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SOME REMARKS ON THE HISTORY OF FUNCTIONAL ANALYSIS

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ABSTRACT. Several information on the beginning of functional analysis as an important and powerful chapter of mathematics, on the results and people, are given.

1. INTRODUCTION

Annals of Functional Analysis is a new established journal devoted to publication of high quality papers mainly in functional analysis and operator theory. As the first paper of the first issue of the journal, it is natural to recall some information on the beginning of functional analysis. Nobody can summarize the whole History of Functional Analysis in a few pages, so that we try only to present some remarks on the history of functional analysis by focusing on Banach spaces and works of Stefan Banach.

2. "STUDIA MATHEMATICA" AND "THÉORIE DES OPÉRATIONS LINÉAIRES"

It is almost always difficult and controversial to decide what period can be regarded as the beginning of a mathematical theory. Of course, to make a theory, several preparations must take place, sometimes for several centuries. Not always several mathematical events lead to the birth of the whole theory.

One of possible definitions of the beginning of a theory may be that the moment of the birth of a theory is when the first fundamental monograph about the theory was published and, moreover, it is the basis to further important research. This

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is quite a good condition in the case of functional analysis. Here, definitely there is no problem in picking the suitable book.

The first monograph on functional analysis was published on 1931. It was "Operacje liniowe" by Stefan Banach ([5]), in 1932 published in French ("Théorie des opérations linéaires", [6]) as the first volume in the series "Mathematical Monographs". For many years it was the most basic book on functional analysis, up to the moment when the famous monograph [22] was published. It should be noted that only in 40s the term "functional analysis" was being used. In Banach times other names were used, especially "the theory of linear operators".

A few years earlier the first international journal in this area of mathematics was created. In 1928 Stefan Banach and Hugo Steinhaus initiated new journal, "Studia Mathematica" which published papers just on functional analysis. Nowadays, majority of journals concentrate on a particular branch (or branches) of mathematics, but for that time it was completely unusual. "Studia Mathematica" was one of the first journals in the history that was specialized only in some areas of mathematics (the very first one was another Polish journal "Fundamenta Mathematicae", devoted mainly to topology). Let us say a little more about this idea.

In 1917 a young Polish mathematician Zygmunt Janiszewski wrote an article entitled "O potrzebach matematyki w Polsce" (On mathematical necessities in Poland, [31]). The article was published in the journal "Nauka Polska" (Polish Science). Janiszewski pointed out the idea of concentrating mathematical research in Poland on the one, new and developing branch of mathematics. He also proposed the editing of a new journal which would publish only papers just connected with some selected areas of mathematics, in particular set theory and topology. This was the origin of the creation of "Fundamenta Mathematicae". Unfortunately, Janiszewski died suddenly in 1920, at age 32, before the first issue of "Fundamenta Mathematicae" was published. When the publication of this journal was planned, several outstanding mathematicians (including Henri Lebesgue and Nicolai Luzin, see ([21])) doubted if such a journal would be on a good level because of the lack of suitable number of papers of good quality. However, after the publication of some issues the journal proved to be of excellent quality which was confirmed also by some, who earlier had been sceptical.

In fact, some Polish authors published in "Fundamenta Mathematicae" research articles on functional analysis. This stopped in 1929, when the first issue of "Studia Mathematica" appeared.

The editors of "Studia Mathematica" were Stefan Banach and Hugo Steinhaus. They were the editors up to 1940, when the volume 9 was published and then there was the break in publishing the journal because of the WWII. Volume 10 was published in 1948. Banach died in 1945, however, in volume 10 two Banach's papers were published. Since 1936 to 1940 the Editors were supported by the Editorial Board: Herman Auerbach, Stanisław Mazur and Władysław Orlicz.

In the first volume of "Studia Mathematica" 14 papers were published. The authors were: S. Banach, Z.W. Birnbaum, L. Fontappié, S. Kaczmarz, S. Mazur, W. Nikliborc, W. Orlicz, S. Saks, J. Schauder and H. Steinhaus.

In volumes 1–9 the majority of 161 papers in "Studia Mathematica" was written by Lvov mathematicians. The greatest number is due to Orlicz: 21 (next Mazur 17 and Banach 16), speaking about mathematicians from other countries, Felix Hausdorff, who published 6 papers is in the first position there. In those years, 56 authors published papers in "Studia Mathematica". The papers were written mainly in German (78) and French (66); 16 papers were in English, one in Italian.

In the first volume, the most important paper were definitely two papers by Banach ([2, 3]). In the first paper Banach proved the theorem now known as the Hahn–Banach Theorem.

Theorem 1 (Hahn–Banach Theorem)**.** *Let X be a real vector normed space, $p : X \to \mathbb{R}$ a function such that $p(\alpha x + (1-\alpha)y) \leq \alpha p(x) + (1-\alpha)p(y)$ for each $x, y \in X$ and $\alpha \in [0,1]$. Assume that $\varphi : Y \to \mathbb{R}$ is a linear functional defined on a vector subspace Y of X with the property $\varphi(x) \leq p(x)$ for all $x \in Y$. Then there exists a linear functional $\psi : X \to \mathbb{R}$ such that $\psi(x) = \varphi(x)$ for all $x \in Y$ and $\psi(x) \leq p(x)$ for all $x \in X$.*

Independently, this theorem was (in a simpler case) slightly earlier discovered by Hans Hahn ([26]), which was a generalization of his result from 1922. Banach did not know about Hahn's paper; nevertheless, Banach's version was stronger as Hahn proved a theorem in the case where X is a Banach space. It should be noted that a simpler version of the theorem (in the case where X is the space of continuous real functions on the compact interval) was published much earlier by Eduard Helly ([27]; see [29]). Neither Banach nor Hahn did not know about Helly's theorem.

The Hahn–Banach theorem is regarded as one of three "pillars" of functional analysis by many authorities. It has several versions (compare [41]). The complex version of the theorem was proved later, in 1938 independently by G.A. Soukhomlinov ([45]) and by F. Bohnenblust and A. Sobczyk ([11]).

In the second paper Banach proved several theorems fundamental for functional analysis. In particular, he proved the theorem that a continuous bijective linear functional is invertible under some additional assumptions. The version presented in the paper was shortly afterwards modified by Banach. Now the modified version is known (presented in another form) as the Banach Open Mapping Theorem. It is by many authorities regarded as one of three fundamental "pillars" of the functional analysis.

In the paper [3] there was also proved a version of another very important theorem on functional analysis, nowadays called frequently the Banach–Alaoglu Theorem. It states that

Theorem 2 (Banach–Alaoglu Theorem)**.** *Let X^* be the dual space of a Banach space X. Then the closed unit ball in X^* is compact in X^* equipped with the weak* topology.*

A proof of this theorem was given in 1940 by Leonidas Alaoglu ([1]). In [3] Banach proved the theorem in the case where X is a separable normed vector space.

The name "Banach space" was probably used for the first time by Maurice Fréchet in 1928. Note that independently such spaces were introduced by Norbert Wiener, however Wiener thought that the spaces would not be of importance and gave them up. Long time later Wiener wrote in his memoirs that the spaces quite justly should be named after Banach alone, as sometimes they were called "Banach–Wiener spaces". However, just in the first volume of "Studia Mathematica" Steinhaus [42] suggested to use the term "Banach space" (in Banach terminology such a space was called a space "of type (B))". The name was very quickly adopted to mathematics. Note that although in the thirties the term "Banach space" was commonly used, Banach to the end of his life always used the name "the space of type (B)".

Let us also mention some other papers published in "Studia Mathematica" in the first volumes. In volume 3 there were published two short papers: [38] and [7]. In 1928 Steinhaus posed the question "how large" is the set of nowhere differentiable real functions in the set of all continuous functions on the interval $[0, 1]$. In the paper [38] Stefan Mazurkiewicz proved that this set is very large, i.e. its complement is of the first Baire category. Thus he gave another proof, very original, for the existence of such a function. A few months later Banach ([7]) proved in a different and much simpler way the theorem obtained by Mazurkiewicz. In volume 3 there were also other papers that used very effectively the Baire Cathegory method, written by Kaczmarz and Auerbach. It is said that this method was brought to power by the Polish School of mathematics

In volume 4 Mazur ([37]) gave several theorems on linear topological spaces. This paper gave an important impulse to the development of this branch of functional analysis. In volume 9 Krein and Milman gave the proof of a theorem about the convex sets in dual Banach spaces ([35]). Later on, Kelley generalized this theorem for locally convex Hausdorff spaces ([34]).

Let us come back to the monograph [6] and summarize very briefly its contents.

In the introduction Banach credits the beginning of the theory of linear operations to Vito Volterra and claims that the theory joins in very good way several mathematical branches like the theory of real functions, integral calculus or variational calculus. He concentrates in the book on the theory of operators defined mainly on the spaces of type (B) that are the generalization of many important particular spaces. What is important, Banach investigates in the book rather not spaces, but operators on spaces. Another remarkable note is that Banach presented a unified treatment of finite and infinite dimensional linear spaces.

The book consists of 12 chapters. The first is about topological groups (then a new mathematical notion, not widely known yet). In the second part vector spaces (called the spaces of type (D)) are investigated, but also here the Hahn–Banach Theorem is given. Among the results of Chapter 3 there is the powerful generalization of another one of three "pillars" of functional analysis, i.e. the Banach–Steinhaus Theorem.

The Banach–Steinhaus Theorem says that

Theorem 3 (Banach–Steinhaus Theorem)**.** *Let X be a Banach space, Y be a normed vector space. Consider the family $\mathcal{F}$ of all linear bounded functions from*

X to Y. If for any $x \in X$ the set $\{||T(x)|| : T \in \mathcal{F}\}$ is bounded, then the set $\{||T|| : T \in \mathcal{F}\}$ is bounded.

The theorem was originally published and proved in [8].

In the monograph, the following generalization, by utilizing the Baire Category Theorem, is presented as well.

Theorem 4. *Let X be a Banach space, Y be a normed vector space and $f : X \to Y$ be a continuous linear operator. Then either $f(X)$ is the set of the first Baire category or $f(X) = Y$.*

As a consequence, the Banach Closed Graph Theorem is proved.

Theorem 5 (Banach Closed Graph Theorem)**.** *Let X and Y be Banach spaces, and T be a linear operator from X to Y. Then T is bounded if and only if the graph of T is closed in $X \times Y$.*

This theorem is closely related to the very important Banach Open Mapping Theorem.

Theorem 6 (Banach Open Mapping Theorem)**.** *Let X and Y be Banach spaces, and T be a linear operator from X onto Y. Then for any open subset U of X the set $T(U)$ is open in Y.*

In Chapters 4–12 we have the theory of Banach spaces, starting form the definition of normed spaces. Several results from the theory, as well obtained by Banach as other authors, and the basis for further investigations are presented. The important element of the book are “Remarks”. Remarks, prepared by Banach with the collaboration of Mazur, contain several historical details, many additional information and open questions.

The more precise description of the contents of the monograph is very well presented in [19]

In the introduction Banach also wrote about his plans of writing the second part of the monograph. This would be about other nonlinear functional operations and topological methods. However, the second part was never written.

3. Stefan Banach

Many mathematicians had a large influence to the creation of functional analysis. However, by many authorities just Stefan Banach is regarded as the creator of the theory. In this chapter, we write a little more about this outstanding and unusual mathematician.

Stefan Banach was born in the Polish city Kraków in March 30, 1892. Banach’s parents were not married. He unusually had a surname after his mother, Katarzyna Banach. His father was Stefan Greczek, a soldier (probably assigned orderly to the officer under whom Katarzyna was a servant), who could not marry Katarzyna because of some military rules. Banach was grown up by the owner of the laundry Franciszka Płowa and her niece Maria.

Banach attended school in Kraków and took there final exams. He was very interested in mathematics. However, he thought that in this area nothing much new can be discovered and he decided to study at the Technical University at Lvov (Politechnika Lwowska). At those times both Kraków and Lvov were at the territory governed by Austro-Hungary. His studying in Lvov was interrupted by the First World War and Banach came back to Kraków. He was excused from military service because of poor vision as well as left-handedness.

As a mathematician, Banach was self-taught. He did not study mathematics, however he attended some lectures at the Jagiellonian University, especially delivered by Stanisław Zaremba. In 1916, very important event took place. Hugo Steinhaus (1887–1972) spent some time in Kraków. In 1916, during his evening walk at the Planty Park in the centre of Kraków, Steinhaus heard the words "Lebesgue integral". In those times it was a very modern mathematical term, so Steinhaus, a little surprised, started to talk with two young men, who were speaking about the Lebesgue measure. These two men were Banach and Otto Nikodym. Steinhaus told them about the problem of the average convergence of Fourier series on which he was currently working, and a few days later Banach visited Steinhaus and presented him a correct solution, cf. [43].

Steinhaus realised that Banach has an incredible mathematical talent. Steinahus was just moving to Lvov, where he got a Chair. He offered Banach a position at the Technical University. Thus Banach started his academic career and teaching students. It is notable that Banach did not graduate from any university. Steinhaus used to say later that the discovery of Banach was his greatest mathematical discovery.

In 1920 Banach received a doctorate from Jan Kazimierz University at Lvov and in 1922 Banach, aged 30, was appointed Professor at this university. After the First World War Poland got back its independence and Kraków and Lvov were again in Poland.

In fact, Banach was interested in nothing but mathematics. He wrote down only a small part of his results. He was speaking about mathematics, introducing new ideas, solving problems all the time. Andrzej Turowicz, who knew Banach very well, used to say that two mathematicians should have followed Banach all the time and written down everything he said. So, only some of his discoveries were published; nevertheless, they present themselves an enormous collection. Several of them were already presented in Section 1. However, here we say more about one of the most important mathematical notion, i.e. Banach spaces.

It was checked what names appeared most frequently in the titles of mathematical and physical papers published in the 20th century. It turned out that it was Banach's name that got the first position. The second position was obtained by Sophus Lie, the third by Bernhard Riemann. Of course, Banach deserves this position mainly because of Banach spaces that appear in many titles of papers. A Banach space is *a normed vector complete space*. Normed spaces and Banach spaces (spaces of type (B)) were formally defined in the paper [4].

There are some points which show why the introduction of Banach spaces was so important. For a variety of reasons function spaces are very useful in many investigations and applications. To a large extent, modern mathematics is concerned with the study of general structures. The essential element is finding the right generalization. Insufficient generality can be too restrictive and a great deal of generality may give rise to a situation, where little material can be proved or applied. The space introduced by Banach, especially pointing out completeness, attests to his genius; he hit the traditional nail on the head.

Banach's great merit was that, in principle, it was thanks to him that the "geometric" way of looking at spaces was initiated. The elements of some general spaces might be functions or number sequences, but when fitted into the structure of a Banach space they were regarded as "points", as the elements of a "space". At times this resulted in remarkable simplification. Let us quote here the famous sentence said by Banach in one of his speeches (cf. [32]: "A mathematician is a person who can find analogies between theorems, a better mathematician is one who can see analogies between proofs and the best mathematician can notice analogies between theories; and one imagine that the ultimate mathematician is one who can see analogies between analogies."

Let us recall here the Banach words from [4], following [33]. "The aim of the present work is to establish certain theorems valid in different functional domains . . . Nevertheless, in order not to have to prove them for each particular domain, which would be painful, I have chosen to take a different route; that is, I will consider in a general sense the sets of elements of which I will postulate certain properties. I will deduce from them certain theorems and then I will prove for each specific functional domain that the chosen postulates are true"

Today the notion of a Banach space remains fundamental in many areas of mathematics. The theory of Banach spaces is being continuously developed and new, interesting and occasionally surprising results are obtained by many researches. In particular, some really important results were obtained in the end of the 20th century by William Timothy Gowers. Some problems that he solved waited for the solution since Banach's times. For his research, Gowers was awarded in 1998 with Fields medal.

The period since the end of the First World War up to 1939 was the gold age for Polish mathematics. Functional analysis at that period could be named "the Polish branch of mathematics". In Lvov, together with Banach, there worked many outstanding mathematicians, mentioning for example H. Steinhaus, S. Mazur, J. Schauder, W. Orlicz, S. Ulam and M. Kac (the last two moved to the USA before the war).

In 1939 Banach was elected as President of the Polish Mathematical Society. In the same year Lvov was captured by the Soviet Union, in 1941 Hitler's soldiers took Lvov for 4 years. Banach spent the Second World War in Lvov, living under extremely difficult conditions. After the war Lvov was taken by the Soviet Union again and Banach planned to go to Kraków, where he would have taken a Chair at the Jagiellonian University. Banach died in 1945 of lung cancer, just a few days before the removal. Banach's grave is in the Lychakov Cemetery in Lvov (cf. [15]).

An important role in mathematicians' life in Lvov and the mathematical results (in particular, functional analysis) was played by the Scottish Café. It was a place of their meetings, where they were eating, drinking, speaking about mathematics, stating problems and solving them. Ulam [46] said "It was difficult to outlast or outdrink Banach during these sessions. We discussed problems proposed right there, often with no solution evident even after several hours of thinking. The next day Banach was likely to appear with several small sheets of paper containing outlines of proofs he had completed."

They used to write solutions on marble tables in the café. However, after each such visit the tables were carefully cleaned by the staff. It is said that some difficult proofs of important theorems disappeared in this way. Therefore after some time Banach's wife, Łucja, bought a special book (called later the Scottish Book) that was always kept by the waiters and given to a mathematician when ordered. The problems, solutions, and rewards were written down in the book. After the war, the Scottish Book was taken to Poland by Łucja Banach and later on translated to English by Steinhaus. Ulam spread out the problems from the book in the United States. In 1981 the book was published in English in the version prepared by Dan Mauldin ([36]). This translation is remarkably valuable as except of the problems and solutions. It includes several comments and remarks about the continuation of the investigations inspired by the problems from the Scottish Book. It is a large and important piece of mathematics.

Let us mention here one problem from the book that became particularly famous because of the reward. On 6 November 1936 Stanisław Mazur stated the following problem (in the Scottish Book the problem had the number 153).

Problem 7. Assume that a continuous function on the square $[0,1]^2$ and the number $\varepsilon > 0$ are given. Do there exist numbers $a_1, \ldots, a_n$, $b_1, \ldots, b_n$, $c_1, \ldots, c_n$, such that

$$\left| f(x,y) - \sum_{k=1}^{n} c_k f(a_k, y) f(x, b_k) \right| \leq \varepsilon$$

for any $x, y \in [0,1]$?

Now it is said that the problem was about the existence of Schauder basis in arbitrary separable Banach space. However, it was not known at that time. It was only in 1955 when Alexandre Grothendieck showed ([25]) that the existence of such numbers is equivalent to so called "the approximation problem", i.e. the problem if *every compact linear operator T from a Banach space X into a Banach space Y is a limit in norm of operators of finite rank*. The problem was especially attractive as Mazur offered a prize: a live goose. The approximation problem (and, consequently, the original Mazur's problem) was solved only in 1972 by a Swedish mathematician Per Enflo (then 28 years old) ([23]), who shortly after giving a solution came to Warsaw and got from Mazur the prize.

There are still two significant theorems named after him:

The Banach Fixed Point Theorem, which reads as follows.

Theorem 8 (Banach Fixed Point Theorem)**.** *Suppose that (X, d) is a complete metric space and $f : X \to X$ be a contraction, i.e., there is a nonnegative real number $c < 1$ such that $d(f(x), f(y)) \leq c\, d(x, y)$ for all $x, y \in X$. Then f admits a unique fixed point in X, i.e. a point $x_0 \in X$ such that $f(x_0) = x_0$.*

This theorem was published in [4].

The Banach–Tarski paradox appeared in 1924 in Fundamenta Mathematicae [9]:

Theorem 9 (Banach–Tarski Paradox)**.** *Let X and Y are bounded subsets of the Euclidean space $\mathbb{R}^3$ having nonempty interiors. Then there exist a natural number n and partitions $\{X_i : 1 \le i \le n\}$ and $\{Y_i : 1 \le i \le n\}$ of X and Y, respectively, such that for each $1 \le i \le n$, the sets X_i and Y_i are congruent.*

Roughly speaking, the paradox says that a ball in the ordinary Euclidean space can be divided up into two balls each identical to the first.

For more details about Banach, the reader is referred particularly to [10, 12, 14, 13, 16, 19, 18, 30, 32].

4. The early years

The wonderful unification and the creation of general spaces would not be possible without the significant work of many mathematicians for many years. Indeed, several examples must have been considered, several spaces and functions investigated, several results proved in particular cases to come to the general theory. Without that, probably nobody would take care of abstract spaces.

The preparation for the general theory may take many years. That was also in the case of functional analysis.

The full and perfect analysis of those results can be found in [17]. In particular, in [17] there is a precise description of mathematical results mentioned in a sequel. Here, we present a very brief selection of them.

The fundamental basis for functional analysis was given in the beginning of the XXth century. Then, four mathematicians obtained the results that made a great influence for the future.

The first one was a Swedish mathematician Ivar Fredholm (1866–1927). In the end of the XIXth century he presented his first results on integral equations and published them in 1900. The results were completed a couple of years later and published in [24]. Here, Fredholm considered some general equations

$$f(s) = g(s) + \lambda \int_a^b K(s,t) g(t) dt \, ,$$

where K is bounded and piecewise continuous in $[a,b]^2$ and g is continuous in $[a,b]$. They were later given (by Hilbert) the name *integral equations of the second kind* (or *Fredholm integral equations of the second kind*).

Fredholm reduced the problem of solving those equations to the solution of linear algebraic equation.

The second mathematician was a Frenchman Henri Lebesgue (1875–1941). In 1902 he wrote his thesis about integration. The importance of Lebesgue integral for mathematics is by all means obvious.

Between 1904 and 1906 a German mathematician David Hilbert (1862–1943) published six papers on integral equations. Later they were collected together in a single volume [28]. Some further papers were published in the next years. In these papers the assumption that the square of the function is integrable was of fundamental meaning. Also, the functions and sequences were investigated there as the points of space. This point of view was crucial to further research and gave the beginning for Hilbert spaces. One can notice that here we have the official birth of spaces $\mathcal{L}^2$ and l^2. The axiomatic definition of the concept of Hilbert space was given by John von Neumann (1903–1957).

The fourth mathematician, who made a great contribution here was Maurice Fréchet (1878–1973) from France. He presented the form of a linear functional in an L^2-space. In his thesis in 1906 he also defined metric and abstract metric spaces. Independently, metric was defined by Felix Hausdorff in 1914.

These results led to the introducing the spaces $\mathcal{L}^p$ and l^p for $p \in (1, +\infty)$. This was done in 1910–1913 by a Hungarian mathematician Frigyes Riesz (1880–1956). Riesz discovered the natural duality between the different spaces $\mathcal{L}^p$ and $\mathcal{L}^q$, where $p + q = 1$. Riesz's research gave the next impulse to the creation of general theory.

It must be stressed, that also the results mentioned above "had to be prepared". It is impossible to point out all important researches giving the base to those results. However, some names of mathematicians, who obtained crucial results in partial differential equations should be mentioned, like Dirichlet, Laplace, Schwarz (a paper on the theory of minimal surfaces and a particular equation, where as the main tool he used an inequality now named by him) and Poincaré. Also, some contributions of Fourier were very important.

It should also be noted that a very important (in fact, one could say necessary) element for the creation of functional analysis was the suitable preparation from linear algebra and topology. Linear algebra developed in the second half of the XIXth century, topology developed rapidly in the beginning of the XXth century, especially after WWII. In the twenties they gave the sufficient material to functional analysis.

Ending this chapter, it would by useful to quote the opinion of Jean Dieudonné from [17]. He says: "If one were to reduce this complicated history to a few key words, I think the emphasis should fall on the evolution of two concepts: *spectral theory* and *duality*. Both of course stem from very concrete problems encountered in the solution of linear equations (or systems of linear equations), where the unknown are *functions*.

Many additional information may be found in [17, 20, 33, 39, 44, 40].

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