume that π is a Greek plane. Then the q-arc $\pi \cap C$ can be extended to a conic by adjoining one point P and all Greek and Latin planes on P have positive deficiency (Lemma 5.4).

Consider the q lines $t_1, \ldots, t_q$ of π that are tangents to C_π in a point of $\pi \cap C = C_\pi \backslash \{P\}$. By Lemma 5.1 (b), the sum of the deficiencies of the Latin planes that meet π in a line is $q + d \leqslant 2q - 1$. As the $q+1$ Latin planes on P have positive deficiency, we see that at least two of lines t_i have the property that the Latin planes on it have deficiency zero.

As C is a maximal cap, the set $C \cup \{P\}$ is not a cap. Hence, there exists a line g of $Q^+(5,q)$ on P that meets C in two points. This line does not lie in π. Let τ be the Latin plane on g. Then π and τ meet in a line l on P. This line meets $\pi \cap C$ in at most one point. Thus, at most one of the lines t_i meets l in a point of C. From the above arguments, it follows therefore that we find one line $t = t_i$ such that the point $R := t \cap l$ is not in C and such that the Latin plane τ' on t has deficiency zero. Then $\tau' \cap C$ is a conic and, as $|t \cap C| = 1$, the point R is an external point of this conic.

We can therefore apply Proposition 4.1 and obtain a quadric Q_R of $R^\perp$ that satisfies one of the conclusions (a), (b), (c) of Proposition 4.1, since (d) is excluded by Lemma 5.2. This shows that for every Greek and Latin planes σ on R that meets C in more than three points, the set $\sigma \cap Q_R$ is a conic that contains $\sigma \cap C$. Thus $\pi \cap Q_R$ is a conic and therefore $\pi \cap Q_R = (\pi \cap C) \cup \{P\}$. This shows that $P \in Q_R$.

If conclusion (b) of Proposition 4.1 holds for R, then we are done by Lemma 5.5. Now we complete the proof by showing that conclusions (a) and (c) of Proposition 4.1 for the point R cannot occur.

Assume it is (a). Then $R^\perp \cap C \subseteq Q_R$. As $|g \cap C| = 2$ and as the point P lies in $g \cap Q_R$ but not in C, then g has three points in Q_R. But in the situation of 4.1 (a), every Greek and Latin plane on R (and thus the plane τ) meets Q_R in a conic, a contradiction.

Assume that it is (c), that is R lies on a Latin plane τ_0 and on a Greek plane π_0 that meet Q_R each in exactly one point, and every other Greek or Latin plane σ on R meets Q_R in a conic and satisfies $\sigma \cap C \subseteq \sigma \cap Q_R$. This implies that every point of $R^\perp \cap C$ that is not on the line $\pi_0 \cap \tau_0$ lies in Q_R. The line g of the plane τ contains the point P of Q_R and two more points of C. Thus it is not true that $\tau \cap C \subseteq \tau \cap Q_R$ and that $\tau \cap Q_R$ is a conic. Then τ is not one of the planes σ, so we must have $\tau_0 = \tau = \langle R, g \rangle$. Then P is the unique point of $\tau_0 \cap Q_R$. Then the two points of $g \cap C$ lie in $R^\perp \cap C$ but not in Q_R, so they must both lie on the line $\pi_0 \cap \tau_0$. Hence $\pi_0 \cap \tau_0 = g$. But this is not possible, as the line $\pi_0 \cap \tau_0$ contains R and the line g does not. □

Lemma 5.7. *If $d_0 > 1$, then $|C| = q^3 + q^2 + 2$ and C satisfies the conclusion* (b) *of Theorem* 1.2.

Proof. As $d_0 > 1$, then $d_0 = d$ by Lemma 5.4. Thus every plane σ of $Q^+(5, q)$ has the property that $\sigma \cap C$ is a conic, or that $|\sigma \cap C| = q + 1 - d$. Lemma 5.1 (a) shows then that there are exactly $q + 1$ Latin and $q + 1$ Greek planes that have deficiency d. Let $\pi_0, \dots, \pi_q$ and $\tau_0, \dots, \tau_q$ be the Greek and Latin planes of deficiency d. Lemma 5.1 (b) implies that every plane π_i meets every plane τ_j in a line.

Lemma 5.1 (d) shows that every point of π_0 lies on a Latin plane of positive deficiency. Thus, the $q + 1$ lines $\pi_0 \cap \tau_j$, $j = 0, \dots, q$, cover all points of π_0. Thus, the $q + 1$ lines $\pi_0 \cap \tau_j$ pass through a common point P of π_0 (this can be seen as follows: if P is the point of intersection of two of the lines, then every line l of π_0 on P must be one of the lines, since otherwise q more lines would be needed to cover l). Then all Latin planes τ_j pass through P. Lemma 5.1 (d) shows then that also all Greek planes π_i pass through P. Hence all Greek and Latin planes on P have deficiency d while the other Greek and Latin planes meet C in a conic.

We have $|\pi_0 \cap C| = q + 1 - d \geqslant 2$. Let $X \in \pi_0 \cap C$ with $X \neq P$. Then X lies on $d + 1 \geqslant 2$ lines of π_0 that contain no further point of C. Let l be a line of π_0 on X that contains no further point of C and that does not pass through P. As $|\pi_0 \cap C| = q + 1 - d \leqslant q$, there exists a line h of π_0 on P that contains no point of C except possibly P (we do not know whether or not P is in C). Let R be the point in which l and h meet. We may assume that τ_0 is the Latin plane on h.

The Latin plane on l does not contain P and meets C therefore in a conic. Also, since $l \cap C = X$, the point R is an external point of this conic. Then we can apply Proposition 4.1 and obtain a quadric Q_R of $R^\perp$. For each of the $2q$ Greek and Latin planes σ on R different from π_0 and τ_0 we know that $\sigma \cap C$ is a conic (since $P \notin \sigma$) and that $\sigma \cap C = \sigma \cap Q_R$ (Proposition 4.1). This gives $(\pi_0 \backslash h) \cap C = (\pi_0 \backslash h) \cap Q_R$. Recall that $h \cap C = \emptyset$ or $h \cap C = \{P\}$. Also the point X lies in $(\pi_0 \backslash h) \cap C$ and thus X belongs to Q_R. As Q_R is a quadric with $R \notin Q_R$, then $\pi_0 \cap Q_R$ is a point, a line, or a conic. We show that the first case gives what we want, and that the two other cases lead to a contradiction.

Case 1. $\pi_0 \cap Q_R$ is a point. This point must be X. Also $\pi_0 \cap C = (h \cap C) \cup \{X\}$, since $(\pi_0 \backslash h) \cap C = (\pi_0 \backslash h) \cap Q_R$. As $|\pi_0 \cap C| \geqslant q + 1 - d \geqslant 2$, this gives the following. We have $|\pi_0 \cap C| = 2$, the point P lies in C, and the two points of $\pi_0 \cap C$ are X and P. Hence $d = q - 1$ and $P \in C$. This is what we wanted to show.

Case 2. $\pi_0 \cap Q_R$ is a line. Since $R \notin Q_R$, this line is not h. Hence q points of this line lie in $\pi_0 \backslash h$ and thus in $(\pi_0 \backslash h) \cap Q_R = (\pi_0 \backslash h) \cap C$. But C does not contain three collinear points, a contradiction.

Case 3. $\pi_0 \cap Q_R$ is a conic. At least $q-1$ points of this conic lie in $\pi_0 \backslash h$ and hence in C. Thus $|\pi_0 \cap C| \geqslant q-1$ and hence $d \leqslant 2$. As $d = d_0 > 1$, then $d = 2$ and $|\pi_0 \cap C| = q-1$. Thus $\pi_0 \cap C$ can be extended to the conic C_{π_0} by adjoining two points. Let R' be one of these two points and such that $R' \neq P$. Then R' lies on $q-1$ lines of π_0 that meet $\pi_0 \cap C$. Let l' be a line of π_0 on R' such that $P \notin l'$ and such that l' meets $\pi_0 \cap C$ in a (necessarily unique) point X'. As above, the Latin plane on l' meets C in a conic and R' is an external point of this conic. Thus, we can again apply Proposition 4.1 and obtain a quadric $Q_{R'}$ of $R'^{\perp}$ with $R' \notin Q_{R'}$. As $d = 2$ (so that every Greek and Latin plane meets C in at least $q-1$ points), only conclusion (a) of 4.1 is possible. Hence $R'^{\perp} \cap C \subseteq Q_{R'}$ and all Greek and Latin planes on R' meet $Q_{R'}$ in a conic. As $R' \in C_{\pi_0}$ and $R' \notin Q_{R'}$, then $\pi_0 \cap Q_{R'}$ and C_{π_0} are distinct conics that share the $q-1$ points of $\pi_0 \cap C$. This contradicts Result 3.3. □

The preceding two lemmas show that $|C| = q^3 + q^2 + 2$ and that C satisfies one of the two conclusions of Theorem 1.2. This proves Theorem 1.1 and Theorem 1.2.

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K. Metsch, Mathematisches Institut, Arndtstrasse 2, 35392 Giessen, Germany
Email: klaus.metsch@math.uni-giessen.de