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MÖBIUS MODULUS OF RING DOMAINS IN $\overline{\mathbf{R}}^n$

Zair Ibragimov

University of Michigan, Department of Mathematics
Ann Arbor, MI 48109, U.S.A.; ibragim@umich.edu

Abstract. We introduce a new Möbius invariant modulus for ring domains R in $\overline{\mathbf{R}}^n$ which coincides with the usual modulus whenever R is a Möbius annulus, i.e.,

$$f(R) = \{x \in \mathbf{R}^n : 1 < |x| < t\}$$

for some Möbius transformation f of $\overline{\mathbf{R}}^n$ and some $t > 1$. We obtain a sharp upper bound for the Möbius modulus of a ring R which separates two pairs $\{a, b\}$ and $\{c, d\}$ of distinct points in $\overline{\mathbf{R}}^n$. Our result proves a conjecture made by M. Vuorinen in 1992 [14].

1. Introduction

Notation. We denote by $\mathbf{R}^n$ the n-dimensional Euclidean space and by $\{e_1, e_2, \dots, e_n\}$ its standard basis. The one-point compactification $\mathbf{R}^n \cup \{\infty\}$ of $\mathbf{R}^n$ is denoted by $\overline{\mathbf{R}}^n$. The open and closed balls of radius $r > 0$ and centered at $x \in \mathbf{R}^n$ are denoted by $B^n(x, r)$ and $\overline{B}^n(x, r)$, respectively. $S^{n-1}(x, r)$ is a sphere of radius $r > 0$ and centered at $x \in \mathbf{R}^n$. The closed segment between $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^n$ is denoted by $[x, y]$. For $x \in \mathbf{R}^n$, $x \neq 0$, we set

$$[x, \infty] = \{tx : t \geq 1\} \cup \{\infty\}.$$

The group of all Möbius transformations of $\overline{\mathbf{R}}^n$ is denoted by $\text{Möb}(\overline{\mathbf{R}}^n)$.

A *ring* is a domain $R \subset \overline{\mathbf{R}}^n$ whose complement is the union of two disjoint non-degenerate compact connected sets. A ring with complementary components C_1 and C_2 is denoted by $R(C_1, C_2)$. A ring $R(C_1, C_2)$ is said to *separate* the sets E and F if $E \subset C_1$ and $F \subset C_2$. Hence a ring $R(C_1, C_2)$ separates the complementary components of a ring $R(E, F)$ if $E \subset C_1$ and $F \subset C_2$.

If R is a ring in $\overline{\mathbf{R}}^2$, then R can be mapped conformally onto a circular annulus

$$\{z \in \mathbf{C} : 1 < |z| < t\}$$

and the *modulus* of R is defined to be $\log t$.

In $\overline{\mathbf{R}}^n$, $n > 2$, by Liouville's theorem the Möbius transformations are the only conformal mappings in $\overline{\mathbf{R}}^n$. A ring R is said to be a *Möbius annulus* if

$$f(R) = \{x \in \mathbf{R}^n : 1 < |x| < t\} \quad \text{for some } f \in \text{Möb}(\overline{\mathbf{R}}^n) \text{ and } t > 1.$$

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The points $f^{-1}(0)$ and $f^{-1}(\infty)$ are called *relative centers* of the Möbius annulus R. The modulus of such a ring R is defined as $\log t$.

In general, the modulus of a ring $R = R(C_1, C_2)$ is defined as follows. Let Γ be the family of all curves joining C_1 and C_2 in R and let $F(\Gamma)$ be the set of all non-negative Borel functions

$$\varrho\colon \mathbf{R}^n \to \overline{\mathbf{R}}^1 \qquad \text{such that } \int_\gamma \varrho\, ds \ge 1$$

for every locally rectifiable curve $\gamma \in \Gamma$. Then the modulus of the curve family Γ is defined as

$$M(\Gamma) = \inf_{\varrho \in F(\Gamma)} \int_{\mathbf{R}^n} \varrho^n \, dm.$$

Observe that since C_1 and C_2 are non-degenerate, $0 < M(\Gamma) < \infty$ by [13, 11.5 and 11.10]. The modulus of R is defined as

$$\operatorname{mod} R = \left[\frac{\omega_{n-1}}{M(\Gamma)} \right]^{1/(n-1)}.$$

See, for instance, [1, 8.30]. Here ω_{n-1} is the surface area of the unit sphere in $\mathbf{R}^n$.

The rings of Grötzsch and Teichmüller play an important role in the theory of quasiconformal mappings. The complementary components of Grötzsch ring $R_G(s)$, $s > 1$, are $\overline{B}^n(0,1)$ and $[se_1, \infty]$ while those of Teichmüller ring $R_T(t)$, $t > 0$, are $[-e_1, 0]$ and $[te_1, \infty]$.

For the convenience of the reader we recall some properties of the modulus of Grötzsch and Teichmüller rings. The following functional relation holds.

$$\operatorname{mod} R_T(t) = 2 \operatorname{mod} R_G\big(\sqrt{t+1}\,\big). \tag{1.1}$$

See [4] and [1, 8.32 and 8.37(1)]. The function

$$\operatorname{mod} R_T(t) - \log(t+1) \tag{1.2}$$

is a nondecreasing function in $(0, \infty)$ and

$$\lim_{t \to \infty} \big(\operatorname{mod} R_T(t) - \log(t+1) \big) = \log \lambda_n^2 < \infty. \tag{1.3}$$

Here λ_n is a constant which depends only on n. See, for instance, [4] and [1, 8.38].

Our main focus in this paper is the following extremal problem of Teichmüller. Let a, b, c, d be distinct points in $\overline{\mathbf{R}}^n$. Among all the rings which separate the sets $\{a, b\}$ and $\{c, d\}$ it is required to find one with the largest modulus.

For $n = 2$, this problem was considered by O. Teichmüller [12] in 1938 and a complete solution was given by M. Schiffer [11] in 1946. In this case the points

a, b, c, d can be normalized so that $a=-1$, $b=1$, $c=\xi$, $d=-\xi$, where $\xi\in\overline{B}^2(0,1)$ is a unique point with

$$\frac{c-a}{c-b}\cdot\frac{d-b}{d-a}=\left(\frac{\xi+1}{\xi-1}\right)^2.$$

If $\mathscr{R}(\xi)=R\big(C_1(\xi),C_2(\xi)\big)$ is an extremal ring, i.e., a ring with the largest modulus, then

$$h_1\big(C_1(\xi)\big)=C_1(\xi),\qquad h_1\big(C_2(\xi)\big)=C_2(\xi)$$

and

$$h_2\big(C_1(\xi)\big)=C_2(\xi),\qquad h_2\big(C_2(\xi)\big)=C_1(\xi),$$

where

$$h_1(z)=-z\qquad\text{and}\qquad h_2(z)=\frac{\xi}{z}.$$

In particular,

$$0\in C_1(\xi)\qquad\text{and}\qquad\infty\in C_2(\xi). \tag{1.4}$$

See [8, pp. 199–200].

For $n>2$, Teichmüller's problem is solved only when the points a, b, c, d lie on a circle or a line in this order. In this case the points a, b, c, d can be normalized so that

$$a=-e_1,\ b=0,\ c=te_1,\ d=\infty,\qquad\text{where } t=\frac{|b-c|\,|d-a|}{|b-a|\,|d-c|}.$$

Then by means of a spherical symmetrization one shows that Teichmüller's ring $R_T(t)$ is an extremal ring. See, for instance, [4], [10] and [1, Theorem 8.46].

When the points a, b, c, d do not lie on a circle or a line in this order, the problem is still open. In general the points a, b, c, d can be normalized so that

$$a=0,\ b=e_1,\ c=x,\ d=\infty\quad\text{where } x\in\mathbf{R}^n\quad\text{and}\quad |x|=\frac{|a-c|\,|b-d|}{|a-b|\,|c-d|}.$$

M. Vuorinen has considered a ring R_0 whose complementary components are some circular arc joining the points 0 and e_1 and some ray emanating from the point x [14]. This ring coincides (up to a Möbius transformation) with Teichmüller's ring when the points a, b, c, d lie on a circle or a line in this order. It follows from Theorem 3.16 [14] that there exists a Möbius annulus A separating the complementary components of R_0 and such that

$$\operatorname{mod}A=\operatorname{arccosh}\left(|x|+|x-e_1|\right).$$

Due to the monotonicity of the modulus we then have

$$\operatorname{arccosh}\left(|x|+|x-e_1|\right)=\operatorname{mod}A\le\operatorname{mod}R_0.$$

In order to assure that $\operatorname{arccosh}\left(|x|+|x-e_1|\right)$ is the sharpest lower bound for $\operatorname{mod}R_0$ obtained in this manner, Vuorinen was led to the following conjecture.

Conjecture 1.5. *For* $x \in \mathbf{R}^n \setminus [0, e_1]$,

$$\max_A \operatorname{mod} A = \operatorname{arccosh}(|x| + |x - e_1|),$$

where the maximum is taken over all Möbius annuli A *which separate the sets* $\{0, e_1\}$ *and* $\{x, \infty\}$.

In this paper we consider a new measure for ring domains called the Möbius modulus. Our main result, Theorem 3.8 below, shows that Teichmüller's problem has a complete solution when considered with respect to the Möbius modulus. As a corollary to Theorem 3.8 we settle the conjecture of Vuorinen. After this paper was submitted the referee pointed out that an alternative proof of the conjecture was also given in [3].

2. Some results on the cross-ratio

The main result of this section is Theorem 2.16. The *cross-ratio* of a quadruple a, b, c, d of points in $\overline{\mathbf{R}}^n$ with $a \neq b$ and $c \neq d$ is defined as follows. If $a, b, c, d \in \mathbf{R}^n$, then

$$|a, b, c, d| = \frac{|a - c|\,|b - d|}{|a - b|\,|c - d|}. \tag{2.1}$$

Otherwise we omit the terms containing ∞. For example,

$$|a, b, c, \infty| = \frac{|a - c|}{|a - b|}. \tag{2.2}$$

A homeomorphism $f\colon \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ belongs to $\text{Möb}(\overline{\mathbf{R}}^n)$ if and only if

$$|f(a), f(b), f(c), f(d)| = |a, b, c, d| \tag{2.3}$$

for all quadruples a, b, c, d in $\overline{\mathbf{R}}^n$. See [2, Theorem 3.2.7]. For a quadruple a, b, c, d in $\overline{\mathbf{R}}^n$ we put

$$\sigma(a, b, c, d) = |a, b, c, d| + |b, a, c, d|. \tag{2.4}$$

Hence

$$\sigma(a, b, c, d) = \frac{|a - c|\,|b - d| + |a - d|\,|b - c|}{|a - b|\,|c - d|} \geq 1 \tag{2.5}$$

with equality if and only if the points a, c, b, d lie on a circle or a line in this order. A simple computation shows that

$$|a, b, c, d| = \frac{\sigma(a, b, c, d) + 1}{\sigma(a, c, b, d) + 1} \qquad \text{and} \qquad \sigma(a, b, c, d) = \frac{|a, d, c, b| + 1}{|d, a, c, b|}. \tag{2.6}$$

It follows from (2.3) and (2.6) that a homeomorphism $f\colon \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ belongs $\text{Möb}(\overline{\mathbf{R}}^n)$ if and only if

$$\sigma\big(f(a), f(b), f(c), f(d)\big) = \sigma(a, b, c, d) \tag{2.7}$$

for all quadruples a, b, c, d in $\overline{\mathbf{R}}^n$.

The following two lemmas are used in the proof of Theorem 2.16. The first lemma is an immediate consequence of Corollary 7.25 [1].

Lemma 2.8. *Let* $a,b,c,d \in \overline{\mathbf{R}}^n$ *be distinct points. Let* f *be a Möbius transformation such that*

$$f(a) = -e_1,\ f(b) = e_1,\ f(c) = -w \quad \text{and} \quad f(d) = w$$

for some $w \in \mathbf{R}^n$ *with* $|w| \geq 1$. *Then*

$$|w| = p + \sqrt{p^2-1}\,, \qquad \text{where } p = \sigma(a,b,c,d).$$

Proof. Let $g \in \mathrm{Möb}(\overline{\mathbf{R}}^n)$ be the inversion in $S^{n-1}(0,1)$. Then by applying Corollary 7.25 [1] to the composition $g \circ f$ and using (2.6) we obtain

$$\frac{1}{|w|} = \frac{|d,a,c,b|}{1+|a,d,c,b|+\sqrt{(1+|a,d,c,b|)^2-(|d,a,c,b|)^2}} = \frac{1}{p+\sqrt{p^2-1}}$$

as required. □

The next lemma is an extension of a special case of Lemma 2.12 [6].

Lemma 2.9. *If* a_1, a_2, a_3, a_4 *are distinct points in* $\mathbf{R}^n$ *with*

$$|a_1| = |a_2| = s < t = |a_3| = |a_4|, \tag{2.10}$$

then

$$\frac{s^2+t^2}{2st} \leq \sigma(a_1,a_2,a_3,a_4). \tag{2.11}$$

Equality holds if and only if

$$a_1+a_2 = a_3+a_4 = 0. \tag{2.12}$$

Proof. Define c_{ij} as the cosine of the angle $\angle(a_i 0 a_j)$ and set

$$g(u) = \frac{N(u)}{D}, \qquad \text{where } D = (1-c_{12})^{1/2}(1-c_{34})^{1/2}$$

and

$$N(u) = (u-c_{13})^{1/2}(u-c_{24})^{1/2} + (u-c_{14})^{1/2}(u-c_{23})^{1/2}.$$

Note that $D \leq 2$ and that $D = 2$ if and only if (2.12) holds. Next

$$\frac{d}{du}\big((u-c_{ij})^{1/2}(u-c_{kl})^{1/2}\big) = \frac{1}{2}\left(\frac{(u-c_{kl})^{1/2}}{(u-c_{ij})^{1/2}} + \frac{(u-c_{ij})^{1/2}}{(u-c_{kl})^{1/2}}\right) \geq 1,$$

whence

$$g'(u) = \frac{N'(u)}{D} \geq 1 \qquad \text{for } u > 1. \tag{2.13}$$

Since

$$g(1) = \sigma\left(a_1, a_2, \frac{s}{t}a_3, \frac{s}{t}a_4\right) \geq 1,$$

we have

$$g(u) = \int_1^u g'(r)\,dr + g(1) \geq u \qquad \text{for all } u \geq 1. \tag{2.14}$$

In particular,

$$\sigma(a_1, a_2, a_3, a_4) = g\left(\frac{s^2+t^2}{2st}\right) \geq \frac{s^2+t^2}{2st} \tag{2.15}$$

which completes the proof of (2.11).

Assume next that (2.11) holds with equality and set

$$v = \frac{s^2+t^2}{2st}.$$

Then (2.15) implies that $g(v) = v$. Using the differentiability of g along with (2.14) we obtain

$$g'(v) = \lim_{u \to v} \frac{g(v) - g(u)}{v - u} \leq \lim_{u \to v} \frac{v-u}{v-u} = 1$$

and hence using (2.13) we conclude that $g'(v) = 1$. Since $N'(v) \geq 2$ and $D \leq 2$, the equality $g'(v) = 1$ implies $D = 2$ which, as noted above, implies (2.12) as required.

Finally, a simple computation shows that (2.12) implies that (2.11) holds with equality. □

Theorem 2.16. *Let $a, b, c, d \in \overline{\mathbf{R}}^n$ be distinct points and $p = \sigma(a,b,c,d)$. Then for all distinct pairs $\{u, v\}$ of points in $\overline{\mathbf{R}}^n \setminus \{a,b,c,d\}$*

$$\min\big\{|u,a,c,v|, |u,a,d,v|, |u,b,c,v|, |u,b,d,v|\big\} \leq p + \sqrt{p^2-1}\,. \tag{2.17}$$

Moreover, if $p > 1$, then there exists a unique pair $\{u, v\}$ for which the equality holds.

Proof. Let $u, v \in \overline{\mathbf{R}}^n \setminus \{a,b,c,d\}$ be distinct points. Since (2.17) is invariant under the elements of $\text{Möb}(\overline{\mathbf{R}}^n)$, we can assume that $u = 0$ and $v = \infty$. Then

$$\min\big\{|u,a,c,v|, |u,a,d,v|, |u,b,c,v|, |u,b,d,v|\big\} = \frac{\min\{|c|,|d|\}}{\max\{|a|,|b|\}} = \frac{t}{s}.$$

Since $p + \sqrt{p^2-1} \geq 1$, there is nothing to prove if $s \geq t$. Hence we may assume that $s < t$. Then (2.17) is equivalent to

$$\frac{s^2+t^2}{2st} \leq \sigma(a,b,c,d). \tag{2.18}$$

Let

$$S^{n-1}(v_1,r_1) \subset \overline{B}^n(0,s) \quad \text{and} \quad S^{n-1}(v_2,r_2) \subset \overline{\mathbf{R}}^n \setminus B^n(0,t)$$

be spheres such that $a, b \in S^{n-1}(v_1,r_1)$ and $c, d \in S^{n-1}(v_2,r_2)$. Then we have $R_1 \subset R_2$, where

$$R_1 = R\big(\overline{B}^n(0,s), \overline{\mathbf{R}}^n \setminus B^n(0,t)\big)$$

and

$$R_2 = R\big(\overline{B}^n(v_1,r_1), \overline{\mathbf{R}}^n \setminus B^n(v_2,r_2)\big).$$

In particular,

$$\operatorname{mod} R_1 \leq \operatorname{mod} R_2. \tag{2.19}$$

Choose $h \in \text{Möb}(\overline{\mathbf{R}}^n)$ that maps $S^{n-1}(v_1,r_1)$ and $S^{n-1}(v_2,r_2)$ onto concentric spheres $S^{n-1}(0,s')$ and $S^{n-1}(0,t')$, respectively. Then

$$0 < |h(a)| = |h(b)| = s' < t' = |h(c)| = |h(d)|$$

and we have

$$\log\frac{t}{s} = \operatorname{mod} R_1 \leq \operatorname{mod} R_2 = \log\frac{t'}{s'}. \tag{2.20}$$

Using (2.11) we now have

$$\frac{s^2+t^2}{2st} \leq \frac{{s'}^2+{t'}^2}{2s't'} \leq \sigma\big(h(a),h(b),h(c),h(d)\big) = \sigma(a,b,c,d)$$

which proves (2.18) and hence the first part of the theorem.

Notice that equality in (2.18) implies that $|a| = |b| = s$ and $|c| = |d| = t$.

To prove the second part of the theorem, we let $f \in \text{Möb}(\overline{\mathbf{R}}^n)$ be such that

$$f(a) = -e_1, \quad f(b) = e_1, \quad f(c) = -w \quad \text{and} \quad f(d) = w$$

for some $w \in \mathbf{R}^n$ with $|w| \geq 1$. Since $p > 1$, we have $|w| = p + \sqrt{p^2 - 1} > 1$ by Lemma 2.8. Then the equality in (2.17) holds if we take $u = f^{-1}(0)$ and $v = f^{-1}(\infty)$. This establishes the existence of the pair $\{u, v\}$.

To prove the uniqueness, we assume that $\{u_1, v_1\}$ is another pair for which the equality in (2.17) holds and show that $u_1 = u$ and $v_1 = v$. Let $g \in \text{Möb}(\overline{\mathbf{R}}^n)$ be such that

$$g(u_1) = 0, \quad g(v_1) = \infty \quad \text{and} \quad g(a) = -e_1.$$

By our assumption we have

$$\min\{|0, -e_1, g(c), \infty|, |0, -e_1, g(d), \infty|, |0, g(b), g(c), \infty|, |0, g(b), g(d), \infty|\}$$
$$= \frac{\min\{|g(c)|, |g(d)|\}}{\max\{|g(a), |g(b)|\}} = p + \sqrt{p^2 - 1}.$$

Hence

$$|g(a)| = |g(b)| < |g(c)| = |g(d)|$$

as we have noted above. Then using (2.12) we have

$$g(a) + g(b) = g(c) + g(d) = 0.$$

Hence

$$g(a) = -e_1, \quad g(b) = e_1, \quad g(c) = -z \quad \text{and} \quad g(d) = z$$

for some $z \in \mathbf{R}^n$ and by Lemma 2.8 we have $|z| = p + \sqrt{p^2 - 1} = |w|$. Since

$$|f(a), f(b), f(c), f(d)| = |a, b, c, d| = |g(a), g(b), g(c), g(d)|$$

and

$$|f(b), f(a), f(c), f(d)| = |b, a, c, d| = |g(b), g(a), g(c), g(d)|,$$

we have

$$|w - e_1| = |z - e_1| \qquad \text{and} \qquad |w + e_1| = |z + e_1|.$$

Hence the angle $\angle(w0e_1)$ is equal to the angle $\angle(z0e_1)$. By means of a preliminary rotation about the e_1-axis if necessary, we can assume that $w = z$. Hence f^{-1} and g^{-1} agree on a set $\{-e_1, e_1, -w, w\}$ and consequently they agree on a 2-dimensional (1-dimensional, if w lies on the e_1-axis) subspace of $\mathbf{R}^n$ containing these points. In particular,

$$u = f^{-1}(0) = g^{-1}(0) = u_1 \qquad \text{and} \qquad v = f^{-1}(\infty) = g^{-1}(\infty) = v_1$$

as required. □

Remark 2.21. The hypothesis $p > 1$ in the uniqueness part of Theorem 2.16 cannot be removed. We now show that if $p = 1$ the uniqueness part fails.

Indeed, if $a = -e_1$, $b = te_1$, $c = -te_1$, $d = e_1$ for some $0 < t < 1$, then $\sigma(a, b, c, d) = 1$. But equality in (2.17) holds for all points $\{u, v\}$ with

$$|u - a| = |v - a|, \quad |u - b| = |v - b| \qquad \text{and} \qquad |u - c| = |v - c|, \quad |u - d| = |v - d|.$$

3. Möbius modulus of ring domains

In this section we define the Möbius modulus of rings. Our main result is Theorem 3.8 which gives a solution to Teichmüller's extremal problem for the Möbius modulus. As a corollary to Theorem 3.8 we settle the conjecture of Vuorinen. We will then establish a relationship between the usual and the Möbius moduli of rings and compute the Möbius modulus of Grötzsch and Teichmüller rings.

Definition 3.1. Let $R = R(C_1, C_2)$ be a ring in $\overline{\mathbf{R}}^n$. The quantity

$$\operatorname{mod}_M R = \max_{u,v\in\overline{\mathbf{R}}^n} \min_{x\in C_1,\, y\in C_2} \left| \log \frac{|u-y|\,|x-v|}{|u-x|\,|y-v|} \right| \tag{3.2}$$

is called the *Möbius modulus* of R.

Observe that if R is any ring and A is a Möbius annulus separating the complementary components of R, then

$$\operatorname{mod}_M R \geq \operatorname{mod} A > 0. \tag{3.3}$$

Indeed, we can assume that

$$A = B^n(0,t) \setminus \overline{B}^n(0,1)$$

for some $t > 1$. Then

$$\operatorname{mod}_M R \geq \min_{x\in C_1,\, y\in C_2} \left| \log \frac{|0-y|\,|x-\infty|}{|0-x|\,|y-\infty|} \right| = \min_{x\in C_1,\, y\in C_2} \left| \log \frac{|y|}{|x|} \right| \geq \operatorname{mod} A.$$

On the other hand, if $\operatorname{mod}_M R > 0$, then there exists a Möbius annulus A separating the complementary components of R such that

$$\operatorname{mod} A = \operatorname{mod}_M R. \tag{3.4}$$

Indeed, let $u, v \in \overline{\mathbf{R}}^n$ be a pair with

$$\operatorname{mod}_M R = \min_{x\in C_1,\, y\in C_2} \left| \log \frac{|u-y|\,|x-v|}{|u-x|\,|y-v|} \right| > 0.$$

We can assume that $u = 0$ and $v = \infty$. Then

$$\operatorname{mod}_M R = \min_{x\in C_1,\, y\in C_2} \left| \log \frac{|y|}{|x|} \right| = \log \frac{s}{t},$$

where

$$s = \min\{|x| : x \in C_2\} \qquad \text{and} \qquad t = \max\{|x| : x \in C_1\}.$$

Then the ring

$$A = \{x \in \mathbf{R}^n : t < |x| < s\}$$

is the required Möbius annulus.

Thus we have the following remark to Definition 3.1.

Remark 3.5. Let R be any ring with $\operatorname{mod}_M R > 0$. Then

$$\operatorname{mod}_M R = \max_A \operatorname{mod} A, \tag{3.6}$$

where the maximum is taken over all Möbius annuli A which separate the complementary components of R. In particular, if R is a Möbius annulus, then

$$\operatorname{mod}_M R = \operatorname{mod} R. \tag{3.7}$$

Ring domains with separating euclidean or Möbius annuli were studied in [7] ($n = 2$) and [14] ($n \geq 2$), respectively.

Theorem 3.8. *Let $a, b, c, d \in \overline{\mathbf{R}}^n$ be distinct points. Then*

$$\max_R \operatorname{mod}_M R = \operatorname{arccosh}\big(\sigma(a, b, c, d)\big), \tag{3.9}$$

where the maximum is taken over all rings R which separate the sets $\{a, b\}$ and $\{c, d\}$.

Proof. Let $R = R(C_1, C_2)$ be a ring with $a, b \in C_1$ and $c, d \in C_2$ and assume that $\operatorname{mod}_M R > 0$. Let $u, v \in \overline{\mathbf{R}}^n$ and $x_0 \in C_1$, $y_0 \in C_2$ be such that

$$\operatorname{mod}_M R = \big|\log |u, x_0, y_0, v|\big|.$$

By performing a preliminary Möbius transformation we can assume that $u = 0$, $v = \infty$ and $|x_0| < |y_0|$. Then the Möbius annulus

$$A = B^n(0, |y_0|) \setminus \overline{B}^n(0, |x_0|)$$

separates C_1 and C_2. In particular, we have

$$\max\{|a|, |b|\} \leq |x_0| < |y_0| \leq \min\{|c|, |d|\}.$$

Hence by Theorem 2.16 we have

$$\operatorname{mod}_M R = \log\frac{|y_0|}{|x_0|} \leq \log\left(\min\left\{\frac{|c|}{|a|}, \frac{|d|}{|a|}, \frac{|c|}{|b|}, \frac{|d|}{|b|}\right\}\right) \leq \operatorname{arccosh}\big(\sigma(a, b, c, d)\big).$$

The equality holds for the ring $R_0 = R(C_1, C_2)$ where

$$C_1 = f^{-1}\big([-w, \infty] \cup [w, \infty]\big) \qquad \text{and} \qquad C_2 = f^{-1}\big([-e_1, e_1]\big)$$

and f is a Möbius transformation such that

$$f(a) = w,\ f(b) = -w, \quad f(c) = e_1 \quad \text{and} \quad f(d) = -e_1$$

for some $w \in \mathbf{R}^n$ with $|w| \geq 1$. Indeed, for $u = f^{-1}(\infty)$ and $v = f^{-1}(0)$ we have

$$\begin{aligned}\operatorname{mod}_M R_0 = \operatorname{mod}_M f(R_0) &\geq \min\big\{\big|\log|\infty, x, y, 0|\big| : x \in C_1,\ y \in C_2\big\}\\ &= \log|w| = \operatorname{arccosh}\big(\sigma(a, b, c, d)\big)\end{aligned}$$

using Lemma 2.8. Hence

$$\operatorname{mod}_M R_0 = \operatorname{arccosh}\big(\sigma(a, b, c, d)\big).$$

completing the proof. □

Next we have the following two corollaries of Theorem 3.8. The first one settles the conjecture of Vuorinen.

Corollary 3.10. *For* $x \in \mathbf{R}^n \setminus [0, e_1]$,

$$\max_A \operatorname{mod} A = \operatorname{arccosh}(|x| + |x - e_1|), \tag{3.11}$$

where the maximum is taken over all Möbius annuli A *which separate the sets* $\{0, e_1\}$ *and* $\{x, \infty\}$.

Proof. Since

$$|x| + |x - e_1| = \sigma(0, e_1, x, \infty),$$

Theorem 3.8 along with (3.6) imply that

$$\max_A \operatorname{mod} A = \operatorname{mod}_M R_0 = \operatorname{arccosh}\big(\sigma(0, e_1, x, \infty)\big) = \operatorname{arccosh}(|x| + |x - e_1|),$$

where $R_0 = R\big([0, e_1], [x, \infty]\big)$. □

The next corollary gives a characterization of Möbius transformations of $\overline{\mathbf{R}}^n$ in terms of the Möbius modulus of rings. A similar type of characterization in terms of the modulus of rings is given in [5].

Corollary 3.12. *A homeomorphism* $f\colon \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ *belongs to* $\text{Möb}(\overline{\mathbf{R}}^n)$ *if and only if*

$$\operatorname{mod}_M f(R) = \operatorname{mod}_M R$$

for all rings R *in* $\overline{\mathbf{R}}^n$.

Proof. The necessity part follows from (2.3) and Definition 3.1. For the sufficiency part it is enough to show that

$$\sigma\big(f(a), f(b), f(c), f(d)\big) = \sigma(a, b, c, d)$$

for all quadruples $a, b, c, d \in \overline{\mathbf{R}}^n$. Given a, b, c, d, we let $R = R(C_1, C_2)$ be a maximal ring, i.e., a ring with

$$\operatorname{mod}_M R = \operatorname{arccosh}\big(\sigma(a, b, c, d)\big).$$

Then using Theorem 3.8 and the fact that $f(a), f(b) \in f(C_1)$ and $f(c), f(d) \in f(C_2)$ we get

$$\begin{aligned}\operatorname{arccosh}\big(\sigma(a, b, c, d)\big) &= \operatorname{mod}_M R\big(f(C_1), f(C_2)\big)\\ &\le \operatorname{arccosh}\Big(\sigma\big(f(a), f(b), f(c), f(d)\big)\Big)\end{aligned}$$

which implies

$$\sigma(a, b, c, d) \le \sigma\big(f(a), f(b), f(c), f(d)\big).$$

By applying the same argument to f^{-1} we get

$$\sigma\big(f(a), f(b), f(c), f(d)\big) \le \sigma(a, b, c, d)$$

as required. □

We have the following relation between the modulus and the Möbius modulus of a ring R.

Lemma 3.13. *For any ring $R = R(C_1, C_2)$ in $\overline{\mathbf{R}}^n$ we have*

$$\operatorname{mod}_M R \le \operatorname{mod} R \le \operatorname{mod}_M R + c(n), \tag{3.14}$$

where $c(n)$ is a constant depending only on n.

Proof. It follows from (1.2) and (1.3) that

$$\operatorname{mod} R_T(t) \le \log\bigl(\lambda_n^2(t+1)\bigr) \qquad \text{for all } t > 0.$$

The first inequality in (3.14) follows from the monotonicity of the modulus along with Corollary 3.5. To show the second inequality in (3.14), let $\log|x, u, v, y| = \operatorname{mod}_M R$ and

$$\log|x', u', v', y'| = \max_{u\in C_1,\, v\in C_2} \min_{x\in C_1,\, y\in C_2} \left|\log \frac{|u-y|\,|x-v|}{|u-x|\,|y-v|}\right|.$$

Then $|x', u', v', y'| \le |x, u, v, y|$ and by Theorem 8.46 [1] we have

$$\begin{aligned}\operatorname{mod} R &\le \operatorname{mod} R_T(|x', u', v', y'|) \le \operatorname{mod} R_T(|x, u, v, y|)\\ &\le \log(2\lambda_n^2 |x, u, v, y|) = \operatorname{mod}_M R + \log(2\lambda_n^2).\end{aligned}$$

Hence the lemma holds with $c(n) = \log(2\lambda_n^2)$. □

Finally, we compute the Möbius modulus of Grötzsch and Teichmüller rings.

Example 3.15. For $s > 1$

$$\operatorname{mod}_M R_G(s) = \operatorname{arccosh}(s).$$

Proof. Since $-e_1, e_1 \in \overline{B}^n(0,1)$ and $se_1, \infty \in [se_1, \infty]$, Theorem 3.8 implies that

$$\operatorname{mod}_M R_G(s) \le \operatorname{arccosh}(s).$$

Equality holds for

$$u_0 = \bigl(s - \sqrt{s^2-1}\,\bigr)e_1 \qquad \text{and} \qquad v_0 = \bigl(s + \sqrt{s^2-1}\,\bigr)e_1.$$

See (3.2). □

Example 3.16. For $t > 0$

$$\operatorname{mod}_M R_T(t) = \operatorname{arccosh}(2t+1).$$

Proof. Since $-e_1, 0 \in [-e_1, 0]$ and $te_1, \infty \in [te_1, \infty]$, Theorem 3.8 implies that

$$\operatorname{mod}_M R_T(t) \le \operatorname{arccosh}(2t+1).$$

Equality holds for

$$u_0 = \big(-\sqrt{t(t+1)} + t\big)e_1 \qquad \text{and} \qquad v_0 = \big(\sqrt{t(t+1)} + t\big)e_1.$$

See (3.2). □

Remark 3.17. We have the same relation between the Möbius modulus of Grötzsch and Teichmüller rings as in (1.1), namely

$$\operatorname{mod}_M R_T(t) = 2\operatorname{mod}_M R_G\big(\sqrt{t+1}\,\big). \tag{3.18}$$

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