

Scalar curvature of definable CAT-spaces

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Abstract. We study the scalar curvature measure for sets belonging to o -minimal structures (e.g. semialgebraic or subanalytic sets) from the viewpoint of metric differential geometry. Theorem: Let S be a compact connected definable pseudo-manifold with curvature bounded from above, then the singular part of the scalar curvature measure is non-positive. The topological restrictions cannot be removed, as is shown in examples.

1 Introduction

1.1 Plan of the paper and main results. One of the most important and most difficult problems in subanalytic geometry is to understand the induced length metric of subanalytic sets. For instance, the behavior of geodesics remains completely mysterious. Apart from a theoretical interest, this question has applications even outside mathematics, e.g. in robotics. This paper is devoted to the study of subanalytic sets by means of integral geometry and metric differential geometry.

In a preceding paper ([2]), we defined a scalar curvature measure for singular spaces. More precisely, we associated to each compact connected set belonging to some o -minimal structure (e.g. semialgebraic or subanalytic sets) a signed measure, called scalar curvature measure, which shares many of the properties of the usual scalar curvature of Riemannian manifolds. One of the main results of [2] was to relate the scalar curvature measure to curvature bounds in the sense of metric differential geometry. It has been shown that a lower bound κ on the curvature of a compact connected definable set of dimension m implies the lower bound $\kappa m(m-1) \text{vol}(-)$ for the scalar curvature measure. This theorem generalizes in a non-trivial way the easy fact from differential geometry that positive sectional curvature implies positive scalar curvature.

In this paper, we carry on the study of scalar curvature measure of definable sets. We will show that the analogous result in the case of negative sectional curvature is valid under some minor topological restrictions.

The main theorem is the following (see 1.2 for definitions):

Theorem 1.1 (Main Theorem). *Let S be a compact connected definable pseudo-manifold with curvature bounded from above, then the singular part of $\text{scal}(S, -)$ is non-positive.*

Remarks. If the dimension of S is m and the upper curvature bound κ , then we get an inequality between signed measures:

$$\text{scal}(S, -) \leq \kappa m(m-1) \text{vol}(S, -).$$

This generalizes the classical upper bound for the scalar curvature $s \leq \kappa m(m-1)$ on m -dimensional Riemannian manifolds with sectional curvature bounded from above by κ .

If S is a pseudo-manifold, then there exists a stratification with the property that each $m-1$ -stratum is contained in the boundary of exactly two m -strata. This is what we really need.

The topological assumption made in the theorem is in general necessary. For instance, take the closed unit ball in Euclidean space. Being convex, it has non-positive curvature in the metric sense, but its total scalar curvature is strictly positive because of the boundary contribution. Another example will be presented in Section 4.

Since everything is done locally, the theorem remains true for closed connected definable sets S . In this case, $\text{scal}(S, -)$ is still a difference of non-negative Borel measures and $\text{scal}(S, U)$ is well-defined for precompact U .

The paper is organized as follows. After giving the basic definitions, we will show some results about geodesics on definable spaces, which are of independent interest, see Section 2. The proof of the main theorem is contained in Section 3. It consists of two steps, one for strata of codimension 1, one for strata of codimension 2. The proofs in these cases are different, although they share some common features, as the use of the triangle inequality for angles and the extension property for geodesics. A consequence of one of the obtained formulas is the invariance of the scalar curvature measure under isometries. Such a result (actually a more general version) was conjectured by J. Fu. This invariance property is obtained in Section 4. It is the generalization of the classical fact that the scalar curvature on a Riemannian manifold is an inner quantity.

The idea behind the proof of the main theorem is to use the fact that, in the situation we will consider, each geodesic can be extended beyond its endpoints, by properties of CAT-spaces. In codimension 2, the triangle inequality for angles then yields that the density at each point of the stratum is bounded from below by 1. In codimension 1, a careful investigation of geodesics and angles between them is necessary. Again, we will conclude by the triangle inequality for angles that the scalar curvature measure is non-positive.

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1.2 Recall of definitions. First, we will introduce spaces with curvature bounded from above and give some properties that will be needed in the following proof. There is an excellent book, [4], where the reader can find a systematic treatment of spaces with curvature bounded from above. The following definitions are taken from Part II of that book.

Definition 1.2. A geodesic triangle in an inner metric space consists of three points A, B, C and geodesic segments $[A, B]$, $[B, C]$ and $[C, A]$ between them. We say that it satisfies the $\text{CAT}(\kappa)$ inequality ($\kappa \in \mathbb{R}$), if for all points $P \in [A, B]$ and $Q \in [A, C]$, we have $d(P, Q) \leq d(\tilde{P}, \tilde{Q})$, where \tilde{P} and \tilde{Q} are points on the sides $[\tilde{A}, \tilde{B}]$ respectively $[\tilde{A}, \tilde{C}]$ of a comparison triangle $\tilde{A}, \tilde{B}, \tilde{C}$ in the 2-dimensional space form M_κ with constant curvature κ with $d(\tilde{A}, \tilde{P}) = d(A, P)$, $d(\tilde{A}, \tilde{Q}) = d(A, Q)$.

A metric space is called *d geodesic* if all pairs of points a distance less than d apart can be joined by a (not necessarily unique) geodesic.

We denote by D_κ the diameter of M_κ , that is $D_\kappa = \infty$ for $\kappa \leq 0$ and $D_\kappa = \frac{\pi}{\sqrt{\kappa}}$ for $\kappa > 0$.

Definition 1.3. a) An inner metric space X is called $\text{CAT}(\kappa)$ space if it is a D_κ geodesic space and all geodesic triangles of perimeter less than $2D_\kappa$ satisfy the $\text{CAT}(\kappa)$ inequality.

b) A metric space X is said to be of curvature $\leq \kappa$ if it is locally a $\text{CAT}(\kappa)$ -space, i.e. for every $x \in X$ there exists $r_x > 0$ such that the ball $B(x, r_x)$, endowed with the induced metric, is a $\text{CAT}(\kappa)$ -space.

Remark 1.4. The initials C, A, T stand for Cartan, Alexandrov and Toponogov.

Recall that in a metric space X , the angle $\angle(A, B, C)$ is defined as the angle at A of a comparison triangle $\tilde{A}, \tilde{B}, \tilde{C}$ in Euclidean space. The Alexandrov angle between two geodesics c, c' with $c(0) = c'(0) = x$ is defined as $\angle(c, c') := \limsup_{t, t' \rightarrow 0} \angle(x, c(t), c'(t))$.

The next few statements are taken from [4]. We will need them in the course of the proof of our main theorem.

Proposition 1.5. *Suppose X is a $\text{CAT}(\kappa)$ -space. Then:*

- *The Alexandrov angle between the sides of any geodesic triangle of perimeter smaller than $2D_\kappa$ in X with distinct vertices is not greater than the angle between the corresponding sides of its comparison triangle in M_κ^2 . Moreover, every D_κ -geodesic metric space with this property is actually a $\text{CAT}(\kappa)$ -space.*
- *Every local geodesic in X of length at most D_κ is a geodesic.*
- *If X is a topological manifold, then each geodesic $\gamma : [a, b] \rightarrow X$ can be extended to a geodesic $\gamma : [a - \varepsilon, b + \varepsilon] \rightarrow X$ for some $\varepsilon > 0$. It follows that, if X is complete, any geodesic can be extended to a geodesic of length D_κ .*

Next, we recall the definition of *o*-minimal structures. Most propositions about semialgebraic sets can be deduced from a short list of properties, including the Tarski–Seidenberg principle (which states that the projection of a semialgebraic set is semialgebraic). Each class of sets satisfying these axioms shares automatically many useful properties with semialgebraic sets, e.g. boundedness of number of connected components. The definition is the following:

Definition 1.6. An *o-minimal structure* is a sequence $\sigma = (\sigma_n)_{n=1,2,3,\dots}$ such that:

- a) σ_n is a Boolean algebra of subsets of \mathbb{R}^n .
- b) Algebraic subsets belong to σ .
- c) If $S \in \sigma_n$ and $S' \in \sigma_m$, then $S \times S' \in \sigma_{n+m}$.
- d) If $S \in \sigma_{n+1}$ then $\pi(S) \in \sigma_n$ where $\pi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ is the projection on the first n coordinates.
- e) σ_1 consists exactly of finite unions of points and intervals.

Examples 1.7. The smallest example of an *o-minimal structure* is the set of semi-algebraic sets.

Globally subanalytic sets form an *o-minimal structure*. A set is called globally subanalytic if its closure in the projective space is subanalytic.

Sets definable over $\mathbb{R}_{\text{exp}} = (\mathbb{R}, +, \cdot, <, \text{exp})$ yield another example of an *o-minimal structure*.

The basic reference for *o-minimal structures* is [21], see also [10]. In the rest of the paper, we fix an *o-minimal system* and refer to its elements as *definable sets*.

Definition 1.8. Let $S \subset \mathbb{R}^n$ be a compact connected definable set. Then the restriction of the Euclidean metric of \mathbb{R}^n defines a metric on S , also called *Euclidean metric* and denoted by d_e . In general, d_e is not a length metric, but it induces a unique length metric on S , which we will denote by d_i . It is called *inner metric* of S .

Let $\delta(T, T') := \sup_{v \in T, \|v\|=1} d(v, T')$ denote the “distance” of vector subspaces of \mathbb{R}^n . Then T is contained in T' if and only if $\delta(T, T') = 0$. It is easy to see that $\delta(T, T') = \|\Pi_T^\perp \Pi_{T'}\|$, where Π_T denotes the projection on T and Π_T^\perp the projection on the orthogonal complement of T .

Definition 1.9. A stratification $S = \bigcup_i X^i$ of a closed subset of \mathbb{R}^n is called *Verdier-stratification* if for any pair X^i, X^j of strata with $X^i \subset \partial X^j$ and each $x \in X^i$ we have the following *Verdier condition*: There are a real number $C > 0$ and a neighborhood U of x such that $\delta(T_p X^i, T_q X^j) \leq C \|p - q\|$ for $p \in X^i \cap U, q \in X^j \cap U$.

Given a Whitney-stratification $S = \bigcup X^i \subset \mathbb{R}^n$ of a compact definable set, we denote by $\text{Nor}_e S \subset \mathbb{R}^n \times S^{n-1} \subset \mathbb{R}^{2n}$ the set of unit vectors normal to some stratum. $\text{Nor}_e T_{X^i} X^j, X^i \subset \overline{X^j}$ denotes the set of limits of unit normal vectors of X^j with foot-points tending to points X^i . A more detailed description of these sets and their relations can be found in [16].

Definition 1.10. We call a Whitney-stratification $S = \bigcup X^i \subset \mathbb{R}^n$ of a compact definable set *tame* if there is a stratification $\bigcup N^i$ of the unit normal space $\text{Nor}_e S \subset \mathbb{R}^n \times S^{n-1} \subset \mathbb{R}^{2n}$ satisfying the following two conditions:

- a) $\{N^\mu\}$ is compatible with the sets $\text{Nor}_e T_{X^i} X^j$, where $X^i \subset \overline{X^j}$. This means that each of the latter sets is a union of strata.
- b) The projection $\pi : \text{Nor}_e S \rightarrow S, (x, v_e) \mapsto x$ is a submersion on each stratum.

Remark 1.11. Let S be a compact definable set. Then it admits stratifications of each of the above types, as was proven by Ta Lê Loi ([18]) for Verdier stratifications and by Bröcker–Kuppe ([5], [16]) for tame stratifications.

If one defines scalar curvature measure in the integral geometric setting, then one needs tame stratifications from the very beginning. However, in this paper, we define scalar curvature measure by an explicit formula and will need tame stratifications only for some technical arguments in the proof of the main theorem.

We now come to the definition of the scalar curvature measure of definable sets. At first look, this seems to be an ad hoc definition, but it is shown in Theorem 1.2 of [2] that this definition coincides with a more natural definition coming from integral geometry. Since we do not want to go into details on integral geometry (such as Lipschitz–Killing curvatures), we define the scalar curvature measure from the very beginning by the formula below.

Definition 1.12. Let S be a compact connected definable set of dimension m with a stratification $S = \bigcup_i X^i$. Then we define a (signed) Borel measure by setting for each Borel subset $U \subset S$

$$\begin{aligned} \text{scal}(S, U) := & \int_{U \cap X^m} s(x) d\text{vol}_m(x) + 2 \int_{U \cap X^{m-1}} \sum_{i=1}^k \text{tr } II_{w_i} d\text{vol}_{m-1}(x) \\ & + 4\pi \int_{U \cap X^{m-2}} \left(\frac{1}{2} + \frac{(-1)^m}{2} \chi_{\text{loc}}(S, x) - \theta_m(S, x) \right) d\text{vol}_{m-2}(x). \end{aligned}$$

Here, w_1, w_2, \dots, w_k denote the normal vectors of X^{m-1} in direction of the highest dimensional strata and $\text{tr } II_{w_i}$ is the trace of the second fundamental form of X^{m-1} in direction w_i ($II_{w_i} = -\nabla w_i$). By $\chi_{\text{loc}}(S, x) := \chi(H_*(S, S \setminus \{x\}))$ we denote the local Euler-characteristic of S at x with respect to Borel–Moore homology. $\theta_m(S, x)$ denotes the m -dimensional density of S at x .

Remark 1.13. The scalar curvature measure is independent of the chosen stratification. See [16] or [5] for details.

Examples 1.14. A Riemannian manifold has positive or negative scalar curvature measure if and only if its usual scalar curvature is positive or negative respectively.

A Riemannian manifold with boundary has positive or negative scalar curvature measure if and only if its scalar curvature is positive or negative and the mean curvature of the boundary is positive or negative respectively.

For a compact Riemannian manifold S , $\text{scal}(S, S)$ is called the total scalar curvature. The Einstein–Hilbert functional of S is the functional that associates to a metric the total scalar curvature.

The scalar curvature measure of the surface of a cube is concentrated in its vertices. The scalar curvature measure of a vertex is $4\pi(1 - \frac{3}{4}) = \pi$. The total scalar curvature equals $8\pi = 4\pi\chi$. This is the Gauss–Bonnet-formula, which remains true in this setting. See [5] for details.

2 Law of reflection

Not much is known about the behavior of geodesics on definable sets. Approaching the boundary of a highest dimensional stratum, the differential equation for geodesics becomes singular. However, using some metric arguments and sufficiently good stratifications, we are able to show a law of reflection that will be important later on.

Proposition 2.1. *Let S be a compact definable set of dimension m with a fixed tame stratification, X^{m-1} and X^m strata of dimension $m - 1$ respectively m with $X^{m-1} \subset \overline{X^m}$. Then there is a definable subset E of X^{m-1} of dimension smaller than $m - 1$ such that for each point $P \in X^{m-1} \setminus E$ the following two conditions are satisfied:*

- *The tangent map extends continuously from X^m to P , i.e. there is a unique m -dimensional space T such that $T_Q X^m \rightarrow T$ for $Q \rightarrow P$.*
- *Near P , Verdier's condition is satisfied.*

Proof. We can refine the given stratification to a stratification satisfying Verdier's condition. Therefore, if we remove some strata of smaller dimension, Verdier's condition will be satisfied on X^{m-1} .

On the other hand, the set of limit tangent spaces of X^m has dimension less than m in the Grassmannian of pointed $m - 1$ dimensional affine subspaces of \mathbb{R}^n (see [16]). We denote for each $x \in \overline{X^{m-1}} \setminus X^{m-1}$ the set of limit tangent spaces at x by $T_{\lim, x} X^m$. Then $T_{\lim} X^m = \bigcup_{x \in \overline{X^m} \setminus X^m} T_{\lim, x} X^m$. We therefore see that the dimension of $T_{\lim, x} X^m$ has to be 0 almost everywhere. As a consequence of Hardt's Theorem (see for instance [10], Theorem 5.22), this dimension is a definable function of x . It follows that it is 0 outside a set of smaller dimension. With X^m being locally connected, $T_{\lim, P} X^m$ is connected for each $P \in X^{m-1}$. Therefore, if it is 0-dimensional, it consists only of a single space which implies that the tangent map extends continuously to P . \square

Proposition 2.2. *Let X^m and X^{m-1} be two strata as above. Suppose that the tangent map extends continuously from X^m to P with limit tangent space T . Then we can describe X^m (locally at P) in the following way: There are an orthonormal base of \mathbb{R}^n with associated coordinates $\{x_1, \dots, x_n\}$ and smooth functions g, f such that:*

$$T_P X^{m-1} = \{x_m = x_{m+1} = \dots = x_n = 0\}.$$

$$T = \{x_{m+1} = x_{m+2} = \dots = x_n = 0\}.$$

X^m is the graph of f over the open set $U = \{(x_1, \dots, x_m) \in T : x_m > g(x_1, \dots, x_{m-1})\}$.

The derivatives of f , $\frac{\partial f}{\partial x_i}(Q)$ tend to 0 for $Q \rightarrow P$.

Proof. Remember that X^{m-1} is smooth. The tangent space $T_P X^{m-1}$ is included in T , hence the projection of X^{m-1} to T will be a smooth hypersurface of T which can be written as the graph of a smooth function g . If we choose first a coordinate system on $T_P X^{m-1}$, extend it to a coordinate system of T and finally to one of \mathbb{R}^n , we get automatically the first two conditions.

By our assumption, the tangent space $T_Q X^m$ will be close to T in the Grassmannian. Therefore, the projection from X^m to T is a bijective smooth map. By simple topological reasons, its image must be one of the sets $\{x_m > g(x_1, \dots, x_{m-1})\}$ or $\{x_m < g(x_1, \dots, x_{m-1})\}$. Changing the signs of the coordinate x_m and of g , we can suppose that this image is $U = \{x_m > g(x_1, \dots, x_{m-1})\}$. Since X^m is smooth, we get some smooth function f such that X^m is the graph of f over U . The partial derivatives of f tend to 0 as follows easily from the uniqueness of the limit tangent space. \square

Remark. We can extend f to a continuous function (also denoted by f) on $\bar{U} = U \cup \text{graph } g$. The graph of f over the set $\text{graph } g$ is nothing else than X^{m-1} . Since X^{m-1} is smooth, $f|_{\text{graph } g}$ is smooth. We know furthermore that $T_P X^{m-1} \subset T$ which yields that all derivatives of $f|_{\text{graph } g}$ vanish at 0.

Proposition 2.3. *Same situation as above. Let $A \in X^m$, $P \in X^{m-1}$ be points such that $d_i(A, P) \leq d_i(A, P')$ for all points $P' \in X^{m-1}$. Further suppose that the tangent map extends continuously from X^m to P . Choose a geodesic γ between P and A , parameterized by arclength and with $\gamma(0) = P$. Let $w \in T$ be the vector which is given in our coordinate system by $(0, \dots, 0, 1)$. Then*

$$\lim_{\alpha \rightarrow 0} \frac{\gamma(\alpha) - P}{\alpha} = w.$$

Proof. We argue by contradiction. Suppose there exist a vector $w' \neq w$ and a sequence of real numbers $\alpha_1 > \alpha_2 > \dots \rightarrow 0$ such that

$$\lim_{i \rightarrow \infty} \frac{\gamma(\alpha_i) - P}{\alpha_i} = w'.$$

Set $r_i := \left\| \frac{\gamma(\alpha_i) - P - \alpha_i w'}{\alpha_i} \right\|$. Then $r_i \rightarrow 0$. By the triangle inequality we get

$$r_i \alpha_i \geq d_e(\gamma(\alpha_i), P) - \alpha_i \|w'\|.$$

By some easy arguments using Whitney's condition B, we see that $\lim_{i \rightarrow \infty} \frac{d_e(\gamma(\alpha_i), P)}{\alpha_i} = \lim_{i \rightarrow \infty} \frac{d_i(\gamma(\alpha_i), P)}{\alpha_i} = 1$. So we easily get $\|w'\| \geq 1$.

On the other hand, we always have $d_i \geq d_e$. It follows $\|\gamma(\alpha_i) - P\| \leq \alpha_i$ and

$$\|w'\| \leq \left\| w' - \frac{\gamma(\alpha_i) - P}{\alpha_i} \right\| + \left\| \frac{\gamma(\alpha_i) - P}{\alpha_i} \right\| \leq r_i + 1.$$

Since $r_i \rightarrow 0$, it follows $\|w'\| \leq 1$ and finally $\|w'\| = 1$.

Next, T is the unique limit tangent space and therefore $w' \in T$. Since $w' \neq w$, some of the first $m - 1$ coordinates of w' must be non-zero. Suppose without loss of generality that $w' = (w_1, \dots, w_m, 0, \dots, 0)$ with $w_1 \neq 0$.

Now, $\gamma(\alpha_i)$ has coordinates $(h_1, \dots, h_m, 0, \dots, 0)$ with $h_j = \alpha_i w_j + o(\alpha_i)$. The line L_i that joins this point with the point $P' = (h_1, \dots, h_{m-1}, g(h_1, \dots, h_{m-1}), 0, \dots, 0)$ has length $l_i = g(h_1, \dots, h_{m-1}) - h_m$. Since the derivatives of g at 0 vanish, it follows that

$$\lim_{i \rightarrow \infty} \frac{l_i}{\alpha_i} = |w_m| < 1.$$

For i sufficiently big, the function f will have arbitrarily small derivatives on L_i . It follows that the length of the pre-image of L_i under the projection to T , which is a curve in X^m , has a length which is close to the length of L_i , hence strictly smaller than α_i . This is a contradiction, since we have supposed that there is no point on X^{m-1} with smaller distance to A than P , but P' would be such a point. \square

A similar proof will yield the following:

Proposition 2.4 (Law of reflection). *Let S be a compact connected definable set of dimension m with a fixed tame stratification and X^{m-1} an $m - 1$ stratum neighboring exactly two m -strata X_1^m, X_2^m . Let γ be a geodesic such that $\gamma(\alpha) \in X_1^m$ for $\alpha < 0$, $P = \gamma(0) \in X^{m-1}$ and such that γ can be extended beyond P . We assume that the tangent map extends continuously from X_1^m, X_2^m to P . We choose for each stratum X_i^m a representation as in Proposition 2.2. Since both T_1 and T_2 contain $T_P X^{m-1}$, we can suppose that the coordinates x_1, \dots, x_{m-1} coincide on $T_P X^{m-1}$. Let $w_1 = (a_1, \dots, a_m, 0, \dots, 0)_1$ be a tangent vector of $\gamma|_{<0}$. Then, if w_1 is not contained in $T_P X^{m-1}$, $\gamma|_{>0}$ has a unique tangent vector w_2 at P which is given by $w_2 = (-a_1, \dots, -a_{m-1}, a_m, 0, \dots, 0)_2$.*

Remarks. a) If w_1 is contained in $T_P X^{m-1}$, the same holds true for w_2 , for otherwise we could apply the proposition with the reversed geodesic.

b) If w_1 is not contained in $T_P X^{m-1}$, both sides of γ have unique tangent vectors. (Just apply the proposition twice.) It follows that $\gamma(\alpha) = P + \alpha w_1 + o(\alpha)$ and $\gamma(-\alpha) = P + \alpha w_2 + o(\alpha)$ for small positive α .

Proof of the proposition. Assume first that there is a sequence $\alpha_1 > \alpha_2 > \dots \rightarrow 0$ such that $\gamma(\alpha_i) \in \bar{X}_1^m$ for each i . Then the set

$$\left\{ \frac{\gamma(\alpha_i) - P}{\alpha_i} \mid i = 1, 2, \dots \right\}$$

has an accumulation point w' of norm 1 (this follows from arguments as in Proposition 2.3). By choosing a subsequence of $\{\alpha_i\}$ (which for simplicity we call again α) we can thus suppose that

$$\gamma(\alpha_i) = P + \alpha_i w' + o(\alpha_i).$$

Let w' be given in coordinates by $w' = (a'_1, a'_2, \dots, a'_m, 0, \dots, 0)$. Let Π_1 be the projection of \mathbb{R}^n to T_1 . Since all derivatives of g at 0 vanish, the line between the points

$\Pi_1\gamma(\alpha_i)$ and $\Pi_1\gamma(-\alpha_i)$ lies entirely in U for α_i small enough. Its length is given by $\alpha_i\|w' - w\| + o(\alpha_i)$. The pre-image of this line yields a curve between $\gamma(\alpha_i)$ and $\gamma(-\alpha_i)$ of length bounded by

$$(1 + o(1))(\alpha_i\|w' - w\| + o(\alpha_i)) = \alpha_i\|w' - w\| + o(\alpha_i).$$

This follows from the fact that the derivatives of f tend to 0 for $\alpha_i \rightarrow 0$.

On the other hand, γ is a geodesic between $\gamma(\alpha_i)$ and $\gamma(-\alpha_i)$ which shows

$$d_i(\gamma(\alpha_i), \gamma(-\alpha_i)) = 2\alpha_i.$$

From both inequalities we easily deduce $\|w' - w\| \geq 2$. This is a contradiction, since w' and w both have norm 1 and have coordinates $a_m > 0$ respectively $a'_m \geq 0$. Our assumption was wrong, therefore $\gamma(\alpha) \in X_2^m$ for α in an interval $(0, \varepsilon)$, $\varepsilon > 0$.

Take w'' and a sequence $\alpha_1 > \alpha_2 > \dots \rightarrow 0$ with

$$\gamma(\alpha_i) = P + \alpha_i w'' + o(\alpha_i).$$

We shall show that $w'' = w_2$.

Again, $\|w''\| = 1$ by easy arguments.

Consider in \mathbb{R}^m the points (a_1, \dots, a_m) and $(a'_1, \dots, a'_{m-1}, -a'_m)$. Since $a_m > 0$ and $a'_m \geq 0$, the line between these points intersects the set $\{x_m = 0\}$ in a point $b = (b_1, \dots, b_{m-1}, 0)$.

For α_i small, the line in T_1 between $\Pi_1\gamma(-\alpha_i)$ and $(b, g_1(b), 0, \dots, 0)_1$ lies in U_1 and has a certain length l_1 , while the line between $\Pi_2\gamma(\alpha_i)$ and $(b, g_2(b), 0, \dots, 0)_2$ lies in U_2 and has a certain length l_2 . From the fact that g is smooth with vanishing derivatives at 0, we deduce that

$$l_1 + l_2 = \alpha_i\|(a_1 - a'_1, \dots, a_{m-1} - a'_{m-1}, a_m + a'_m)\| + o(\alpha_i).$$

Again, we can lift the union of these lines by Π_1 and Π_2 to get a curve joining $\gamma(\alpha_i)$ and $\gamma(-\alpha_i)$ whose length is bounded by

$$\alpha_i\|(a_1 - a'_1, \dots, a_{m-1} - a'_{m-1}, a_m + a'_m)\| + o(\alpha_i).$$

This cannot be smaller than $d_i(\gamma(\alpha_i), \gamma(-\alpha_i)) = 2\alpha_i$ since γ is a geodesic between both points. We deduce that

$$\|(a_1 - a'_1, \dots, a_{m-1} - a'_{m-1}, a_m + a'_m)\| \geq 2.$$

Finally, it follows that $a'_1 = -a_1, a'_2 = -a_2, \dots, a'_{m-1} = -a_{m-1}, a'_m = a_m$. \square

Proposition 2.5. *Let S be a compact definable set of dimension m with a fixed tame stratification. Suppose S has the geodesic extension property and X^{m-1} is a stratum neighboring exactly two strata X_1^m, X_2^m of highest dimension. Then at a dense set of points $P \in X^{m-1}$ there is a geodesic γ passing through both sides of X^{m-1} that has unique tangent vectors at P which are furthermore orthogonal to $T_P X^{m-1}$.*

Proof. Choose some point $A \in X_1^m$ near X^{m-1} and the point P which is the nearest to A in the inner metric. It is clear that there is a dense subset of points P arising in this way. So we can furthermore assume that P satisfies both conditions of Proposition 2.1 and we can apply Propositions 2.4 and 2.3.

Choose a geodesic between A and P and extend it beyond P to a geodesic γ . We can reparameterize by arclength such as to have $\gamma(0) = P$, $\gamma(-d_i(A, P)) = A$. By Proposition 2.4, $\gamma(\alpha) \in X_2^m$ for small positive α . We know by Proposition 2.3 that $\gamma|_{<0}$ has a unique tangent vector $w_1 = (0, \dots, 0, 1)_1$ (the index refers to the coordinate system). We can therefore apply Proposition 2.4 to see that $\gamma|_{>0}$ must have $w_2 = (0, \dots, 0, 1)_2$ as unique tangent vector. This finishes the proof of the proposition. \square

3 Proof of the Main Theorem

We recall that the Main Theorem states that a compact connected definable set of dimension m which is a pseudo-manifold with curvature bounded from above by κ has scalar curvature bounded from above by $\kappa m(m-1)$, which means that $\text{scal}(S, -) \leq \kappa m(m-1) \text{vol}(S, -)$. Before turning to the proof, we will give two easy examples.

Example 3.1. Let S be a connected two-dimensional piecewise linear space. Suppose that S is a topological manifold. Then the following statements are equivalent:

- a) S is a metric space of curvature ≤ 0 .
- b) For every vertex v of S , the link of S at v is a CAT(1)-space.
- c) The density at each vertex is at least 1.
- d) $\text{scal}(S, -) \leq 0$.

The equivalence between a) and b) is remarked in [4], Theorem 5.2., the equivalence between b) and c) comes from the fact that a one-dimensional space is CAT(1) if and only if there are no loops of length smaller than 2π . The equivalence between c) and d) is clear by definition.

Example 3.2. Let us generalize this example to the higher dimensional case. S is now a piecewise linear manifold which is supposed to be a topological manifold and to have curvature bounded from above by 0.

Let m denote the dimension of S . The scalar curvature measure is concentrated on simplices of dimension $m-2$. Locally at a point P of such a simplex, S is the product of the normal section and \mathbb{R}^{m-2} . Consequently, the normal section is (at least locally) a convex subset of S and thus a CAT(0)-space in a neighborhood of P (see also Example II 1.15 of [4]). Since it is a two-dimensional space, this is by the preceding example equivalent to the density at P being at least 1, which implies non-positive scalar curvature measure by Definition 1.12.

Proof of Theorem 1.1. Let S be a compact connected definable pseudo-manifold of dimension m which has curvature bounded from above by κ . Choose a tame stratification $S = \cup X^i$ of S . By the topological condition, refining this stratification if nec-

essary, we can assume that each stratum of codimension 1 lies in the boundary of exactly two m -strata, hence in Formula 1.12 we have $k = 2$.

Furthermore, by Thom's Isotopy Lemma ([20], [14]), we get for each point $x \in X^{m-2}$ that $\chi_{\text{loc}}(S, x) = (-1)^m$. In view of Definition 1.12, we have to show that for each Borel subset $U \subset S$, the expression

$$\begin{aligned} \text{scal}(S, U) &= \int_{U \cap X^m} s(x) \, d\text{vol}_m(x) + 2 \int_{U \cap X^{m-1}} \text{tr} II_{w_1+w_2} \, d\text{vol}_{m-1}(x) \\ &\quad + 4\pi \int_{U \cap X^{m-2}} (1 - \theta_m(S, x)) \, d\text{vol}_{m-2}(x) \end{aligned}$$

is bounded from above by $\kappa m(m-1) \text{vol}(U)$.

This can be done for each stratum dimension separately, the codimension 0 case being trivial (the $\text{CAT}(\kappa)$ -condition implies $s(x) \leq \kappa m(m-1)$ on the smooth part). For strata of codimension 1 and 2, we will show the non-positivity of the scalar curvature measure in Propositions 3.3 and 3.6. \square

3.1 Strata of codimension 1. The aim of this section is the proof of the following:

Proposition 3.3. *If S is a compact connected definable set of dimension m which is a $\text{CAT}(\kappa)$ -space for some κ (with respect to the inner metric) and X^{m-1} an $m-1$ -stratum on the boundary of exactly two m -strata, then $\text{scal}(S, -)|_{X^{m-1}} \leq 0$*

We recall that $\text{scal}(S, -)|_{X^{m-1}}$ is given by integration of the definable function $h := 2 \text{tr} II_{w_1+w_2}$ on X^{m-1} . To show that $\text{scal}(S, -)|_{X^{m-1}} \leq 0$, it is therefore enough to establish that $h(P) \leq 0$ on a dense subset of X^{m-1} . We are going to prove that $h(P) \leq 0$ for all points $P \in X^{m-1}$ such that:

- a) The tangent map can be extended continuously from both X_1^m, X_2^m to P .
- b) The stratification satisfies Verdier's condition near P .
- c) There is a geodesic passing through P as in Proposition 2.5.

Let S have curvature $\leq \kappa$ and let $S = \cup X^i$ be a tame stratification of S and X^{m-1} a stratum neighboring exactly two m -strata. Then by Propositions 2.1 and 2.5, the set of points $P \in X^{m-1}$ satisfying the above three conditions is a dense subset of X^{m-1} .

Let us fix such a point P . We find a geodesic γ consisting of two geodesic arcs γ_1, γ_2 such that γ_i lies entirely in X_i^m and such that

$$\gamma_i(\alpha) = P + \alpha w_i + r_i(\alpha)$$

for small positive α and where r_i denotes a function with $\lim_{\alpha \rightarrow 0} \frac{r_i(\alpha)}{\alpha} = 0$. Furthermore, $w_i \perp T_P X^{m-1}$ and $\|w_i\| = 1$.

Choose real functions $\varepsilon_1, \varepsilon_2 : (0, \infty) \rightarrow (0, \infty)$ such that for $i = 1, 2$

- a) ε_i is monotonically increasing,
- b) $\lim_{\alpha \rightarrow 0} \varepsilon_i(\alpha) = 0$,
- c) $\left\| \frac{r_i(\alpha)}{\alpha} \right\| \leq \varepsilon_i(\alpha)$.

We carry out the following construction for both X_1^m and X_2^m . In order to simplify the notation, we omit indices. We write γ in place of γ_i and so on. The well defined limit tangent space of X^m at P is denoted by T .

Apply Proposition 2.2 to the stratum X^m and the point P . This yields a system of coordinates x_1, \dots, x_n , a set

$$U = \{(x_1, \dots, x_m) \in \mathbb{R}^m : x_m > g(x_1, \dots, x_{m-1})\} \subset T = \{x_{m-1} = \dots = x_n = 0\}$$

and a function $f : U \rightarrow \mathbb{R}^{n-m}$ such that X^m is the graph of f and f has derivatives that converge to 0 if we approach P . In coordinates, f can be written as $f = (f_1, \dots, f_{n-m})$. Since P is contained in T , we get $f(0, \dots, 0) = 0$.

Lemma 3.4. *There are a neighborhood V_P of P and a constant $C > 0$ such that the function f satisfies for each $j = 1, \dots, n - m$ the following estimate:*

$$\left| \frac{\partial f_j}{\partial x_i}(Q) \right| \leq C d_e(Q, P) \quad \text{for } i = 1, \dots, m - 1 \text{ and } Q \in V_P.$$

Furthermore,

$$\left| \frac{\partial f_j}{\partial x_m}(Q) \right| \rightarrow 0 \quad \text{for } Q \rightarrow P.$$

Proof of the lemma. The second assertion follows from the fact that the tangent map can be extended continuously from X^m to P (see Proposition 2.2).

Let us sketch the proof of the first assertion. It will be a consequence of Verdier's condition. Remark that the tangent space of X^m at Q is generated as a vector space by the vectors

$$\left(1, 0, \dots, 0, \frac{\partial f_1}{\partial x_1}, \dots, \frac{\partial f_{n-m}}{\partial x_1}\right), \dots, \left(0, \dots, 0, 1, \frac{\partial f_1}{\partial x_m}, \dots, \frac{\partial f_{n-m}}{\partial x_m}\right).$$

Since the tangent spaces $T_Q X^m$ converge to T for $Q \rightarrow P$, we see that all derivatives converge to 0. We can suppose without loss of generality that among the vectors $\frac{\partial f}{\partial x_i}(Q)$, $i = 1, \dots, m - 1$, the first one is the longest. We denote its length by $L = L(Q)$. We have to show that $L(Q) \leq C d_e(P, Q)$ for some constant $C > 0$.

Let Π_P denote the orthogonal projection to $T_P X^{m-1}$ and Π_Q^\perp the orthogonal projection to the orthogonal complement of $T_Q X^m$. From Verdier's condition we know that $\|\Pi_Q^\perp \Pi_P\| \leq C d_e(P, Q)$ for some constant $C > 0$.

Consider the unit vector $(1, 0, \dots, 0) \in T_P X^{m-1}$. Its projection $\Pi_Q(1, 0, \dots, 0)$ to $T_Q X^m$ is given by

$$\left(a_1, a_2, \dots, a_m, a_1 \frac{\partial f_1}{\partial x_1} + \dots + a_m \frac{\partial f_1}{\partial x_m}, \dots, a_1 \frac{\partial f_{n-m}}{\partial x_1} + \dots + a_m \frac{\partial f_{n-m}}{\partial x_m}\right)$$

for some real numbers (a_1, \dots, a_m) . The condition of orthogonality yields a system of linear equations for the unknowns a_1, \dots, a_m . It has the form

$$(\text{Id} + o(L))(a_1, \dots, a_m)^T = (1, 0, \dots, 0)^T$$

where Id_m denotes the identity matrix of type (m, m) and $o(L)$ is a matrix that tends faster to 0 than L .

Resolution gives us $a_1 = 1 + o(L), a_2 = o(L), \dots, a_m = o(L)$ where $o(L)$ stands for real numbers that tend faster to 0 than L . Hence

$$\Pi_Q^\perp \Pi_P(1, 0, \dots, 0) = \left(o(L), o(L), \dots, o(L), \frac{\partial f_1}{\partial x_1} + o(L), \dots, \frac{\partial f_{n-m}}{\partial x_1} + o(L) \right)$$

and therefore

$$\|\Pi_Q^\perp \Pi_P(1, 0, \dots, 0)\| = \left\| \left(0, \dots, 0, \frac{\partial f_1}{\partial x_1}, \dots, \frac{\partial f_{n-m}}{\partial x_1} \right) \right\| + o(L) = L + o(L).$$

We know from Verdier's condition that this must be bounded by $Cd_e(P, Q)$ for some constant $C > 0$. Since we already know that L tends to 0 for $Q \rightarrow P$, we see that L is bounded by $2Cd_e(P, Q)$ for Q sufficiently near P . This proves the lemma. \square

Next, choose some real function $\varepsilon_3 : (0, \infty) \rightarrow (0, \infty)$ such that:

- a) ε_3 is monotonically increasing,
- b) $\lim_{s \rightarrow 0} \varepsilon_3(s) = 0$,
- c) for all $s > 0$, $Q \in B_e(P, s)$ and $j = 1, \dots, n - m$ we have $\left| \frac{\partial f_j}{\partial x_m}(Q) \right| \leq \varepsilon_3(s)$.

Remember that there are two m -strata neighboring X^{m-1} . We define ε_4 in the same way as we did for ε_3 , but this time for the other m -strata. The existence of such functions follows from the second assertion of the lemma.

We denote by Π_T the orthogonal projection from S to T .

Let s be a positive real number which we will let tend to 0. Set

$$\begin{aligned} \varepsilon(s) &:= \max\{s, \varepsilon_1(s), \varepsilon_2(s), \varepsilon_3(s), \varepsilon_4(s)\}^{1/2}, \\ \alpha(s) &:= s^2/\varepsilon(s), \end{aligned}$$

$$A = A(s) := x(s) \in X^{m-1}, \quad \bar{A} := \Pi_T A \in \text{graph } g \subset \bar{U} \subset T,$$

$$B = B(\alpha) := \gamma(\alpha), \quad \bar{B} := \Pi_T(B) \in U \subset T.$$

Note that ε is monotonically increasing and $\lim_{s \rightarrow 0} \varepsilon(s) = 0$. We easily see that $\alpha \leq s$ for $s \leq 1$.

Remember that $\gamma(\alpha) = P + \alpha w + r(\alpha)$ where the vector $w \in T$ is given in our coordinate system by $(0, \dots, 0, 1)$ and where $r(\alpha) = o(\alpha)$. Then \bar{B} is given by $(0, \dots, 0, \alpha) + \bar{r}(\alpha)$. Hence, $\frac{\|\bar{r}(\alpha)\|}{\alpha}$ tends to 0 for $\alpha \rightarrow 0$.

Since $x_1(s)$ is a geodesic on X^{m-1} , $x_1''(0)$ is orthogonal to $T_P X^{m-1}$. From $x_1(s) = P + sx_1'(0) + \frac{s^2}{2}x_1''(0) + O(s^3)$ we get

$$\bar{A}(s) = \left(s, 0, \dots, 0, \frac{s^2}{2} \langle x_1''(0), w \rangle \right) + O(s^3).$$

Lemma 3.5. *For sufficiently small s , the line between \bar{A} and \bar{B} lies entirely in U .*

Remark. Here is where we need that α is strictly bigger than s^2 . On the other hand, we will need later that the quotient $\frac{s^2}{\alpha}$ should very slowly tend to 0 in order to erase some superfluous terms. This is why we defined α in such a complicated way. Compare also with the situation for Alexandrov spaces with curvature bounded from below ([2]), where a similar definition for α was necessary.

Proof. Suppose there are arbitrarily small positive s such that the line does not lie entirely in U . The slope of this line in direction x_m is of order $\alpha/s = s/\varepsilon(s)$. If the line does not lie in U , it must cut the graph of g at some point between \bar{A} and \bar{B} . Then, at some other point, the slope of the graph of g would equal the slope of the line.

Remember that g is smooth with vanishing derivatives at 0. Therefore the derivatives of g behave like $O(s)$. This is a contradiction, since $\varepsilon(s)$ tends to 0 for $s \rightarrow 0$. This shows the lemma. \square

Let \bar{L} denote the line between \bar{A} and \bar{B} . It is given by

$$\begin{aligned} \bar{L}(t) &= (1-t)\bar{A} + t\bar{B} = \left((1-t)s, 0, \dots, 0, t\alpha + (1-t)\frac{s^2}{2}\langle x_1''(0), w \rangle \right) \\ &\quad + t\bar{r}(\alpha) + O(s^3) \quad 0 \leq t \leq 1. \end{aligned}$$

Let L denote the pre-image of \bar{L} under Π . Since \bar{L} lies in U , L is well defined and yields a curve in S between A and B . In our coordinate system, L is given by

$$L(t) = (\bar{L}(t), f_1(\bar{L}), f_2(\bar{L}), \dots, f_{n-m}(\bar{L})).$$

We will compute its length in order to get an upper bound for $d_i(A, B)$. First, note the following estimates:

$$\begin{aligned} |\bar{B}_1 - \bar{A}_1| &= |-s + \bar{r}_1(\alpha)| \leq O(s), \\ |\bar{B}_k - \bar{A}_k| &= |\bar{r}_k(\alpha)| \leq O(\alpha) \leq O(s), \quad k = 2, \dots, m-1, \\ |\bar{B}_m - \bar{A}_m| &= \left| -\frac{s^2}{2}\langle x_1''(0), w \rangle + \alpha + \bar{r}_m(\alpha) + O(s^3) \right| \leq O(\alpha) \leq O(s). \end{aligned}$$

On the other hand, since the first component of $\bar{B} - \bar{A}$ is $-s + \bar{r}_1(\alpha)$, we have $\|\bar{B} - \bar{A}\| \geq C_2s$ for some constant C_2 .

$$L'(t) = \left(\bar{L}'(t), \frac{\partial}{\partial t} f_1(\bar{L}), \frac{\partial}{\partial t} f_2(\bar{L}), \dots, \frac{\partial}{\partial t} f_{n-m}(\bar{L}) \right).$$

Obviously, $\bar{L}'(t) = \bar{B} - \bar{A}$. We calculate the other derivatives. It suffices to do the calculation for $\frac{\partial}{\partial t} f_1(\bar{L})$:

$$\frac{\partial}{\partial t} f_1(\bar{L}) = \sum_{k=1}^m \frac{\partial f}{\partial x_k} \frac{\partial \bar{L}_k}{\partial t}.$$

We claim that this is an $O(s^2)$ -term. For $k = 1, 2, \dots, m-1$, we have $\left| \frac{\partial f}{\partial x_k} \right| \leq O(s)$ and $\left| \frac{\partial \bar{L}_k}{\partial t} \right| \leq O(s)$. For $k = m$ and small s we have $\left| \frac{\partial f}{\partial x_k} \right| \leq \varepsilon_3(s) \leq \varepsilon(s)$ and $\left| \frac{\partial \bar{L}_k}{\partial t} \right| \leq O(\alpha)$. The product is $\leq O(\varepsilon(s)\alpha) = O(s^2)$. This shows the claim.

It follows that

$$\|L'(t)\|^2 \leq \|\bar{L}'(t)\|^2 + O(s^4) = \|\bar{B} - \bar{A}\|^2 + O(s^4) \leq (1 + O(s^2))\|\bar{B} - \bar{A}\|^2.$$

From this we can deduce

$$d_i(A, B) \leq l(L) = \int_0^1 \|L'(t)\| dt \leq (1 + O(s^2))\|\bar{B} - \bar{A}\|.$$

The inequality $d_i(A, B) \geq d_e(A, B) \geq d_e(\bar{A}, \bar{B}) = \|\bar{B} - \bar{A}\|$ implies that in fact

$$d_i(A, B) = (1 + O(s^2))\|\bar{B} - \bar{A}\|.$$

In our coordinate system, $\bar{B} - \bar{A}$ is represented as

$$\bar{B} - \bar{A} = \left(-s + \bar{r}_1(\alpha), \bar{r}_2(\alpha), \dots, \bar{r}_{m-1}(\alpha), \alpha + \bar{r}_m(\alpha) - \frac{s^2}{2} \langle x_1''(0), w \rangle \right) + O(s^3).$$

It follows that

$$\begin{aligned} \|\bar{B} - \bar{A}\|^2 &= s^2 - 2s\bar{r}_1(\alpha) + O(\bar{r}_1(\alpha)^2) + \dots + O(\bar{r}_m(\alpha)^2) \\ &\quad + \alpha^2 + O(\alpha\bar{r}_m(\alpha)) - \alpha s^2 \langle x_1''(0), w \rangle + O(\bar{r}_m(\alpha)s^2) + O(s^4) \\ &= s^2 - 2s\bar{r}_1(\alpha) + \alpha^2 - \alpha s^2 \langle x_1''(0), w \rangle \\ &\quad + O((\alpha\varepsilon_1(\alpha))^2) + O(\alpha^2\varepsilon_1(\alpha)) + O(\alpha\varepsilon_1(\alpha)s^2) + O(s^4). \end{aligned}$$

Next we have for $s \rightarrow 0$:

$$\begin{aligned} \frac{\alpha^2\varepsilon_1(\alpha)^2}{\alpha s^2} &\leq \frac{\alpha\varepsilon_1(s)^2}{s^2} \leq \frac{\alpha\varepsilon(s)^4}{s^2} = \varepsilon(s)^3 \rightarrow 0, \\ \frac{\alpha^2\varepsilon_1(\alpha)}{\alpha s^2} &\leq \frac{\alpha\varepsilon_1(s)}{s^2} \leq \frac{\alpha\varepsilon(s)^2}{s^2} = \varepsilon(s) \rightarrow 0, \\ \frac{\alpha\varepsilon_1(\alpha)s^2}{\alpha s^2} &= \varepsilon_1(\alpha) \rightarrow 0, \\ \frac{s^4}{\alpha s^2} &= \varepsilon(s) \rightarrow 0. \end{aligned}$$

It follows that

$$\|\bar{B} - \bar{A}\|^2 = s^2 - 2s\bar{r}_1(\alpha) + \alpha^2 - s^2 \langle x_1''(0), w \rangle + o(\alpha s^2).$$

From simple calculations and the fact that γ is a geodesic between P and B we see that

$$s \geq d_i(P, A) \geq d_e(P, A) = s + O(s^3)$$

and

$$d_i(P, B) = \alpha.$$

This enables us to compute the angle $\angle(A, P, B)$ in the case $\kappa = 0$:

$$\begin{aligned} \cos \angle(A, P, B) &= \frac{d(P, A)^2 + d(P, B)^2 - d(A, B)^2}{2d(P, A)d(P, B)} \\ &= \frac{s^2 + \alpha^2 - (1 + O(s^2))(s^2 - 2s\bar{r}_1(\alpha) + \alpha^2 - \alpha s^2 \langle x_1''(0), w \rangle + o(\alpha s^2)) + O(s^4)}{2\alpha(s + O(s^2))} \\ &= \frac{\bar{r}_1(\alpha)}{\alpha} + \frac{s}{2} \langle x_1''(0), w \rangle + o(s). \end{aligned}$$

(*Remark.* In the case $\kappa \neq 0$, we have to apply the corresponding law of cosines, but the resulting asymptotic behavior is the same. This is not surprising, as we consider very small triangles and locally, differences between hyperbolic, spherical and Euclidean space vanish.)

Doing the same calculations with $A = x_1(s)$ replaced by $C = x_1(-s)$ yields

$$\cos \angle(C, P, B) = \frac{-\bar{r}_1(\alpha)}{\alpha} + \frac{s}{2} \langle x_1''(0), w \rangle + o(s).$$

Furthermore, replacing $B = \gamma_1(\alpha_1)$ by $D = \gamma_2(\alpha_2)$ gives us:

$$\begin{aligned} \cos \angle(A, P, D) &= \frac{\bar{r}_2(\alpha_2)}{\alpha_2} + \frac{s}{2} \langle x_1''(0), w_2 \rangle + o(s), \\ \cos \angle(C, P, D) &= \frac{-\bar{r}_2(\alpha_2)}{\alpha_2} + \frac{s}{2} \langle x_1''(0), w_2 \rangle + o(s). \end{aligned}$$

Taking the sum of the cosines of these four angles yields:

$$\begin{aligned} &\cos \angle(A, P, B) + \cos \angle(C, P, B) + \cos \angle(A, P, D) + \cos \angle(C, P, D) \\ &= s \langle x_1''(0), w_1 + w_2 \rangle + o(s) \end{aligned} \tag{1}$$

Let $\gamma_A, \gamma_B = \gamma_1, \gamma_C, \gamma_D = \gamma_2$ denote the geodesics between P and A, B, C, D . By the CAT-inequality, we have

$$\angle_P(\gamma_A, \gamma_B) \leq \angle(A, P, B)$$

and analogous inequalities for the other points. From the triangle inequality for angles (see [4], Proposition 1.14) and the fact that B, P, D lie on a geodesic γ , we get

$$\pi = \angle_P(\gamma_B, \gamma_D) \leq \angle_P(\gamma_B, \gamma_A) + \angle_P(\gamma_A, \gamma_D) \leq \angle(A, P, B) + \angle(A, P, D)$$

Using monotony of the cosine function, we get

$$\cos \angle(A, P, B) + \cos \angle(A, P, D) \leq 0$$

and analogously

$$\cos \angle(C, P, B) + \cos \angle(C, P, D) \leq 0.$$

It follows that

$$s \langle x_1''(0), w_1 + w_2 \rangle + o(s) \leq 0$$

and hence

$$\langle x_1''(0), w_1 + w_2 \rangle \leq 0.$$

In the same way, such an inequality holds true with x_1 replaced by x_i for $i = 1, \dots, m-1$. This shows

$$h(P) = 2(\operatorname{tr} II_{w_1} + \operatorname{tr} II_{w_2}) = 2 \sum_{j=1}^{m-1} \langle x_j''(0), w_1 + w_2 \rangle \leq 0$$

which was to be shown. \square

3.2 Strata of codimension 2.

Proposition 3.6. *Let S be a compact connected definable set of dimension m with a fixed tame stratification. Suppose that S is a space of curvature $\leq \kappa$ for some $\kappa \in \mathbb{R}$ and that each $m-1$ -stratum lies on the boundary of exactly two m -strata. Then $\operatorname{scal}(S, -)|_{X^{m-2}} \leq 0$ for each stratum X^{m-2} of dimension $m-2$.*

Proof. From Thom's Isotopy Lemma ([20], [14]) it follows that $\chi_{\operatorname{loc}}(S, P) = (-1)^m$ for each point $P \in X^{m-2}$. In view of Definition 1.12 we shall show that $\theta(S, P) \geq 1$ at almost each such point P . By the Normal Section Formula of [2], the density of S at P equals almost everywhere the density of the normal section $S_P := (T_P X^{m-2})^\perp \cap S$ at P . It thus suffices to show that $\theta(S_P, P) \geq 1$ almost everywhere.

Fix a point $P \in X^{m-2}$. A neighborhood U_P of P in S_P lies in a tubular neighborhood around X^{m-2} which implies that for each point A in S_P that is sufficiently close to P , the Euclidean distance of A to X^{m-2} equals the Euclidean distance between A and P .

We denote by $S_e(P, r)$ the Euclidean sphere of radius r around P . By hypothesis and Thom's Isotopy Lemma, the sets $S_e(P, r) \cap S_P$ are disjoint unions of sets

$K_1(r), \dots, K_k(r)$ which are homeomorphic images of circles. Their lengths will be denoted by $l_1(r), \dots, l_k(r)$. The density of S_P at P is given by

$$\theta(S_P, P) = \lim_{r \rightarrow 0} \frac{\sum_{i=1}^k l_i(r)}{2\pi r}.$$

This is proven in [2], it follows from easy arguments using Whitney-stratifications.

We will show that each term in this sum is at least 1. It suffices to show this for the first one, so we claim that

$$\lim_{r \rightarrow 0} \frac{l_1(r)}{2\pi r} \geq 1.$$

If this is not the case, choose a real number θ' with $\lim_{r \rightarrow 0} \frac{l_1(r)}{2\pi r} < \theta' < 1$ and $\varepsilon > 0$. For sufficiently small $r > 0$, the following conditions will be satisfied:

- a) $K_1(r)$ lies in a tubular neighborhood of X^{m-2} ,
- b) $0 < l_1(r) \leq 2\pi r \theta'$.

Claim. There exist arbitrarily small $r > 0$ and points $A, B \in K_1(r)$ with the following property: the geodesic between A and B contains a point P' of X^{m-2} .

To prove the claim, we need some topological arguments. Choose a homeomorphism $\varphi : S^1 = \{z \in \mathbb{C} : |z| = 1\} \rightarrow K_1(r)$. We will extend φ to a continuous map $\bar{\varphi} : B^2 \rightarrow S$. To this end, fix some point $A \in K_1(r)$. Then $A' := \varphi^{-1}(A) \in S^1$ and each point C' of B^2 can be represented as

$$C' = A' + t(B' - A')$$

with $0 \leq t \leq 1$ and $B' \in S^1$. If $C' \neq A'$, this representation is unique.

Set $B := \varphi(B')$. Denote the unique geodesic between A and B by $\gamma_{A,B}$. We define

$$\bar{\varphi}(C') := \gamma_{A,B}(td_i(A, B)).$$

Since in $\text{CAT}(\kappa)$ -spaces geodesics of length strictly smaller than D_k depend continuously on their endpoints (see [4]), $\bar{\varphi}$ is a continuous map from B^2 to S .

Suppose that the claim were not fulfilled. Then $\bar{\varphi}$ is a continuous map from B^2 to $S \setminus X^{m-2}$. It follows that the loop $K_1(r)$ is contractible in $S \setminus X^{m-2}$. The image of B^2 under this map lies for r sufficiently small in the neighborhood U_P . Hence $K_1(r)$ is contractible in $(S \setminus X^{m-2}) \cap U_P$.

On the other hand, $K_1(r)$ cannot be contractible in $S_P \setminus \{P\}$, since this set is homotopically equivalent to the union of the sets $K_i(r)$, $i = 1, \dots, k$. By Thom's Isotopy Lemma, $(S \setminus X^{m-2}, P)$ is locally homeomorphic to $((S_P \setminus \{P\}) \times \mathbb{R}^{m-2}, P)$. Therefore $K_1(r)$ cannot be contractible in $S \setminus X^{m-2}$ which is a contradiction. The claim is proved.

Next, choose a sufficiently small r and points A, B as in the claim. The geodesic γ between A and B contains some (not necessarily unique) point $P' \in X^{m-2}$.

The length of the curve $K_1(r)$ is bounded by $2\pi r\theta'$. Since this curve is homeomorphic to a circle, we can join A with B by a curve β of length bounded by $\pi r\theta'$ which stays at a Euclidean distance r away from P . By our assumptions, this curve lies in a tubular neighborhood of X^{m-2} which implies that the Euclidean distance from each point of the curve to P' is at least r .

Take a series of points $A = P_0, P_1, \dots, P_k = B$ on β such that

$$d_i(P_j, P_{j+1}) < \varepsilon, \quad j = 0, 1, \dots, k-1$$

and

$$\sum_{j=0}^{k-1} d_i(P_j, P_{j+1}) \leq l(\beta) \leq \pi r\theta'.$$

Let γ_j be the geodesic between P_j and P' . We deduce from the CAT-inequality and from the triangle inequality for angles (see [4], Proposition 1.14)

$$\pi = \angle_{P'}(\gamma_0, \gamma_k) \leq \sum_{j=0}^{k-1} \angle_{P'}(\gamma_j, \gamma_{j+1}) \leq \sum_{j=0}^{k-1} \angle(P_j, P', P_{j+1}).$$

Let $l_\kappa(r)$ denote the length of the circle of radius r in M_κ^2 .

To estimate the angle $\angle(P_j, P', P_{j+1})$ note that $d_i(P_j, P') \geq r$. It follows that this angle is not greater than the corresponding angle of a comparison triangle in M_κ^2 with side lengths $r, r, d_i(P_j, P_{j+1})$ which is bounded by $\psi(\varepsilon)2\pi \frac{d_i(P_j, P_{j+1})}{l_\kappa(r)}$ where $\psi(\varepsilon)$ tends to 1 for $\varepsilon \rightarrow 0$ (just take a Taylor development of the κ -cosine).

It follows that

$$\pi = \angle_{P'}(\gamma_0, \gamma_k) \leq \sum_{j=0}^{k-1} \angle(P_j, P', P_{j+1}) \leq \psi(\varepsilon) \frac{2\pi}{l_\kappa(r)} \sum_{j=0}^{k-1} d_i(P_j, P_{j+1}) \leq \psi(\varepsilon) \frac{2\pi r}{l_\kappa(r)} \pi\theta'$$

If we let ε tend to 0, we get

$$\pi \leq \frac{2\pi r}{l_\kappa(r)} \pi\theta'$$

For $r \rightarrow 0$, the right hand side tends to $\pi\theta' < \pi$, which is a contradiction. Hence the assumption $\theta < 1$ cannot be fulfilled and Proposition 3.6 is proved. \square

The proof of the main theorem is complete.

4 Further remarks

A natural conjecture would be that the main theorem remains true under the weaker condition that every stratum of codimension 1 lies on the boundary of at least two

top dimensional strata. Surprisingly, this turns out to be false. The first part of the proof of the main theorem can be generalized to this situation, the problem lies in the codimension 2 case.

Take for instance 5 points on a 4-dimensional unit sphere and join each pair of them by a semialgebraic arc of length $\frac{2\pi}{3}$ in such a way that these arcs do not intersect. With respect to the induced length metric, this graph is a CAT(1)-space, since every non-contractible loop has length at least $2\pi = 2D_1$ (see [4], Example II 1.15). The Euclidean cone X over this graph with base point the center P of the sphere is a (semialgebraic) CAT(0)-space by a theorem of Berestovskii (see [4], Theorem II 3.14).

On the other hand, $\chi_{\text{loc}}(X, P) = 1 - 5 + 10 = 6$ and $\theta_2(X, P) = 10 \frac{1}{2\pi} \frac{2\pi}{3} = \frac{10}{3}$ which implies $\text{scal}(X, \{P\}) = \frac{1}{2} + 3 - \frac{10}{3} = \frac{1}{6} > 0$. Hence the scalar curvature measure is strictly positive at P .

Another remark concerns the invariance of the scalar curvature measure under isometries. As Bröcker–Kuppe and Fu have shown, each Lipschitz–Killing curvature measure is preserved under any definable isometry. The scalar curvature measure is one of these curvatures. However, there are simple examples of non-definable isometries between definable sets (for instance between a circle and an ellipse). But it is not clear if there is really a big difference between “isometric” and “definably isometric”. Anyway, the next corollary shows that scalar curvature measure is preserved by any isometry:

Theorem 4.1. *Let S, \tilde{S} be two compact, connected, definable sets and $f : S \rightarrow \tilde{S}$ an isometry between them. Then f induces an isomorphism of the scalar curvature measures, i.e. for each Borel measure $\tilde{U} \subset \tilde{S}$ we have*

$$\text{scal}(\tilde{S}, \tilde{U}) = \text{scal}(S, f^{-1}(\tilde{U})).$$

Proof. The idea is to use Formula (1). The scalar curvature measure on strata of codimension 1 is there expressed only using inner geometric terms: certain geodesics and angles. On m -strata, the scalar curvature is invariant under isometries (classical), on strata of codimension 2 as well (local Euler-characteristic and density are invariant under isometries). Furthermore, the scalar curvature measure is independent of the stratification. These ideas can be easily put together to give a proof of Theorem 4.1. \square

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