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Multiple spread retraction

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Abstract. Translation planes of order q^t and kernel containing $K \cong \mathrm{GF}(q)$ admitting fixed-point-free collineation groups GK^*, each of whose point orbits is the set of nonzero vectors of a 2-dimensional K-subspace, are shown to permit spread-retraction and produce either Baer subgeometry or mixed partitions of a corresponding projective space. When the same translation plane or spread produces a number of partitions of isomorphic projective spaces, we call this *multiple spread-retraction*. This analysis is used to describe triply-retractive spreads, in general, and to consider the triply-retractive spreads of order 16, in particular.

Key words. retraction, multiple, spread.

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1 Introduction

In Mellinger [14], there is a complete enumeration of the mixed partitions in $\mathrm{PG}(3,4)$. Each partition produces a translation plane of order 16. In fact, it is shown, in particular, that these partitions produce all of the translation planes of order 16. However, several inequivalent partitions produce the same translation plane. The question is, why is this so?

In Johnson [12], it is shown that given a translation plane of order q^4 with spread in $\mathrm{PG}(7,q)$ that may be considered a $\mathrm{GF}(q^2)$-vector space, which admits the $\mathrm{GF}(q^2)^*$ group as a collineation group with component orbits of lengths 1 or $q+1$, there is a *retraction* which produces a mixed partition in $\mathrm{PG}(3,q^2)$. In the case of order 16, this would mean that a translation plane of order 16 admitting a $\mathrm{GF}(4)^*$ group as a collineation group would correspond to a mixed partition in $\mathrm{PG}(3,4)$.

In this article, we consider the implication of the results of Mellinger [14] from the standpoint of the translation plane and generalize the ideas for spreads of order 2^n admitting fixed-point-free groups of order 3. In particular, we are able to show that the set of partitions arising from a given translation plane is in bijective correspondence to the set of fixed-point-free groups of order 3 acting on the plane. More

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generally, we consider the action of fixed-point-free collineation groups of specified type and re-formulate some equivalent conditions for the existence of a spread which permits spread-retraction.

A *multiple retraction* refers to a translation plane or a spread which admits retraction in a number of ways. Various planes of order 16 admit double and triple retraction. An infinite class of triply retractive translation planes is constructed. Furthermore, it is shown that quadruply retractive planes of square order cannot exist. This result depends on a general analysis of spread-retraction of which the following result is perhaps the most useful.

Theorem 1. *Let π be a translation plane of order q^t and kernel containing a field K isomorphic to* $\mathrm{GF}(q)$ *which admits a collineation group G such that GK^* is fixed-point-free and $GK^* \cup \{0\}$ is a field of order q^2. Then, the spread associated with π permits spread-retraction.*

2 Fixed-point-free groups

Our discussion will center around the existence of certain groups. In general, we will think of the groups as subgroups of the translation complement of an affine translation plane. The subgroup in question is typically considered as a homology group of an associated translation plane whose center is some affine point often called the *origin*. The origin is clearly fixed by any element of the translation complement. Hence, when discussing groups and orbits, we will think of the groups as acting on all of the affine points except for the origin.

We will also frequently consider groups as acting on the parallel classes of lines of the affine plane. Equivalently, we can think of this as an action on the spread associated with the translation plane via the Bose/André construction. Such parallel classes (or spread elements) will be referred to as *components*.

In Johnson [12], the following result is proved:

Theorem 2 (Johnson [12]). *Let π be a translation plane with spread in* $\mathrm{PG}(4m-1, q)$. *Suppose the associated vector space may be written over a field K isomorphic to* $\mathrm{GF}(q^2)$ *which extends the indicated field* $\mathrm{GF}(q)$ *as a $2m$-dimensional K-vector space.*

If the scalar mappings with respect to K over V_{2m}/K act as collineations of π, assume that the orbit lengths of components are either 1 *or* $q+1$ *under the scalar group of order* q^2-1.

Let δ denote the number of components of orbit length 1 *and let* $(q+1)d$ *denote the number of components of orbit length* $q+1$.

Then one may construct a mixed partition of $\mathrm{PG}(2m-1, q^2)$ *consisting of δ copies of* $\mathrm{PG}(m-1, q^2)$ *and d copies of* $\mathrm{PG}(2m-1, q)$.

Definition 1. Under the above conditions, we shall say that the mixed partition of $\mathrm{PG}(2m-1, q^2)$ is a *retraction* of the spread of π or a *spread-retraction*.

Theorem 3 (Johnson [12]). *Let π be a translation plane of order q^{2m+1} with kernel* *containing* GF(q) *and spread in* PG($4m+1,q$), *whose underlying vector space is a* GF(q^2)*-space and which admits as a collineation group the scalar group of order* q^2-1. *If all orbits of components have length* $q+1$ *corresponding to* $K-\{0\}$, *then a Baer subgeometry partition of* PG($2m,q^2$) *may be constructed.*

Definition 2. A Baer subgeometry partition produced from a spread as above is called a *spread-retraction*.

The question for us here is whether the assumptions on the lengths of orbits or the assumption on the vector space are necessary; or what assumptions are equivalent. We begin by merely making an assumption on the point-orbits.

Theorem 4. *Let* π *be a translation plane of order* q^t *and kernel containing* K *isomorphic to* GF(q). *Let* G *be a collineation group of* π *such that* GK^* *is fixed-point-free* (*where* K^* *denotes the kernel homology group of order* $q-1$).

If all point orbits union the zero vector are 2*-dimensional* K*-subspaces then* GK^* *union the zero mapping is a field isomorphic to* GF(q^2). *Furthermore*,

(1) *Every component orbit under* GK^* *has length* 1 *or* $q+1$.

(2) *Furthermore, all component orbits have length* $q+1$ *if and only if* t *is odd.*

(3) *All component orbits have length* 1 *if and only if* GK^* *is a kernel subgroup of the translation plane.*

(4) *The plane* π *permits spread-retraction and produces a partition of a corresponding projective space.*

Proof. Assume that GK^* has an orbit Γ of components of length $1<t<q+1$. Then, there is a point orbit within Γ which is a 2-dimensional K-subspace and hence intersects with the components of Γ in either a 1 or 2-dimensional K-subspace. It is clear that no such orbit Γ can exist.

Hence, all orbits are of length 1 or $q+1$. Assume that there are no orbits of length 1. Then, $q+1$ must divide q^t+1 implying that t is odd. On the other hand, assume that t is odd, and that there is an orbit of length 1. Then, since every point-orbit of GK^* is a 2-dimensional K-subspace, it follows that q^2-1 must divide q^t-1, which is a contradiction. Hence, there are no component orbits of length 1 if and only if t is odd.

If all orbits of components have length 1, then GK^* is a subgroup of a kernel homology group and hence GK^* union the zero vector is a field since the order of GK^* is q^2-1.

Hence, we may consider that we have an orbit of components Γ of length $q+1$. Since the point-orbits union the zero vector are subspaces, it follows that Γ is a subplane covered net (see Johnson [11]) so that Γ is a regulus net and hence, by the structure of GK^*, a K-regulus net. Now the vector space is a direct sum of subplanes of Γ and by varying the subplanes, we may assume that all are Desarguesian subplanes that are isomorphic K_o^+-modules, for K_o^+ isomorphic to GF(q^2). Since the vector space is a direct sum of K_o^+-modules that are also GK^*-invariant, we may assume that GK^* acts on the K_o^+-vector space. Furthermore, we may consider the vector space as a t-dimensional K_o^+-space which admits the natural *scalar* group $\mathscr{K}^+$ of order $q^2 - 1$. We note that this group is necessarily fixed-point-free, cyclic and contains the $\mathrm{GF}(q)$-scalar group. It follows that the point orbits of this group are Desarguesian affine planes which are 1-dimensional K_o^+-subspaces. Furthermore, the orbits of GK^* in Γ union the zero vector are 1-dimensional K_o^+-subspaces. Writing the vector space as a direct sum of GK^*-invariant K_o^+-modules, implies that GK^* is a block diagonal group with identical entries. Hence, an element of $GK^*\mathscr{K}^+$ fixes a non-zero point of one of the GK^*-orbits in Γ if and only if the element is the identity. Since both groups are transitive, given an element σ in GK^*, there is an element τ_σ in $\mathscr{K}^+$ such that $\sigma\tau_\sigma$ fixes a non-zero point of one of the GK^*-orbits, implying that $\sigma\tau_\sigma = 1$. Hence, it follows that $GK^* = \mathscr{K}^+$. That is, GK^* union the zero vector is a field of order q^2.

Hence, we obtain a spread-retraction exactly as in Johnson [12]. □

There is another purely geometric interpretation for the last theorem. Consider the group G as acting on the spread $\mathscr{S}$ associated with the translation plane π. Then, the orbits of G form lines of the space Π containing $\mathscr{S}$. Because of the action of G, these lines must form a geometric 1-spread of Π. This geometric 1-spread can then be used to retract $\mathscr{S}$ to a mixed partition just as in [12]. See [15] for a complete geometric description.

Note that if we assume that GK^* is cyclic of order $q^2 - 1$ and $GK^* \cup \{0\}$ is a field isomorphic to $\mathrm{GF}(q^2)$, it then follows that GK^* is generated by an element Z and

$$GK^* = \{Z\alpha + \beta : \alpha, \beta \in K; (\alpha, \beta) \neq (0, 0)\}.$$

Let w be any nonzero vector. Then, the 2-dimensional K-subspace generated by w and wZ: $\langle wZ, w \rangle_K$ is $\{w(Z\alpha + \beta) : \alpha, \beta \in K\}$; that is, by similar arguments as above, every point-orbit is a 2-dimensional K-subspace and hence, we may apply the above theorem to obtain the following.

Theorem 5. *Let π be a translation plane of order q^t and kernel containing K isomorphic to* $\mathrm{GF}(q)$ *that admits a collineation group G such that GK^* is fixed-point-free and $GK^* \cup \{0\}$ is a field of order q^2. Then, the spread permits spread-retraction.*

We now assume that GK^* is cyclic.

Theorem 6. *Let π be a translation plane of order q^t and kernel containing K isomorphic to* $\mathrm{GF}(q)$. *Let G be a collineation group of π such that GK^* is cyclic and fixed-point-free* (*where K^* denotes the kernel homology group of order $q - 1$*).

If there exists a set S of $t + 1$ point-orbits which together with the zero vector are 2-*dimensional K-subspaces and any t of these have direct sum π then GK^* union the zero vector is a field and π permits spread-retraction.*

Proof. Since each such point orbit union the zero vector defines a Desarguesian affine plane of order q and we can vary the summands, it follows that we may assume that there is a field K_o^+ isomorphic to $\mathrm{GF}(q^2)$ such that the vector space is a K_o^+-subspace and GK^* acts as a group over this vector space. Since GK^* is cyclic, we may assume that this group acts in $\mathrm{GL}(2, K_o)^+$ and may be diagonalized. Since the group is cyclic, we choose a generator as follows: If the $(1,1)$ entry in the associated matrix is d of order $q^2 - 1$, we may assume that (i,i)-entry is d^{λ_i}, also of order $q^2 - 1$, since the group is fixed-point-free, where $d^{\lambda_i} \in K_o^+$ for all $i = 1, 2, \ldots, t$. We know also that GK^* contains the kernel homology group of order $q - 1$, which implies that when $d^n \in K$, we have $d^{n\lambda_i} = d^n$. This implies that λ_i acts as an automorphism of K so that $\lambda_i = q^{\tau(i)}$ where $\tau(i) = 1$ or q. However, if $g \in GK^*$ then g restricted to a summand lies in a field, thus it follows immediately that GK^* lies in a field and hence GK^* union the zero mapping is a field. Again, we may apply the above theorem to finish the proof. □

Example 1. Let π be a translation plane of order q^{2r} with kernel containing K^+ isomorphic to $\mathrm{GF}(q^2)$ and assume that there is both a right and middle nucleus of a coordinatizing quasifield equal to the kernel K^+. Furthermore, assume that the quasifield is a vector space over the right and/or middle nucleus (for example, this would be the case when the quasifield is a semifield). Furthermore assume that K isomorphic to $\mathrm{GF}(q)$ is in the center of the nuclei.

Then, the plane admits retraction by three distinct groups whose generators are given as follows:

$$(x, y) \mapsto (xa, ya^q),$$
$$(x, y) \mapsto (xa, ya),$$
$$(x, y) \mapsto (ax, ay),$$

where a is a primitive element of K^+. The groups are of the form GK^*, their union with the zero mapping produces a field, and the groups are fixed-point-free. Hence, each group produces a retraction.

In the following sections, we shall be interested in whether the various partitions induced from a given translation plane are isomorphic. Clearly, for mixed partitions, if the partition numbers in the partitions are distinct then the partitions are non-isomorphic. However, potentially, two partitions can be non-isomorphic and have identical partition numbers.

Theorem 7. *Two partitions of a projective space that produce the same translation plane are isomorphic if and only if the corresponding collineation groups in the translation plane are conjugate within the full collineation group of the plane.*

Proof. The collineation of the projective space which maps one partition to the other extends to a collineation of the associated translation plane constructed from either of these partitions. The collineation necessarily conjugates the associated groups (isomorphic to $\mathrm{GF}(q^2)^*$ in the finite case). The converse is immediate. □

3 Partitions of PG(*V*, 4)

This study of retraction actually originated from an analysis of the mixed partitions of $\mathrm{PG}(3,4)$. In particular, this article grew out trying to understand why three particular mixed partitions produce the same translation plane. We note that all translation planes of order 16 may be obtained from one of the three semifield planes by a derivation (see e.g. Johnson [6]). Later, we shall completely determine the semifield planes which produce three partitions of $\mathrm{PG}(3,q)$. Many of the concepts that we shall use in the general study arise from looking at the special case of order 16.

Applying our previous results to $\mathrm{PG}(V,4)$, we obtain:

Theorem 8. *Let π be a finite translation plane of order 2^n and suppose π admits a fixed-point-free collineation of order* 3.

(1) *If n is odd, then there is a corresponding Baer subgeometry partition of* $\mathrm{PG}(n-1,4)$.

(2) *If n is even, then there is a corresponding mixed partition of* $\mathrm{PG}(n-1,4)$. *Also, when $n=2m$, the fixed components correspond to* $\mathrm{PG}(m-1,4)$ *and the orbits of length* 3 *correspond to* $\mathrm{PG}(2m-1,2)$.

Proof. When n is odd, this follows from a theorem in Johnson [12]. But, we shall give a unified proof. We need only show that every fixed-point-free group of order 3 acting over a field of even characteristic defines a field isomorphic to $\mathrm{GF}(4)$.

If all component orbits have length 1 then clearly the group is a kernel homology group and thus corresponds to a field isomorphic to $\mathrm{GF}(4)$. Hence, assume there exists an orbit of components Γ of length 3.

Choose coordinates so that Γ consists of $x=0,\ y=0,\ y=x$ and the group element g of order 3 permutes the previous subspaces in the order indicated. It then follows that g has the following matrix form:

$$g=\begin{bmatrix} A & A \\ A & 0 \end{bmatrix}$$

for some matrix A. Since the order of g is 3, it follows that $A^3=I$, noting that A is non-singular.

Now

$$(x,y)\begin{bmatrix} A & A \\ A & 0 \end{bmatrix}=(x,y)$$

if and only if $y=xA$ and $xA+yA=x$ or, equivalently, $y(I+A)=yA^{-1}=yA^2$, so $y(I+A+A^2)=0$. But note that $(A^3+I)=0=(A+I)(A^2+A+I)$. If we assume that $A+I$ is nonsingular, then $A^2 = A+I$. In this case, there are always fixed points. Hence, $A+I$ is singular.

Also, we note that g has order 3 and satisfies the polynomial $x^3+1 = (x+1)(x^2+x+1)$. It follows that either x^3+1 or x^2+x+1 is the minimal polynomial for g. In the former case, the rational normal form for g has the following form:

$$\begin{bmatrix} I & 0 \\ 0 & T \end{bmatrix}$$

there T is a block diagonal matrix of 2×2 submatrices of the form $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ and where I is a non-trivial identity matrix. However, such an element must have fixed points. Hence, g may be written as a diagonal block matrix with entries $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ which implies that the minimal polynomial for g is x^2+x+1. Hence, we must have

$$\left(\begin{bmatrix} 0 & A^2 \\ A^2 & A^2 \end{bmatrix} = \begin{bmatrix} A & A \\ A & 0 \end{bmatrix}^2\right) + \begin{bmatrix} A & A \\ A & 0 \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = 0,$$

implying that $A+I=0$. So, we may represent the mapping g as $\left[\begin{smallmatrix} I & I \\ I & 0 \end{smallmatrix}\right]$ and since $g^2 = g+1$, it follows that $\langle g \rangle$ union the zero mapping is a field isomorphic to $\mathrm{GF}(4)$. The theorem now follows from Theorem 5. □

Mellinger [14] enumerates all mixed partitions in $\mathrm{PG}(3,4)$ using Magma. In the following sections, we shall consider general situations for translation planes producing several mixed partitions. First, we revisit the conclusions in light of fixed-point-free collineations for translation planes of order 16. For the remainder of the article, we will refer to mixed partitions by their *type*. We will say a mixed partition of $\mathrm{PG}(2m-1,q^2)$ is an (a,b)-*partition* or of *type* (a,b) if the partition contains a copies of $\mathrm{PG}(m-1,q^2)$ and b Baer subgeometries (copies of $\mathrm{PG}(2m-1,q)$).

3.1 The Planes of Order 16. For purposes of identification, we shall denote the eight translation planes of order 16 as the Desarguesian plane, the semifield plane with kernel $\mathrm{GF}(4)$, the semifield plane with kernel $\mathrm{GF}(2)$ and the five derived planes. The plane derived from the Desarguesian plane is the Hall plane. There are three planes derived from the semifield plane with kernel $\mathrm{GF}(4)$, called the Lorimer–Rahilly plane, the Johnson–Walker plane and another plane which we shall simply call the $\mathrm{GF}(4)$-derived plane. The plane derived from the semifield plane with kernel $\mathrm{GF}(2)$ is the Dempwolff plane.

In this setting, K is always isomorphic to $\mathrm{GF}(2)$.

Theorem 9 (see also Mellinger [14]). *All translation planes of order* 16 *admit fixed-point-free groups of order* 3 *which fix a vector subspace of dimension two over* $\mathrm{GF}(2)$ *and hence correspond to mixed partitions of* $\mathrm{PG}(3,4)$. *The ℓ fixed components corre-*

spond to $PG(1,4)$ *and the b orbits of length* 3 *correspond to* $PG(3,2)$, *type* (ℓ, b). *More specifically,*

(1) *If* π *is Desarguesian with group* $\langle (x,y) \mapsto (xa, xa^2) \rangle$, *there is an associated* $(2,5)$-*partition.*

(2) *Derivation of* π *using a derivable net invariant under the group of* (1) *produces a* $(5,4)$-*partition in the Hall plane.*

Let the semifield of order 16 *with the kernel* $GF(4) =$ *right nucleus* $=$ *left nucleus be denoted by* $(S,+,*)$. *Consider the group* $G_j K^* = \langle (x,y) \mapsto (x*a, x*a^{2^j}) \rangle$, *for* j *fixed.*

(3) $j = 0$.
   (a) *Then the semifield plane produces a* $(5,4)$-*partition.*
   (b) *In part* (a), *choose a derivable net as a right* $=$ *middle nucleus net to produce a* $(5,4)$-*partition in the derived semifield plane.*
   (c) *There is another derivation that produces a* $(8,3)$-*partition in the same derived semifield plane using the same group* $G_o K^*$, *as above.*

(4) *Choose* $j = 1$.
   (a) *Then the semifield plane produces a* $(5,4)$-*partition but from the group* $G_1 K^*$.
   (b) *Using the same net as in* (3)(b), *the derived plane produces another* $(5,4)$-*partition using the group* $G_1 K^*$.
   (c) *There is another derivation that produces a* $(8,3)$-*partition in the derived semifield plane using the group* $G_1 K^*$.

(5) *Consider the kernel homology group* $G_3 K^*$: $\langle (x,y) \mapsto (a*x, a*y) \rangle$. *This produces a* $(14,1)$-*partition of the derived plane.*

   Hence, the semifield plane with kernel $GF(4)$ *produces two partitions of type* $(5,4)$ *and the derived plane from the semifield plane with kernel* $GF(4)$ *corresponds to two partitions each of types* $(5,4)$, $(8,3)$ *and a partition of type* $(14,1)$. *The partitions of the same type induce isomorphic planes.*

(6) *The Lorimer–Rahilly and Johnson–Walker planes derived from the semifield plane with kernel* $GF(4)$ *also produce partitions of types* $(14,1)$.

(7) *The Dempwolff plane admits a fixed-point-free group within* $SL(2,4)$ *producing a partition of type* $(8,3)$.

Proof. As noted above, the translation planes are determined as the Desarguesian, Hall, semifield with kernel $GF(2)$, its derived plane which is the Dempwolff plane admitting $SL(2,4)$, the semifield plane with kernel $GF(4)$ and the three derived planes including the Lorimer–Rahilly and Johnson–Walker planes admitting $PSL(2,7)$.

It might be noted that there are various ways that a fixed-point-free group of order 3 could act, thereby changing the configuration of the associated mixed partition. It should be noted that, in all cases, the group of order 3 defines a field isomorphic to $GF(4)$ for all groups arise from a kernel homology group or a middle or right nucleus group of a semifield plane and these groups union the zero vector form fields.

The Desarguesian and Hall Planes. We begin with the Desarguesian plane of order 16 coordinatized by GF(16). Hence, with the standard representation, there is a collineation group of the form $(x, y) \mapsto (xa, xa^2)$ where a has order 3. Note that $y = x^2$ is left invariant and is a 4-dimensional GF(2)-subspace. The group generated by a is $\mathrm{GF}(4)^*$ and $y = x^2$ restricted to GF(4) defines an affine subplane of order 2. This group is clearly fixed-point-free and hence generates a mixed partition in $\mathrm{PG}(3,4)$. There are two fixed components and five orbits of length 3 producing a type $(2,5)$ mixed partition.

Take the standard derivable net in the Desarguesian affine plane and note that the indicated group will fix each Baer subplane incident with the zero vector. Hence, in the derived plane the structure of fixed components and orbits is changed into a type $(5,4)$ mixed partition in the Hall plane.

The Semifield Plane with kernel GF(4) *and its derived planes*. Now consider the semifield plane with kernel GF(4). The kernel, right and middle nuclei are all isomorphic to GF(4) and it is possible to arrange the nuclei so that the left, middle, and right are equal, exactly two are equal, or all three are distinct. In each of these situations, there are derivable nets producing exactly three mutually non-isomorphic translation planes (see e.g. Johnson [6]).

Let $(S, +, *)$ denote the associated semifield.

When the right and middle nuclei are equal, then there is a collineation group of the form $(x, y) \mapsto (x * a, y * a^{2^j})$, for a fixed j, where a has order 3 and is in the right and middle nucleus of the associated semifield coordinatizing the plane. Choosing a right and middle nucleus net to derive, the group fixes each Baer subplane of the associated derivable net, since the group is generated by central collineations. There are five orbits of length 3 of the middle nucleus group which are permuted by the right nucleus group. Hence, there are two orbits (of orbits) which are both fixed and, it turns out, one orbit (of orbits) is of length 3. So, there is an orbit of length 9 and two orbits of length 3 under the product of the two groups; an elementary Abelian 3-group of order 9. So, there is a group of order 3 which fixes three components of one of the orbits. Furthermore, the group is fixed-point-free as it is a product of homology groups.

Therefore, the semifield plane with kernel GF(4) has a fixed-point-free group of order 3 with either exactly 5 or exactly 8 components. For example, we may realize the group in the following form: $(x, y) \mapsto (x * a, y * a)$ where a is in $\mathrm{GF}(4)^*$ and $*$ denotes multiplication in the semifield. We see that a component $y = x * m$ is fixed if and only if $m * a = a * m$. If an element m not in GF(4) has this property then it follows easily that this is true in general, implying that the coordinate structure commutes over GF(4) so that the derivable net in question is a regulus net, which is not the case. Hence, the group fixes exactly 5 components. So, the semifield plane with kernel GF(4) produces a $(5,4)$-partition. Note that the group fixes all Baer subplanes of all of the various right and middle nucleus nets. Hence, upon derivation of the middle nucleus net which contains the five fixed components, the derived semifield plane admits the same sort of partitioning—a $(5,4)$-type.

However, if the orbit of length 3 which is fixed by the group is combined with the axes $x = 0$, $y = 0$ into a derivable net, then the derived plane will admit the three fixed components and the five fixed Baer subplanes as lines thus producing an $(8,3)$-type situation. For example, realizing the group above when $j=1$, we obtain that $y = x * m$ is invariant if and only if $a * m = m * a^2$. Representing $m = tm_1 + m_2$, and using the nucleus properties, we obtain that m is fixed if and only if $m = tm_1$ for all $m_1 \in \mathrm{GF}(4)$. We note that this indicates that apart from $x = 0$, $y = 0$, there are exactly three fixed components in the semifield plane. That is, this is another example of a $(5,4)$-type within the semifield plane, but from different groups. However, this group must fix all of the Baer subplanes of the standard net, which means that within the derived plane, we also obtain an $(8,3)$ situation.

The two remaining planes derived from a semifield plane with kernel GF(4) are the Lorimer–Rahilly and Johnson–Walker planes. These planes correspond to regular parallelisms in PG(3, 2) and hence, according to Jha and Johnson [3] admit SL(2, 2) where the involutions are generated by elations. It follows that the element of order 3 which is the product of two elations from different Sylow 2-groups is fixed-point-free of the form $(x, y) \mapsto (y, x + y)$. Since there are 7 derivable nets containing $x = 0$, $y = 0$, $y = x$ and these components define an orbit under the group above, it follows that there are 14 fixed components and one orbit thus leading to a $(14,1)$ type mixed partition.

We note that the derived plane, the semifield plane with kernel GF(4), admits a $(17,0)$ type from this group but we do not consider this a *mixed* partition. However, the group is the kernel group of the semifield plane and hence, any derived plane; the L-R, J-W and the standard derived plane, all then produce a mixed partition of type $(14,1)$.

Hence, the standard derived plane produces three types of mixed partitions: a $(5,4)$, $(8,3)$ and $(14,1)$ and note that the three groups fix exactly two common components of the plane and arise from the product of the kernel homology group, middle and right nucleus groups all of order 3. This means that we must have three distinct groups to accomplish this sort of *multiple-retraction*. Also, note that there are two partitions each of type $(5,4)$ and $(8,3)$ that produce the same group respectively. The generated group (in the original semifield plane) is of the following form: $\langle (x, y) \mapsto ((a * x) * b, (a * y) * c); a, b, c \in \mathrm{GF}(4)^* \rangle$.

The Semifield Plane with kernel GF(2) *and its derived plane*. The Dempwolff plane is analyzed in Johnson [8] and it is shown that there is a fixed-point-free group of order 3 which fixes exactly 8 components; the components of two Baer subplanes fixed pointwise by involutions of a Sylow 2-subgroup of order 4 of SL(2, 4) that share exactly two components. Hence, the Dempwolff plane produces a mixed partition of type $(8,3)$.

The derived plane is the semifield plane with kernel GF(2) by Johnson [8] and the group of order 3 denoted above inherits as a collineation group of this plane. Note that the derivable net is a subnet of the net defined by the eight fixed components. Furthermore, the group fixes two Baer subplanes and has one orbit of Baer subplanes of length 3. Hence, it follows that in the derived plane (the semifield plane with kernel GF(2)), we obtain a type $(5,4)$ partition. □

Hence, we see that a variety of mixed partitions of different types can correspond to the same translation plane that admit different fixed-point free groups. In the next section, we consider arbitrary translation planes producing several partitions.

4 Normalizing paired partitions and multiple retraction

Definition 3. Let π be a translation plane admitting spread-retraction and let $\mathrm{PG}(\pi)$ denote an associated partition. Two partitions $\mathrm{PG}(\pi_1)$ and $\mathrm{PG}(\pi_2)$ will said to be *paired* if and only if the associated translation planes π_1 and π_2 are isomorphic.

We identify the translation planes π_1 and π_2 and continue to use the term of *paired* partitions of the projective space.

If, in this setting, we allow that the two associated cyclic groups G_iK^*, respectively for π_i, $i = 1, 2$, of order $q^2 - 1$ of the translation plane normalize each other, we shall say that the partitions are *normalizing, paired partitions*.

Furthermore, we shall use the term *double retraction* to describe the construction of a normalizing paired partition. In general, *triple retraction* refers to the construction of three partitions which are mutually normalizing paired partitions and *multiple retraction* (or *k-retraction*) shall refer to the construction of a number of (respectively k) partitions which are mutually normalizing paired partitions.

Lemma 1. *Given a translation plane of order q^t with kernel containing K, two cyclic groups G_iK^*, $i = 1, 2$, of order $q^2 - 1$ corresponding to spread-retractions centralize each other if they normalize each other.*

Proof. Fix one group G_1K^* and note that $G_1K^* \cup \{0\}$ is a field isomorphic to $\mathrm{GF}(q^2)$. Considering that the second group normalizes the first, both groups may be considered within $\Gamma L(t, q^2)$, relative to $G_1K^* \cup \{0\}$. We need to show that the second cyclic group is in $\mathrm{GL}(t, q^2)$, in this representation.

We know that both groups are K-linear since their individual point-orbits are 2-dimensional K-subspaces. Hence, the companion automorphisms of the elements of G_2K^*, say, are either 1 or q. So, it follows immediately that there is a *linear* cyclic subgroup of order divisible by $(q^2 - 1)/(2, q - 1)$. If q is even, we are finished. Thus, assume that q is odd with generator g of the basic form:

$$w \mapsto w^q A,$$

where $w^q = (w_1^q, \ldots, w_t^q)$ and $w_i \in \mathrm{GF}(q^2)$. Since this group is sharply transitive on the non-zero points of some Desarguesian spread, the group is isomorphic to a nearfield group with center K. However the associated nearfield group is now cyclic in this context. Hence, both groups are in $\mathrm{GL}(t, q^2)$. □

Lemma 2. *Let P_1 and P_2 be two distinct paired partitions and let G_1K^* and G_2K^* denote the corresponding groups in the translation plane π. Then $G_1K^* \cap G_2K^* = K^*$.*

Proof. Recall that G_iK is a field of order q^2; a quadratic field extension of K, for $i = 1, 2$. Let t be a common element and assume that this element is not in K. Then

$\{t\alpha + \beta I : \alpha, \beta \in K\}$ is a field of linear transformations and since this is also a collineation group, it must be simultaneously G_1K and G_2K. Since we are assuming that the paired partitions are distinct, the groups cannot be identical, so the result follows. □

We begin with a general consideration of normalizing, paired partitions corresponding to translation planes of order q^t. We have seen that there are several mixed partitions that a given translation plane can produce, so this occurs when t is even. The question is, can this happen when t is odd?

Theorem 10. *Let π be a translation plane of order q^t that admits spread-retraction and produces two distinct normalizing, paired partitions of projective spaces; π permits double retraction. Then either*

(1) *t is even, or*

(2) *t is odd and*
  (a) *q is odd, $(q+1)/2$ is odd and $(q+1)/2$ divides t, or*
  (b) *q is even and $q+1$ divides t.*

Proof. Assume that t is odd. Let the two groups in question be denoted by G_1K^* and G_2K^*. Since t is odd, we have that the spread S for π is a union of reguli Γ_i^j for $j = 1, \dots, \frac{q^t+1}{q+1}$ and $i = 1, 2$ respectively for G_1K^* and G_2K^*. We note that

$$\left(q+1, \frac{q^t+1}{q+1}\right) = (q+1, t).$$

Assume that q is odd. If $(q+1)/2$ is odd and $(q+1)/2$ does not divide t, let u be a prime dividing $(q+1)/2$ but not dividing t. Then any element g of G_2K^* having prime power order u^α dividing $q+1$ must fix at least one of the reguli, say Γ_1^1 corresponding to G_1K^*, as G_1 and G_2 must normalize and hence centralize each other. Similarly, if 4 divides $q+1$ then there exists a subgroup of G_2K^* of order $2^z > 4$ that does not divide t since t is odd. In either of these cases, we may assume without loss of generality that there is an element g of prime power order larger than 2 and whose order does not divide t.

So, g fixes Γ_1^1 and has order > 2. There is a regulus Γ_2^k corresponding to G_2K^* which has at least one component ℓ in Γ_1^1. Since g fixes Γ_1^1 and Γ_2^k, it fixes the intersection $\Gamma_1^1 \cap \Gamma_2^k$. But, g has order > 2, so that the intersection has at least three common components. Since these sets are reguli, it follows that $\Gamma_1^1 = \Gamma_2^k$.

The point-orbits of G_2K^* are the subplanes of order q in the regulus nets Γ_2^k. Hence, it follows that the sets of point-orbits of G_1K^* and of G_2K^* are equal on Γ_1^1.

There are $\frac{q^t-1}{q-1}$ subplanes in Γ_1^1. Let $t - 1 = 2w$. We note that

$$\frac{q^t-1}{q-1} = 1 + q + \cdots + q^{t-1} = (q+1) + q^2(q+1) + \cdots + q^{2(w-1)}(q+1) + q^{2w}.$$

Thus

$$\left(q+1, \frac{q^t-1}{q-1}\right) = 1.$$

Since the two groups centralize each other, the stabilizer subgroup C of order $q+1$ which fixes a component of Γ_1^1 must fix all components of Γ_1^1. Any group element which fixes a subplane π_o of Γ_1^1 then induces a kernel homology on π_o of order dividing $q-1$. It follows that the stabilizer of a subplane in C has order dividing $(2, q-1)$.

First assume that $q+1 \neq 2^a$ for any integer a. Then any element of C of odd order must fix a subplane which then provides a contradiction. If $q+1 = 2^a$ for some integer a, let g be an element of order 2^a. Then, there is a subgroup of order 2^a which fixes a subplane, implying that $a = 1$, a contradiction. This completes the proof of the theorem. □

5 A class of spreads that admit triple retraction

In this section, we shall introduce a class of translation planes of order q^4 and kernel containing $GF(q)$ that produce mixed partitions of $PG(3, q^2)$ of types

$$(q^2+1, q^2(q-1)), \quad (2q^2, (q^2-1)(q-1)) \quad \text{and} \quad (q^4-q^2+2, q-1).$$

First assume that we have a translation plane of order q^4 with spread in $PG(3, q^2)$ such that the middle and right nuclei correspond to fields of order q^2 with respect to a fixed coordinate system and choice of base axes, $x = 0$, $y = 0$, $y = x$. Assume that the nuclei are equal and equal to the kernel and that the subkernel group K of order $q-1$ commutes within the underlying quasifield $Q = (Q, +, *)$.

Then, we obtain a group of the following form:

$$\langle (x, y) \mapsto ((a*x)*b, (a*y)*c); a, b, c \in K^+ \cong GF(q^2) - \{0\} \rangle.$$

We consider the following three groups: H_i for $i = 1, 2, 3$:

$$H_1 = \langle (x, y) \mapsto (a*x, a*y); a \in K^+ \cong GF(q^2) - \{0\} \rangle,$$
$$H_2 = \langle (x, y) \mapsto (x*b, y*b^q); b \in K^+ \cong GF(q^2) - \{0\} \rangle,$$
$$H_3 = \langle (x, y) \mapsto (x*c, y*c); c \in K^+ \cong GF(q^2) - \{0\} \rangle.$$

We know that all groups union the zero mapping produce fields isomorphic to $GF(q^2)$, all groups are fixed-point-free and each group contains the K^*-kernel homology group due to the assumption on the centrality of K in the quasifield. Hence, each group necessarily produces a mixed partition of $PG(2, q^2)$ of the given translation plane but we shall be more interested in the derived planes.

Assume that the middle nucleus defines derivable nets and require that these are not K^+-reguli. Then, the kernel homology group H_1 must fix exactly two Baer subplanes of any such derivable net and permute the remaining Baer subplanes into $q-1$ sets of size $q+1$.

Theorem 11. *The derived plane produces a mixed partition in* $\mathrm{PG}(3,q^2)$ *of type* $(q^4-q^2+2, q-1)$ *using the group* H_1.

Now consider the group H_3. It is clear that the plane admits a mixed partition. Furthermore, $y = x * m$ is fixed if and only if

$$(x*b)*m = (x*m)*b$$

for all x and for all $b \in K^{+*}$. Since b is in the middle and right nucleus, this is equivalent to

$$b*m = m*b.$$

We know that the orbits have length 1 or $q+1$. Let $\{1,t\}$ be a basis for Q over K^+ and write $m = m_1 * t + m_2$ for $m_i \in K^+$. Then, since K^+ is in the kernel of the translation plane, and, as we are assuming that the middle nucleus defines a derivable net, and since all such derivable nets are defined from *right* 2-dimensional vector spaces (see Johnson [11]) we have:

$$\begin{aligned} b*(m_1*t+m_2) &= (bm_1*t+bm_2) = (m_1*t+m_2)*b = (m_1*t)*b+m_2b \\ &= m_1*(t*b)+m_2b. \end{aligned}$$

Hence, if $m_1 \neq 0$, and $b*t = t*b$, then K^+ is in the center of the quasifield, a contradiction unless the middle nucleus defines a K^+-regulus net. Hence, it follows that the only fixed components are $x=0$, $y=xm$ for $m \in K^+$. Now derive the standard middle regulus net, and note that the group is generated from central collineations and such groups fix all Baer subplanes of the derivable net which are incident with the zero vector.

Theorem 12. *The plane produces a* $(q^2+1, q^2(q-1))$*-type mixed partition using* H_3. *In addition, the derived plane also produces a* $(q^2+1, q^2(q-1))$*-type mixed partition using* H_3.

Now consider the group H_2. Since the coordinate structure is 2-dimensional over its middle nucleus, let $\{1,t\}$ be a K^+-basis. Then,

$$b*(tm_1+m_2) = (tm_1+m_2)*b^q.$$

Since the kernel is also K^+, the coordinate structure is a left quasifield, and K^+ is also the right nucleus, we have

$$b * (tm_1 + m_2) = (b * t)m_1 + bm_2 = (tm_1 + m_2) * b^q$$
$$= (t * b^q)m_1 + m_2 b^q.$$

Hence $b * t = t * b^q$ and $m_2 = 0$ or $m_1 = m_2 = 0$. Again, note that the group fixes all Baer subplanes of the original standard derivable net, but this time fixes exactly $q^2 - 1$ additional $y = x * m's$ outside of the derivable net.

Theorem 13. *The group H_3 produces, by derivation, a mixed partition of type $(2q^2, (q^2 - 1)(q - 1))$ provided the quasifield has the required property $b * t = t * b^q$ for a basis $\{1, t\}$ over K^+, for all $b \in K^+$.*

The original plane produces another mixed partition of type $(q^2 + 1, q^2(q - 1))$ which is distinct from the mixed partition of the same type produced using H_2. The two partitions are isomorphic in PG$(3, q^2)$ *if and only if there is a collineation σ of the plane π such that $H_2^\sigma = H_3$ and mapping the components fixed by H_2 onto those fixed by H_3, fixing $x = 0$ and $y = 0$.*

Theorem 14. *Let π be a translation plane of order q^4 with kernel $K^+ \cong$ GF(q^2). Choose a coordinate quasifield Q and assume with respect to that quasifield, the right, middle, and left (kernel) nuclei are all equal. Assume further that the translation plane is derivable with respect to a middle nucleus net which is not a K^+-regulus net.*

(1) *If there is a basis $\{1, t\}$ over K^+ such that with respect to quasifield multiplication, $b * t = t * b^q$ for all $b \in K^+$ then the derived plane produces mixed partitions of* PG$(3, q^2)$ *of types $(q^2 + 1, q^2(q - 1))$, $(2q^2, (q^2 - 1)(q - 1))$, $(q^4 - q^2 + 2, q - 1)$.*

(2) *The original translation plane produces distinct mixed partitions of* PG$(3, q^2)$ *of types: $(q^2 + 1, q^2(q - 1))$, $(q^2 + 1, q^2(q - 1))$, $(q^4 + 1, 0)$.*

(3) *Furthermore, suppressing the $*$-notation, the multiplication is as follows:*
Let $t(t + \gamma) = tf(\gamma) + g(\gamma)$ for functions f, $g : K \mapsto K$ then:

$$(t\alpha + \beta)(t\delta + \gamma) = t(\alpha\delta^q f(\delta^{-q}\gamma) + \beta^q\delta) + (\alpha^q\delta g(\delta^{-q}\gamma) + \beta\gamma)$$

for all $\alpha, \beta, \delta \neq 0$, $\gamma \in K$.

Proof. In our analysis above, it was required that the middle nucleus net is not a K-regulus net. The condition on the basis provides that this is the case. Using the kernel, right and middle nucleus properties, the $*$-multiplication may be shown to be as stated. □

Remark 1. The only known examples of quasifields with the above properties originate from the Hughes–Kleinfeld semifields. Note, for example, when we take the semifield plane of order 16 and kernel GF(4), the derived plane admits triple retraction of types $(5,4)$, $(8,3)$ and $(14,1)$ which are the types listed above when $q=2$.

In general, is every such plane a derived Hughes–Kleinfeld semifield plane?

6 Normalizing, paired partitions in PG$(3,q^2)$

In a mixed partition, we have at least two fixed components, say $x=0$ and $y=0$. Each of these components may be considered as a 2-dimensional GF(q^2)-subspace with respect to a given group.

If two mixed partitions produce isomorphic translation planes, we have two possible situations: (i) the partitions have different numbers or (ii) identifying the translation planes there are two distinct sets of partition subspaces. In either case, identifying the translation plane, we see that we have a translation plane with spread in PG$(7,q)$ such that there are two fields K_1 and K_2 both isomorphic to GF(q^2) and such that the groups of orders q^2-1 intersect exactly in the kernel homology group of order $q-1$. Hence, we obtain a collineation group in the associated translation plane π of order $(q+1)^2(q-1)$ which contains the GF(q)-kernel homology group. Moreover, we are assuming that each group normalizes the other and the collineation group inherited from the partitions is a subgroup of $\Gamma L(4,q^2)$. We have noticed that the groups centralize each other provided they normalize each other.

Proposition 1. *In a translation plane π of order q^t, t even, with kernel containing $K \cong$ GF(q), two distinct normalizing paired partitions produce collineation groups G_1K^* and G_2K^* of π which fix at least two common components.*

Proof. Assume that there are no common fixed components. Since the two groups centralize each other, it follows that the number of fixed components from the first field is a multiple of $q+1$, say $k(q+1)$. Hence, we have $k(q+1)+s(q+1)=q^t+1$ where s is the number of $q+1$-orbits from the first field; a contradiction as $q+1$ does not divide q^t+1 if t is even. Hence, there is at least one common component. Suppose there is exactly one common component L. Then, $q^t+1=1+m(q+1)$, which again is a contradiction. Hence, there are at least two common components. □

Theorem 15. *Let π be a translation plane of order q^4 whose kernel contains K isomorphic to* GF(q). *Assume that the plane is doubly-retractive so that there are two common fixed components in the above situation, say $x=0$, $y=0$, for two distinct groups, G_iK^*, for $i=1,2$. Decompose the vector space relative to the first group G_1K^*, which then acts like a scalar* GF(q^2)*-group acting on $x=0$. On $x=0$, the group G_2K^* must permute the $(q^4-1)/(q^2-1)$ one-dimensional* GF(q^2)*-subspaces and hence must fix two such subspaces.*

Then, no third distinct group producing a mixed partition can be within the group generated by the first two.

Proof. On $x=0$, the group G_2K^* acts fixed-point-free on a 1-dimensional GF(q^2)-subspace X and is sharply transitive on the nonzero vectors. Since we must have a nearfield group action, we have that the group G_2K^* has identical action on X as G_1K^* or it has the regular nearfield action $X \to X^{q^{\lambda(a)}}a$ for $a \in \mathrm{GF}(q^2)$. If q is even, it must be that the group has identical action on X. Moreover, since the second group is cyclic, this implies that the action cannot be the regular nearfield action on X. Hence, the action on X is identical for both groups. That is, we may assume that the underlying field of coefficients is the same for either group, say $K^+ \supseteq K$, for K^+ isomorphic to $\mathrm{GF}(q^2)$.

On each of the fixed components, we consider the direct sum of the fixed 1-dimensional $\mathrm{GF}(q^2)$-subspaces. First assume that there are three fixed components and realize that since the group is cyclic, and belongs to a *field* containing $\mathrm{GF}(q)$, it can only be that the group is generated by an element τ which has the following form

$$(x_1, x_2, y_1, y_2) \mapsto (x_1a, x_2a^{q^c}, y_1a, y_2a^{q^c}),$$

where c is either 0 or 1. Now that since this specifies the action on a 4-dimensional $\mathrm{GF}(q^2)$-space, we have the action acting on the 8-dimensional $\mathrm{GF}(q)$-space. However, this means that G_2K^* must have this form as well. Hence, there can be no third group within the group generated by the first two unless the action is identical to the action of one of the first two groups. Note that we are not claiming this form is the form acting as a collineation group, merely as a linear vector group over a field isomorphic to $\mathrm{GF}(q^2)$.

More generally for two common components, since the groups are cyclic, the union with the zero vector is a field isomorphic to $\mathrm{GF}(q^2)$ and contains the $\mathrm{GF}(q)$-scalar kernel homology group. Consider a component as a Desarguesian affine plane of order q^2. A cyclic group of order $q^2 - 1$ acting on a Desarguesian affine plane of order q^2 fixes at least two *components* of this affine plane. By choosing an appropriate basis for each of the fixed *components*, we may diagonalize the group G_1K^*. Either G_1K^* is the scalar group as acting on say $x = 0$ or the group fixes exactly two components of $x = 0$ as a Desarguesian affine plane. In the latter case, G_2K^* must fix or interchange these two subspaces on $x = 0$ and then must fix both since the group is cyclic. Hence, in either case, we may assume that both groups G_iK^* fix two *components* on $x = 0$. It is claimed that either group then is of the following form on $x = 0$:

$$\langle (x_1, x_2) \mapsto (x_1a, x_2a^k) \rangle, \quad \text{for } j = 0 \text{ or } 1, a \in K^+ \text{ of order } q^2 - 1.$$

and $k = 1$ or q. To see this, we simply note that either group is cyclic of order $q^2 - 1$ and the union of the group with 0 is a field containing the scalar field K isomorphic to $\mathrm{GF}(q)$. Since the group union zero then must be additive, it follows that $k = p^a$ where $q = p^r$. But, if σ is a generator then σ^{q+1} is in the kernel homology group so that $a^{q+1} = a^{p^a(q+1)}$, implying that $p^a = 1$ or q.

Hence, we can have only the following possibilities for the groups other than the scalar $\mathrm{GF}(q^2)$-mappings: (we list only a generator for each possible group)

$$\mathscr{A}_1 : (x_1, x_2, y_1, y_2) \mapsto (x_1 a, x_2 a, y_1 a, y_2 a^q),$$
$$\mathscr{A}_2 : (x_1, x_2, y_1, y_2) \mapsto (x_1 a, x_2 a, y_1 a^q, y_2 a),$$
$$\mathscr{A}_3 : (x_1, x_2, y_1, y_2) \mapsto (x_1 a, x_2 a, y_1 a^q, y_2 a^q),$$
$$\mathscr{A}_4 : (x_1, x_2, y_1, y_2) \mapsto (x_1 a, x_2 a^q, y_1 a, y_2 a),$$
$$\mathscr{A}_5 : (x_1, x_2, y_1, y_2) \mapsto (x_1 a, x_2 a^q, y_1 a, y_2 a^q),$$
$$\mathscr{A}_6 : (x_1, x_2, y_1, y_2) \mapsto (x_1 a, x_2 a^q, y_1 a^q, y_2 a),$$
$$\mathscr{A}_7 : (x_1, x_2, y_1, y_2) \mapsto (x_1 a, x_2 a^q, y_1 a^q, y_2 a^q).$$

Moreover, since these groups commute, either all groups acting on a fixed component fix all of the 1-dimensional K^+-subspaces or fix exactly two, implying, in general that all groups fix the same two K^+-subpsaces on both $x = 0$ and $y = 0$. Hence, we have a representational basis which is valid for all groups simultaneously.

The question is whether any one of these groups together with the GF(q^2)-scalar mappings can generate another of these groups. To see that this cannot occur, take any group which we represent in form (a, a^i, a^j, a^k) and multiply by a scalar element (b, b, b, b) to get $(ab, a^i b, a^j b, a^k b)$. We require that a has order $q^2 - 1$ and that ab has order $q^2 - 1$. Clearly, there exists a situation as follows:

$$a^m b = (ab)^n,$$

where $m \neq n$ are in $\{1, q\}$. If $n = 1$ and $m = q$ then $a^q = a$, a contradiction. Hence, $m = 1$ and $n = q$ so that $(ab)^q = ab$, again a contradiction. Hence, letting $\mathscr{A}_o$ denote the scalar group, we have that it is not possible to obtain a third fixed-point-free group of the type required within $\mathscr{A}_o \mathscr{A}_i$. This clearly generalizes to arbitrary pairs. Hence, we have the proof of the theorem. □

Theorem 16. *If π is a multiply-retractive translation plane of order q^4 for any set of distinct groups $G_i K^*$, $i = 1, 2, \ldots, t$ then there are at least two fixed components.*

Proof. The common fixed components of $G_1 K^*$ and $G_2 K^*$ are permuted by $G_3 K^*$. Let there be s_{12} common fixed components by $G_1 K^*$ and $G_2 K^*$. Let s_1 be the components fixed by $G_1 K^*$. Then there are $s_1 - s_{12}$ components fixed by $G_1 K^*$ which are not fixed by $G_2 K^*$. These components are in orbits of length $q + 1$ under $G_2 K^*$. Hence, $s_1 = s_{12} + k_1(q + 1)$ for some integer k_1. The common components s_{12} are permuted by $G_3 K^*$. If none are fixed by $G_3 K^*$ then $q + 1$ divides s_{12} which is a contradiction since $q + 1$ cannot divide s_1. Hence, there is at least one common fixed component. Suppose there is exactly one common fixed component. Then $q + 1$ divides $s_{12} - 1$. We know that $s_1 \equiv 2 \bmod (q + 1)$. To see this, we note that $s_1 + j(q + 1) = q^{2r} + 1$, for some integer j. Then since $q^{2r} + 1 = q^{2r} - 1 + 2$, the previous assertion follows.

But, in this case, $q + 1$ divides $s_{12} - 1$. Hence, $s_1 = s_{12} + k_1(q + 1) = z(q + 1) + 1 + k_1(q + 1)$, implying that $s_1 \equiv 1 \bmod (q + 1)$. Thus, it follows that there are at least two common components of three groups. A similar argument applies to show this true for any number of groups. □

Corollary 1. *For any number of distinct groups producing mixed partitions in a translation plane of order* q^4 *and kernel containing* $\mathrm{GF}(q)$, *there are at least two common fixed components in the group generated by all groups and no third group can be in the group generated by two distinct groups. Furthermore, any two distinct groups intersect exactly in the kernel homology group of order* $q-1$.

Proof. It remains only to verify the intersection assertion. However, this follows immediately by a re-examination of the proof that the third group cannot be within the group generated by the first two. □

7 Triple and quadruple retraction

We first assume that there are three distinct mixed partitions giving rise to the same translation plane of order q^4; the spread permits triple retraction. Assume that there are at least three common fixed components. Consider one of these fixed components L and decompose L with respect to one of the groups G_1K^* as a $\mathrm{GF}(q^2)$-vector space. Hence, we may consider L as a Desarguesian plane of order q^2. We know that G_2K^* fixes at least two 1-dimensional $\mathrm{GF}(q^2)$-subspaces. If G_2K^* fixes three 1-dimensional $\mathrm{GF}(q^2)$-subspaces then the action on L is identical to that of G_1K^* and since there are three fixed components, this would say that the two groups have identical actions and hence are the same group. Thus, G_2K^* fixes exactly two such 1-dimensional $\mathrm{GF}(q^2)$-subspaces on L and G_3K^* must permute these two subspaces. But, since G_3K^* is cyclic of order q^2-1 acting on a Desarguesian affine plane L, it follows that G_3K^* must fix both of the subspaces fixed by G_2K^*. Furthermore, it now follows that there is a group of order $(q+1)^3(q-1)$ acting on a 1-subspace so that there is a group of order $(q+1)^2$ which fixes this 1-subspace over $\mathrm{GF}(q^2)$ pointwise. Since there are at least three common components, this group must be a Baer group of order $(q+1)^2$. However, Baer groups in translation planes of order q^4 have orders dividing $q^2(q^2-1)$, a contradiction.

Hence, when there are three mixed partitions giving rise to the plane, it follows that there can be no Baer group. Thus, we have that there are exactly two common fixed points.

Furthermore, the above argument shows that there must be a common fixed component L among the three groups and thinking of L as a $\mathrm{GF}(q^2)$-subspace in three ways, it follows that each cyclic group of order q^2-1 fixes two of the 1-dimensional $\mathrm{GF}(q^2)$-subspaces. If one of the groups fixes exactly two then the third group also fixes these two since the group is cyclic. In any case, we have a subgroup on L of $\mathrm{GL}(2,q^2)$ of order $(q^2-1)(q+1)^2$ and admitting an Abelian group of order $(q+1)^2$ in $\mathrm{PGL}(2,q^2)$ (but not necessarily faithfully). By the structure of the subgroups of $\mathrm{PGL}(2,q^2)$, it follows that the faithful part is a cyclic subgroup of order dividing $q^2 \pm 1$. We note that the order of any element of the group generated by the three groups divides q^2-1.

$$((q+1)^2, (q^2+1)) = (2, q-1).$$

Hence, in this case, there must be a homology group with axis L of order divisible by $(q+1)^2/(2, q-1)$. If the group induced in $\mathrm{PGL}(2, q^2)$ has order dividing q^2-1, we have:

$$((q+1)^2, (q^2-1)) = (q+1)(q+1, q-1) = (q+1)(2, q-1).$$

Hence, we must have a homology group of order at least order $(q+1)/(2, q-1)$.

Actually, an alternative argument gives a stronger result. Recall that the group acting on the vector space over one field isomorphic to $\mathrm{GF}(q^2)$ has one of the following forms (not including the G_jK^* corresponding to the field in question):

$$\mathscr{A}_1 : (x_1, x_2, y_1, y_2) \mapsto (x_1a, x_2a, y_1a, y_2a^q),$$
$$\mathscr{A}_2 : (x_1, x_2, y_1, y_2) \mapsto (x_1a, x_2a, y_1a^q, y_2a),$$
$$\mathscr{A}_3 : (x_1, x_2, y_1, y_2) \mapsto (x_1a, x_2a, y_1a^q, y_2a^q),$$
$$\mathscr{A}_4 : (x_1, x_2, y_1, y_2) \mapsto (x_1a, x_2a^q, y_1a, y_2a),$$
$$\mathscr{A}_5 : (x_1, x_2, y_1, y_2) \mapsto (x_1a, x_2a^q, y_1a, y_2a^q),$$
$$\mathscr{A}_6 : (x_1, x_2, y_1, y_2) \mapsto (x_1a, x_2a^q, y_1a^q, y_2a),$$
$$\mathscr{A}_7 : (x_1, x_2, y_1, y_2) \mapsto (x_1a, x_2a^q, y_1a^q, y_2a^q).$$

Since we assume that we have the generic group

$$\mathscr{A}_o : (x_1, x_2, y_1, y_2) \mapsto (x_1a, x_2a, y_1a, y_2a); \quad a \in \mathrm{GF}(q^2)^*$$

acting, we can reduce the possible groups that actually end up acting as collineation group. Note that $\mathscr{A}_o\mathscr{A}_1$ will contain a group generated by the following mapping:

$$(x_1, x_2, y_1, y_2) \mapsto (x_1, x_2, y_1, y_2a^{q-1}).$$

However, such an element would fix all points on $y = 0$ as well as some points on $x = 0$, which cannot occur as a collineation of a translation plane. Hence, $\mathscr{A}_1$ cannot act together with $\mathscr{A}_o$ (which we assume acts generically). Similarly, $\mathscr{A}_2$, $\mathscr{A}_4$ and $\mathscr{A}_7$ cannot act as collineation groups. So, we can have only the following possibilities:

$$\mathscr{A}_3 : (x_1, x_2, y_1, y_2) \mapsto (x_1a, x_2a, y_1a^q, y_2a^q),$$
$$\mathscr{A}_5 : (x_1, x_2, y_1, y_2) \mapsto (x_1a, x_2a^q, y_1a, y_2a^q),$$
$$\mathscr{A}_6 : (x_1, x_2, y_1, y_2) \mapsto (x_1a, x_2a^q, y_1a^q, y_2a).$$

We have two distinct groups G_2K^* and G_3K^* of type $\mathscr{A}_i$ for $i = 3$, 5 or 6 as well as the scalar group $\mathscr{A}_o$. If one of the groups is $\mathscr{A}_3$, we have homology groups of order

$q+1$ with axis (resp. coaxis) $x=0$ and coaxis (resp. axis) $y=0$ in $\mathscr{A}_o\mathscr{A}_3$. For example, we obtain:

$$(x_1, x_2, y_1, y_2) \mapsto (x_1, x_2, y_1 a^{q-1}, y_2 a^{q-1}).$$

Otherwise, we would have $\mathscr{A}_o\mathscr{A}_5\mathscr{A}_6$ which also generate homology groups of order $q+1$ with these same axis and coaxis. Here we obtain from $\mathscr{A}_5\mathscr{A}_6$ the homology represented by:

$$(x_1, x_2, y_1, y_2) \mapsto (x_1, x_2, y_1 a^{1-q}, y_2 a^{q-1}).$$

However, these two groups cannot simultaneously exist since if so then we would have the collineation:

$$(x_1, x_2, y_1, y_2) \mapsto (x_1, x_2, y_1, y_2 a^{2(q-1)}),$$

a contradiction as there are too many fixed points.

Note that if 4-retraction is assumed then we have all four of these groups. Hence, 4-retraction is not possible.

Also, we note that a similar observation shows that in all possible groups generated by three $\mathscr{A}_i$: $\mathscr{A}_o\mathscr{A}_3\mathscr{A}_5$, $\mathscr{A}_o\mathscr{A}_3\mathscr{A}_6$ or $\mathscr{A}_o\mathscr{A}_5\mathscr{A}_6$ there are always Baer groups of order $q+1$. That is, on $x=0$ and $y=0$ there are 1-dimensional $\mathrm{GF}(q^2)$-subspaces that are fixed pointwise by a group of order $q+1$. These subspaces are 2-dimensional $\mathrm{GF}(q)$-subspaces. The 4-dimensional $\mathrm{GF}(q)$-subspace pointwise fixed by the group does not lie on the union of the two common components. Since the group is a collineation group, it follows that the fixed-point-space is a Baer subplane.

Since all fixed-point-free groups then fix each Baer axis of a Baer group, and there are exactly two common fixed components, at least one of the fixed-point-free groups has orbits of length $q+1$ on the remaining infinite points of the Baer axes and these orbits are then $\mathrm{GF}(q)$-reguli and there are $q-1$ of these. The set of such reguli must be permuted by the group generated by the three fixed-point-free groups, and the group action on the line at infinity of each group has orbits of length $q+1$ or 1. We note that the set of reguli on the Desarguesian Baer subplane must form a linear set (André set) with respect to the two fixed infinite points. A fixed-point-free group has orbits of length 1 or $q+1$ which implies that every such regulus is fixed by the full group generated by the three fixed-point-free groups. So, either there is a fixed-point-free group which fixes all q^2+1 infinite points of the net, which cannot occur, or all three groups have the same orbits of length $q+1$ on these $q-1$ reguli.

Note that we have a homology group $\mathscr{H}_1$ of order $q+1$ with axis $x=0$ and coaxis $y=0$, a homology group $\mathscr{H}_2$ of order $q+1$ with axis $y=0$ and coaxis $x=0$. By looking at the possible choice of groups, it follows that we must have four Baer groups $\mathscr{B}_i$ for $i=1,2,3,4$. These groups are generated by choosing one of the exactly two 1-dimensional $\mathrm{GF}(q^2)$-subspaces (relative to one of the fixed-point-free groups) on each of the two fixed components $x=0$ and $y=0$. Incidently, we assert from Jha and Johnson [4] that we may assume that the four Baer subplanes π_i for $i=1,2,3,4$ may be partitioned into pairs π_1, π_2 and π_3, π_4 such that π_1 and π_2 are in the same net D_{12}

of degree $q^2 + 1$ and π_3 and π_4 are in the same net of degree D_{34} of degree $q^2 + 1$ where the two nets D_{12} and D_{34} share exactly the common components say $x = 0$ and $y = 0$. Putting this in a different manner, we know that given a Baer group of order $q + 1$, there is a second Baer subplane which is fixed by this Baer group and which lies within the net defined by the Baer axis of the Baer group. Since this is the unique Baer subplane of this net which is fixed by the Baer group, it follows that the intersection of this Baer subplane with L and M must be one of the two invariant 1-dimensional $\mathrm{GF}(q^2)$-subspaces. Hence, the Baer groups can only fix pointwise one of the possible four Baer subplanes generated by choosing one 1-dimensional $\mathrm{GF}(q^2)$-subspace on each of L and M.

Following the discussion above, we have the following result.

Theorem 17. (1) *If three distinct mixed partitions of* $\mathrm{PG}(3, q^2)$ *produce the same translation plane then there are exactly two copies of* $\mathrm{PG}(1, q^2)$ *which belong to all three mixed partitions. That is, there are exactly two components in the translation plane which are fixed by the three groups.*

(2) *For each common* $\mathrm{PG}(1, q^2)$, *there is always a homology group of the associated translation plane of order divisible by* $q + 1$ *whose axis is that common line and whose coaxis the remaining common line.*

(3) *For the two common components L and M, there are four Baer groups of order* $q + 1$, *fixing L and M. Furthermore, there are two nets of degree* $q^2 + 1$ *sharing L and M and each net contains exactly two of the four Baer subplanes fixed pointwise by Baer groups.*

In addition, in each Baer net, there are $q - 1$ $\mathrm{GF}(q)$*-reguli that form a linear subset relative to a Baer axis of a Baer group. Each such regulus is either a* $q + 1$ *orbit for a given fixed-point-free group or is fixed linewise by that group.*

(4) *For q even, the group generated by the fixed-point-free groups may be generated by the two homology groups of order* $q + 1$, *any one of the Baer groups and the kernel homology group of order* $q - 1$, *or generated by any three Baer groups and the kernel homology group.*

(5) *Quadruple spread-retraction cannot occur.*

Proof. It remains to prove Part (4). However, this follows immediately since $(q + 1, q - 1) = 1$. □

8 Open questions

We have given examples of semifield planes of order q^4 that produce three mixed partitions. The following questions are open:

Problem 1. *If* π *is a translation plane of order* q^4 *with kernel containing* $\mathrm{GF}(q)$ *that admits triple retraction, is* π *a Hughes–Kleinfeld semifield plane or a derived Hughes–Kleinfeld semifield plane? If* π *is a semifield plane, can the plane be classified?*

Problem 2. *Let π be a translation plane of order q^4 with kernel containing $\mathrm{GF}(q)$ that produces mixed partitions of the following types*:

$$(q^2+1, q^2(q-1)), \quad (2q^2, (q^2-1)(q-1)), \quad (q^4-q^2+2, q-1).$$

Is π derivable? And, if π is derivable, is it a derived semifield plane? Furthermore, if π is a derived semifield plane, is the semifield plane a Hughes–Kleinfeld plane?

Similarly, we may ask:

Problem 3. *If π is a derivable translation plane of order q^4 and kernel containing $\mathrm{GF}(q)$ which produces mixed partitions of types*

$$(q^2+1, q^2(q-1)), \quad (2q^2, (q^2-1)(q-1)), \quad (q^4-q^2+2, q-1)$$

and the derived plane produces mixed partitions of types

$$(q^2+1, q^2(q-1)), \quad (q^2+1, q^2(q-1)), \quad (q^4+1, 0),$$

classify π.

Problem 4. *Let π be any translation plane of order q^t, where t is odd. If $(q+1)/(t, q+1) < 2$, show that double retraction cannot occur.*

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