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MAPPINGS OF FINITE DISTORTION: COMPACTNESS

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Abstract. We study mappings $f\colon \Omega \to \mathbf{R}^n$ whose distortion functions $\mathscr{K}_l(x,f)$, $l = 1,2,\dots,n-1$, are in general unbounded but subexponentially integrable. The main result is the weak compactness principle. It asserts that a family of mappings with prescribed volume integral $\int_\Omega J(x,f)\,dx$, and with given subexponential norm $\|\sqrt[l]{\mathscr{K}_l}\,\|_{\mathrm{Exp}\mathscr{A}}$ of a distortion function, is closed under weak convergence. The novelty of this result is twofold. Firstly, it requires integral bounds on the distortions $\mathscr{K}_l(x,f)$ which are weaker than those for the usual outer distortion. Secondly, the category of subexponential bounds is optimal to fully describe the compactness principle for mappings of unbounded distortion, even when outer distortion is used.

1. Introduction

We consider here an interesting class of Sobolev mappings $f \in W^{1,1}_{\mathrm{loc}}(\Omega,\mathbf{R}^n)$, $f = (f^1,\dots,f^n)\colon \Omega \to \mathbf{R}^n$, with unbounded distortion. It is convenient to begin with the usual two postulates defining such mappings:

(i) *The Jacobian determinant* $J(x,f) = \det Df(x)$ *is locally integrable*;
(ii) *there exists a measurable function* $K_O(x) \ge 1$, *finite almost everywhere, such that*

$$|Df(x)|^n \le K_O(x) J(x,f) \qquad a.e. \tag{1}$$

We call such a mapping f a mapping of finite distortion. Above we used the operator norm of the differential matrix. There are several distortion functions that are each of considerable interest in geometric function theory [8]. The principal feature of those distortions is, roughly speaking, that they provide some control on the lower order minors of the differential matrix in terms of the determinant. Let

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$\bigwedge^l f(x)$ denote the $\binom{n}{l}\times\binom{n}{l}$-matrix of all $l\times l$-minors of $Df(x)$, $l=1,2,\dots,n-1$. Inequality (1) yields

$$|\bigwedge^l f(x)|^n \le \mathscr{K}_l(x)\big[J(x,f)\big]^l \qquad \text{a.e.} \tag{2}$$

where $1\le \mathscr{K}_l(x)\le \big[K_O(x)\big]^l$. On the other hand, notice that (i) and (2) with some $l\ge 2$ and $\mathscr{K}_l(x)\ge 1$, finite a.e., do not guarantee that f be a mapping of finite distortion; consider e.g. $f(x_1,\dots,x_n)=(x_1,0,\dots,0)$. The smallest $\mathscr{K}_l\ge 1$ for which (2) holds will be denoted by $\mathscr{K}_l(x,f)$ and called the l-th distortion function. Of particular interest is the inner distortion $K_I(x,f)=\mathscr{K}_{n-1}(x,f)$. In this case we denote by $D^\sharp f(x)$ the $n\times n$-matrix of cofactors of Df. Thus (2) reads as

$$|D^\sharp f(x)|^{n/(n-1)} \le \sqrt[n-1]{K_I(x,f)}\, J(x,f). \tag{3}$$

In order to work effectively with such mappings we must impose certain integrability conditions on the distortion functions. This will equip us, via Postulate (i), with a sufficient degree of integrability of the lower order subdeterminants. Here we shall combine these facts with most recent advances in the theory of Jacobians [3], [9], [14] and [4] to obtain basic regularity results such as higher integrability and the weak compactness principle of bounded families of such mappings. To illustrate this, let us assume that the function $\sqrt[l]{\mathscr{K}_l(x)}$ lies in the Orlicz space $\operatorname{Exp}(\Omega)$, that is:

$$\int_\Omega e^{\lambda\sqrt[l]{\mathscr{K}_l(x)}}\,dx<\infty \tag{4}$$

for some $\lambda>0$. The same then holds for $\sqrt[n-1]{K_I(x)}$ in place of $\sqrt[l]{\mathscr{K}_l(x)}$. Applying Hölder's inequality to (3) yields

$$\int_{\Omega'} |D^\sharp f(x)|^{n/(n-1)}\log^{-1}\big(e+|D^\sharp f(x)|\big)\,dx<\infty \tag{5}$$

on compact subsets $\Omega'\subset\Omega$. Furthermore this implies, via recent results in [3, Theorem 12.1], that the Jacobian determinant actually belongs to $L\log\log L(\Omega')$ on compact subsets $\Omega'\subset\Omega$. Precisely, we have

$$\int_{\Omega'} J(x,f)\,\log\log\big[e+J(x,f)\big]\,dx<\infty. \tag{6}$$

Repeated application of Theorem 12.1 in [3] leads to further improvements on the integrability of the Jacobian. This seemingly insignificant improvement turns out to be critical in the study of compactness principles. Is there any better motivation? As a matter of fact, we gain more from Theorem 1.3 in [3] than higher integrability. Namely, the Jacobian determinants obey the rule of integration by parts, and this is vital in order to conclude with weak monotonicity properties of mappings of finite distortion [7]. The exponential integrability of $\sqrt[l]{\mathscr{K}_l(x)}$, as described in (4), is not the weakest possible in the sense that it essentially suffices only to assume that

$$\int_\Omega e^{\mathscr{A}(\sqrt[l]{\mathscr{K}_l(x)})}\,dx < \infty \tag{7}$$

where

$$\int_1^\infty \frac{\mathscr{A}(t)\,dt}{t^2} = \infty. \tag{8}$$

In that case, we say that $\sqrt[l]{\mathscr{K}_l}$ lies in the subexponential Orlicz class $\operatorname{Exp}\mathscr{A}(\Omega)$. We wish to warn the reader that conditions (7) and (8) do not require $\sqrt[l]{\mathscr{K}_l}$ even to be integrable and thus additional technical assumptions on $\mathscr{A}$ have to be posed. To fill up this gap we assume that

$$\lim_{t\to\infty} t\,\mathscr{A}'(t) = \infty. \tag{9}$$

Another minor technical condition on $\mathscr{A}$ will be needed in the proofs of Theorem 1.1 and Theorem 1.2 below, namely we will assume that

$$\textit{the function } t \to e^{\mathscr{A}(\sqrt[l]{t})} \textit{ is convex for } t \ge 1. \tag{10}$$

Theorem 1.1. *Let $n > 2$. Fix $l \in \{1,\dots,n-1\}$ and $p \in [n-1,n)$ so that $p = n-1$ in the case $l \in \{1,\dots,n-2\}$ and $p > n-1$ in the case $l = n-1$. Assume that an Orlicz function $\mathscr{A}$ satisfies (8), (9) and (10), and $A, B \ge 0$. Let $\mathscr{F}$ be the family of mappings $f\colon \Omega \to \mathbf{R}^n$ in the Sobolev class $W^{1,p}_{\mathrm{loc}}(\Omega,\mathbf{R}^n)$ such that*

$$\int_\Omega J(x,f)\,dx \le A, \tag{11}$$

and

$$\big|{\textstyle\bigwedge^l} f(x)\big|^n \le \mathscr{K}_l(x)\big[J(x,f)\big]^l \qquad \text{a.e.} \tag{12}$$

with

$$\int_\Omega e^{\mathscr{A}(\sqrt[l]{\mathscr{K}_l(x)})}\,dx \le B. \tag{13}$$

Then $\mathscr{F}$ is closed under weak convergence in $W^{1,p}(\Omega,\mathbf{R}^n)$.

For mappings of finite distortion we can improve on Theorem 1.1, see Section 8 for further results.

Theorem 1.2. *Let $n \geq 2$. Fix $l \in \{1, \dots, n-1\}$. Assume that an Orlicz function $\mathscr{A}$ satisfies* (8), (9) *and* (10), *and $A, B \geq 0$. Let $\mathscr{F}$ be the family of mappings $f\colon \Omega \to \mathbf{R}^n$ of finite distortion for which*

$$\int_\Omega J(x,f)\,dx \leq A \tag{14}$$

and

$$\int_\Omega e^{\mathscr{A}(\sqrt[l]{\mathscr{K}_l(x)})}\,dx \leq B, \tag{15}$$

for some $l = 1, 2, \dots, n-1$. Then for each $1 \leq p < n$ we have
(i) $\|Df\|^n_{L^p(\Omega)} \leq C_p(n, \mathscr{A}, B) \int_\Omega J(x,f)\,dx$
and
(ii) *$\mathscr{F}$ is closed under weak convergence in $W^{1,p}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$.*

Theorem 1.2 is proven in the case $n = 2$ in [8].

As practical examples, Theorem 1.1 and Theorem 1.2 allow for

$$\mathscr{A} = \lambda t,\ \frac{\lambda t}{\log(e+t)},\ \frac{\lambda t}{\log(e+t)\log\log(e^e+t)},\dots$$

for any string of iterated logarithms and every $\lambda > l-1$. Regarding the sharpness, we will show, in particular, that

$$\mathscr{A} = \frac{\lambda t}{t^\varepsilon},\ \frac{\lambda t}{\log^{1+\varepsilon}(e+t)},\ \frac{\lambda t}{\log(e+t)\log^{1+\varepsilon}\log(e^e+t)},\dots$$

are not sufficient, for any $\varepsilon > 0$ and for every $\lambda > 0$. This easily follows from our next result.

Theorem 1.3. *Let $\mathscr{B}$ be a strictly increasing non-negative function such that*

$$\int_1^\infty \frac{\mathscr{B}(t)}{t^2}\,dt < \infty. \tag{16}$$

Then there exists a sequence of mappings $F_j\colon (-1,1)^n \to \mathbf{R}^n$, $j = 1, 2, \dots$, of finite distortion and a continuous mapping $F \in W^{1,1}\big((-1,1)^n, \mathbf{R}^n\big)$ such that

$$\int_{(-1,1)^n} J(x,F_j) + e^{\mathscr{B}(\sqrt[l]{\mathscr{K}_l(x)})} \leq C \tag{17}$$

for all $l = 1, 2, \dots, n-1$, and each j, for each $1 \leq p < n$

$$F_j \to F \qquad \text{weakly in } W^{1,p}\big((-1,1)^n, \mathbf{R}^n\big),$$

and there is a set $E \subset (-1,1)^n$ with positive measure such that $\big|\bigwedge^l F(x)\big| = 1$ and $J(x,F) = 0$ for all $x \in E$ and every $l = 1, 2, \dots, n-1$. In particular, (2) *cannot hold for F with any $1 \leq \mathscr{K}_l(x)$ finite a.e. and F is not a mapping of finite distortion.*

The paper is organized as follows. In Section 2 we discuss Orlicz spaces and introduce an additional Orlicz function based on $\mathscr{A}$ that will be needed later on. Section 3 contains a short discussion on distortion functions and in Section 4 we introduce a mollifying technique. The crucial tools for the proofs of Theorem 1.1 and 1.2, the isoperimetric inequality and higher integrability of Jacobians, are dealt with in Section 5 and Section 6. Here we employ recent results from [3], [5]. The proofs of the above theorems are given in Section 7, and we point out a certain extension of Theorem 1.2 in Section 8.

2. Subexponential Orlicz spaces

In the notation $L^P(\Omega)$ of an Orlicz space we always assume that $P\colon [0,\infty) \to [0,\infty)$ is continuously increasing from $P(0) = 0$ to $P(\infty) = \lim_{t\to\infty} P(t) = \infty$, and is infinitely differentiable on $(0,\infty)$. The reader will notice that we are not assuming P to be convex. The relation $h \in L^P(\Omega)$ signifies that Ω is an open subset of $\mathbf{R}^n$ and there is $k = k(h) > 0$ such that $P\big(|h|/k\big) \in L^1(\Omega)$. A definition of the Luxemburg functional is expressed by the equation

$$\|h\|_P = \|h\|_{L^P(\Omega)} = \inf\left\{k : \int_\Omega P\left(\frac{|h(x)|}{k}\right) dx \le 1\right\}. \tag{18}$$

Thus, in particular

$$\int_\Omega P\left(|h(x)|\right) dx = 1 \qquad \text{if } \|h\|_P = 1. \tag{19}$$

If P is not convex the Luxemburg functional does not comply with the criteria of a norm (the triangle inequality fails). Nevertheless, it plays an especially basic role in handling estimates in Orlicz spaces. These are always linear complete metric spaces.

In this paper need will arise for generating functions P to have nearly linear growth. There is no genuine definition of such functions but we have in mind examples of the type

$$P(t) = t\log^{\alpha_1}(e+t)\big[\log\log(e^e+t)\big]^{\alpha_2}\cdots\Big[\log\cdots\log(e^{e^{\cdot^{\cdot^{\cdot}}}}+t)\Big]^{\alpha_k} \tag{20}$$

where $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbf{R}$. The characteristic property of nearly linear functions can be expressed by saying that for every $\varepsilon > 0$

$$\lim_{t\to\infty} t^{-1-\varepsilon} P(t) = 0 \qquad \text{and} \qquad \lim_{t\to\infty} t^{-1+\varepsilon} P(t) = \infty. \tag{21}$$

When no confusion is likely, we shall often write

$$L\log^{\alpha_1} L\log\log^{\alpha_2} L\cdots\log\log\cdots\log^{\alpha_k} L \tag{22}$$

in place of L^P. In certain cases we must prevent P from being too far below the identity function. This will be done by imposing the so-called *divergence condition*

$$\int_1^\infty \frac{P(s)\,ds}{s^2} = \infty. \tag{23}$$

To illustrate, this condition tells us that in (20) all of the exponents $\alpha_1, \alpha_2, \dots, \alpha_k$ are smaller than or equal to -1. It is convenient when dealing with functions valued in $\mathbf{R}^n$, or in the space $\mathbf{R}^{n\times n}$ of $n \times n$-matrices, to write $L^P(\Omega, \mathbf{R}^n)$ or $L^P(\Omega, \mathbf{R}^{n\times n})$, respectively. As will be seen subsequently in this paper, and as is illustrated by Proposition 2.2, a somewhat dual category of Orlicz spaces enters into our study. These spaces are generated by the Orlicz function $e^{\mathscr{A}(t)} - 1$, where $\mathscr{A}$ satisfies the divergence condition

$$\int_1^\infty \frac{\mathscr{A}(s)\,ds}{s^2} = \infty. \tag{24}$$

As before, $\mathscr{A}\colon [0,\infty) \to [0,\infty)$ is continuously increasing from $\mathscr{A}(0) = 0$ to $\mathscr{A}(\infty) = \lim_{t\to\infty} \mathscr{A}(t) = \infty$ and is smooth on $(0,\infty)$. It will be convenient to make the technical requirement

$$\mathscr{A}(s) \ge 4\log(s+1) \qquad \text{for } s \ge 0 \tag{25}$$

as this is not necessarily a consequence of (24).

A subexponential class of functions is an Orlicz space generated by $e^{\mathscr{A}(t)} - 1$. As a matter of notation $h \in \operatorname{Exp}\mathscr{A}(\Omega)$ simply means that

$$\int_\Omega \bigl[e^{\mathscr{A}(|h(x)|/k)} - 1\bigr]\,dx < \infty \qquad \text{for some } k = k(h) > 0. \tag{26}$$

We now associate with $\mathscr{A}(t)$ an Orlicz function $P = P_{\mathscr{A}}\colon [0,\infty) \to [0,\infty)$ by the rule

$$P\bigl(te^{\mathscr{A}(t)/4}\bigr) = \int_0^t \frac{d\bigl[se^{\mathscr{A}(s)/4}\bigr]}{s+1} = \frac{te^{\mathscr{A}(t)/4}}{t+1} + \int_0^t \frac{se^{\mathscr{A}(s)/4}\,ds}{(s+1)^2}, \qquad t \ge 0. \tag{27}$$

Then

$$P\bigl(t\,e^{\mathscr{A}(t)/4}\bigr) \le \frac{te^{\mathscr{A}(t)/4}}{t+1} + te^{\mathscr{A}(t)/4}\int_0^t \frac{ds}{(s+1)^2} = te^{\mathscr{A}(t)/4}, \tag{28}$$

and in particular,

$$P(s) \le s \qquad \text{for all } s \ge 0. \tag{29}$$

Similarly one can show that

$$\lim_{s\to\infty}\frac{P(s)}{s}=0. \tag{30}$$

Next we record the identity

$$P'\big[te^{\mathscr{A}(t)/4}\big]=\frac{1}{t+1}\qquad\text{for all } t>0, \tag{31}$$

which yields

$$\begin{aligned}\int_{e^{\mathscr{A}(1)/4}}^{\infty}\frac{P'(s)\,ds}{s}&=\int_1^{\infty}\frac{P'\big[te^{\mathscr{A}(t)/4}\big]\,d\big[t\,e^{\mathscr{A}(t)/4}\big]}{te^{\mathscr{A}(t)/4}}\\&=\int_1^{\infty}\frac{dt}{t(t+1)}+\frac14\int_1^{\infty}\frac{\mathscr{A}'(t)\,dt}{t+1}\\&\geq\frac18\int_1^{\infty}\frac{\mathscr{A}'(t)\,dt}{t}=\frac18\frac{\mathscr{A}(t)}{t}\bigg|_1^{\infty}+\frac18\int_1^{\infty}\frac{\mathscr{A}(t)\,dt}{t^2}=\infty.\end{aligned}$$

This together with (29) leads us to the divergence condition for P:

$$\int_1^{\infty}\frac{P(s)\,ds}{s^2}=\infty. \tag{32}$$

Another preliminary bound for P follows from (28) by applying inequality (25). This gives $se^{\mathscr{A}(s)/4}\leq e^{\mathscr{A}(t)/2}$ and

$$P\big(te^{\mathscr{A}(t)/4}\big)\leq\frac{te^{\mathscr{A}(t)/4}}{t+1}+e^{\mathscr{A}(t)/2}\int_0^t\frac{ds}{(s+1)^2}. \tag{33}$$

Hence

$$P\big(te^{\mathscr{A}(t)/4}\big)\leq\frac{t}{t+1}e^{\mathscr{A}(t)/2}+e^{\mathscr{A}(t)/2}-1\qquad\text{for all } t\geq 0. \tag{34}$$

We formulate our next estimate as a lemma.

Lemma 2.1. *For $J\geq 0$ and $K\geq 0$ it holds that*

$$P(JK)\leq J+\big[e^{\mathscr{A}(K)/2}-1\big]. \tag{35}$$

Proof. We consider the function $J\to P(JK)-JK/(K+1)$ defined for $0\leq J<\infty$. It suffices to show that

$$P(JK)-\frac{JK}{K+1}\leq e^{\mathscr{A}(K)/2}-1. \tag{36}$$

This is certainly true at $J = 0$ and also for sufficiently large values of J, due to (30). It remains to check (36) at the critical points, if such points exist at all. To this end we must solve the equation

$$P'(JK) = \frac{1}{K+1} \tag{37}$$

for $J > 0$. It follows from the identity (31) that P' is decreasing. Thus (37) admits at most one solution. It is not difficult to guess from (31) that the solution really exists and is given by

$$J = e^{\mathscr{A}(K)/4}. \tag{38}$$

Substituting it into (36) and using (34) we arrive at

$$P(JK) - \frac{JK}{K+1} = P\big(Ke^{\mathscr{A}(K)/4}\big) - \frac{K}{K+1}e^{\mathscr{A}(K)/4} \le e^{\mathscr{A}(K)/2} - 1 \tag{39}$$

as claimed.

By combining Lemma 2.1 with the definition of Luxemburg's functional we obtain our principal Hölder-type inequality.

Proposition 2.2. *Let assumptions* (24), (25) *hold, and let* P *be defined by* (27). *Then*

$$\|KJ\|_{L^P(\Omega)} \le 2\,\|J\|_{L^1(\Omega)}\,\|K\|_{\operatorname{Exp}\mathscr{A}(\Omega)} \tag{40}$$

for all $K \in \operatorname{Exp}\mathscr{A}(\Omega)$ *and* $J \in L^1(\Omega)$.

Proof. It involves no loss of generality in assuming that $\|J\|_{L^1(\Omega)} = \frac{1}{2}$ and $\|K\|_{\operatorname{Exp}\mathscr{A}(\Omega)} = 1$. Applying (35) we can write

$$\int_\Omega P(KJ) \le \int_\Omega J + \int_\Omega \big[e^{\mathscr{A}(K)/2} - 1\big] \le \frac{1}{2} + \frac{1}{2}\int_\Omega \big[e^{\mathscr{A}(K)} - 1\big] = 1 \tag{41}$$

which means that $\|KJ\|_P \le 1$, as desired.

In this paper it is important that the function P will be nearly linear in the sense of equation (21). We wish to warn the reader that this is not necessarily a consequence of (8) (see [12, Remark 2.2]) and so something should be added to the condition (8). We will not exclude important examples of Orlicz functions if we assume that $\mathscr{A}$ satisfies (9) i.e.

$$\lim_{t\to\infty} t\mathscr{A}'(t) = \infty.$$

Among power-like functions $\mathscr{A}(t) = t^\alpha$, (8) corresponds to $\alpha < 1$, while (9) is true for all $\alpha > 0$. This explains in what sense we regard (9) to be only a minor technical assumption.

Lemma 2.3. *Assume that* $\mathscr{A}$ *is an Orlicz function satisfying* (9) *and* $\delta \in (0,1)$. *Then there exists* $s_0 \in (0,\infty)$ *such that the function*

$$h\colon s \to \frac{P(s)}{s^\delta}$$

is increasing on (s_0,∞).

Proof. Because the function $g(t) = te^{\mathscr{A}/4}$ is increasing, it suffices to show that

$$\left[\frac{P\big(g(t)\big)}{g(t)^\delta}\right]' \geq 0$$

for large values of t. For this we need only to check that

$$P'\big(g(t)\big)g(t) \geq \delta P\big(g(t)\big). \tag{42}$$

Choosing $a_1 \in (0,\infty)$ such that $s\mathscr{A}'(s) \geq (1-\delta)/\delta$ for all $s \in (a_1,\infty)$ we have

$$P\big(g(t)\big) = \int_0^t \frac{d\big[se^{\mathscr{A}(s)/4}\big]}{s+1}\,ds \leq \frac{1+\delta}{\delta}e^{\mathscr{A}(t)/4}. \tag{43}$$

Next we choose $a \geq a_1$ such that $a/(1+a) \geq \frac{1}{2}(1+\delta)$ and so by (31) we conclude that

$$P'\big(g(t)\big)g(t) = \frac{t}{t+1}e^{\mathscr{A}(t)/4} \geq \frac{1+\delta}{2}e^{\mathscr{A}(t)/4} \tag{44}$$

for every $t > a$. Inequality (42) follows from (43) and (44) for all $t \in (a,\infty)$ and so the function $h\colon s \to P(s)/s^\delta$ is increasing on (s_0,∞), where $s_0 = g(a)$.

Next we show that, under the assumption $\lim_{t\to\infty} t\mathscr{A}'(t) = \infty$, the property (25) is satisfied (for large s): Choose $s_0 \geq 2$ such that $\mathscr{A}'(t)\cdot t \geq 16$ and $\mathscr{A}(t) \geq 1$ for all $t \geq s_0$. Then

$$\int_{\sqrt{s}}^s \mathscr{A}'(t)\,dt \geq 16\int_{\sqrt{s}}^s \frac{dt}{t}$$

for all $s \geq s_0^2$ and

$$\mathscr{A}(s) \geq \mathscr{A}(s) - \mathscr{A}\big(\sqrt{s}\,\big) \geq 8\log s \geq 4\log(s+1)$$

for every $s \geq s_0^2$. Replacing the function $\mathscr{A}$ by the function

$$\tilde{\mathscr{A}}(t) = \begin{cases} \mathscr{A}(t), & t \geq s_0^2, \\ \mathscr{A}(s_0^2)t, & 0 \leq t \leq s_0^2, \end{cases}$$

we can assume that the condition (25) holds under the assumption (9).

Later in this paper we will use the following lemma.

Lemma 2.4. *Assume that $\mathscr{A}$ is an Orlicz function satisfying* (9) *and let* $\varepsilon > 0$. *Then there exists* $s_0 \in (0,\infty)$ *such that the function*

$$h\colon t \to P(t^{1+\varepsilon})$$

is convex on (s_0,∞).

Proof. Because the function $g(t) = te^{\mathscr{A}(t)/4}$ is increasing and

$$P'(t^{1+\varepsilon}) = (1+\varepsilon)\frac{t^{\varepsilon}}{g^{-1}(t^{1+\varepsilon})+1},$$

we only need to check that the function $\big(g^{\varepsilon/(1+\varepsilon)}(t)\big)/(t+1)$ is increasing for large values of t. This follows for all $t \in (a,\infty)$, where a is chosen so that the inequality

$$\frac{\varepsilon}{1+\varepsilon}\left(1+\frac{t}{4}\mathscr{A}'(t)\right) \geq 1$$

is true for every $t \in (a,\infty)$ and so the function $h\colon t \to P(t^{1+\varepsilon})$ is convex on (s_0,∞), where $s_0 = g(a)$.

3. Distortion functions

Distortion functions are designed to control almost everywhere the minors of the differential matrix of the mapping $f\colon \Omega \to \mathbf{R}^n$ by means of the Jacobian determinant. We begin with the distortion functions of linear mappings, also regarded as matrices. The space of all $n\times n$-matrices will be denoted by $\mathbf{R}^{n\times n}$, and those with positive determinant by $\mathbf{R}^{n\times n}_+$. It will be convenient to include the zero matrix and denote such extended class of matrices by $\mathbf{R}^{n\times n}_+ \cup \{0\}$. The commonly used distortion functions on matrices $A \in \mathbf{R}^{n\times n}_+$ are:

The outer distortion: $K_O(A) = |A|^n/\det A$;

the inner distortion: $K_I(A) = K_O(A^{-1}) = |A^\sharp|^n/(\det A)^{n-1}$;

the linear distortion: $H(A) = \sqrt[n]{K_O(A)K_I(A)} = |A||A^{-1}|$.

Note that the operator norm $|A| = \max\big\{|Ah| : |h| = 1\big\}$ is being used here, and $A^\sharp$ is the adjoint matrix, made of cofactors of A. In what follows all distortion functions of the zero matrix are assumed to be equal to 1. There are in fact many more distortion functions which are readily defined in terms of the lower order subdeterminants of the matrix A. For each integer $1 \leq l \leq n$ we denote by $\bigwedge^l A$ the $\binom{n}{l}\times\binom{n}{l}$-matrix of all $l\times l$-minors of A. This, of course, includes A as $\bigwedge^1 A$, $A^\sharp$ as $\bigwedge^{n-1}A$, and $\det A$ as $\bigwedge^n A$.

The following distortion functions will interest us most as they have the very important property of being polyconvex

$$\mathscr{K}_l(A) = \frac{|\bigwedge^l A|^n}{(\det A)^l} = \mathscr{K}_{n-l}(A^{-1}) \qquad \text{for } l = 1,\dots,n-1. \tag{45}$$

Polyconvexity simply means that $\mathscr{K}_l\colon \mathbf{R}_+^{n\times n} \to \mathbf{R}_+$ can be expressed as a convex function of all possible minors of A. Precisely, we have

$$\mathscr{K}_l(A) = \mathscr{P}\Big(\bigwedge\nolimits^l A, \det A\Big) \tag{46}$$

where $\mathscr{P}\colon \mathbf{R}^{\binom{n}{l}\times\binom{n}{l}}\times\mathbf{R} \to [1,\infty)$ is convex, see [8, Section 6] for a fuller discussion.

Having examined these distortion functions for matrices we set for orientation preserving mappings (i.e. $J(x,f)\ge 0$ a.e.) $f\in W^{1,1}_{\mathrm{loc}}(\Omega,\mathbf{R}^n)$:

$$\bigwedge\nolimits^l f(x) = \bigwedge\nolimits^l \big[Df(x)\big]. \tag{47}$$

We define the pointwise distortion functions by setting

$$\mathscr{K}_l(x,f) = \mathscr{K}_l\big[Df(x)\big] = \begin{cases} \dfrac{|\bigwedge^l f(x)|^n}{J(x,f)^l}, & J(x,f)>0,\\ 1, & \big|\bigwedge^l f(x)\big| = 0,\\ \infty, & J(x,f)=0 \text{ and } \big|\bigwedge^l f(x)\big|\ne 0.\end{cases} \tag{48}$$

All these functions are coupled by the inequalities

$$\begin{aligned} \sqrt[n-1]{K_I(x,f)} = \sqrt[n-1]{\mathscr{K}_{n-1}(x,f)} &\le \cdots \\ \le \sqrt[l]{\mathscr{K}_l(x,f)} \le \mathscr{K}_1(x,f) &= K_O(x,f). \end{aligned} \tag{49}$$

Let us also note for later use the reverse estimate

$$K_O(x,f) \le K_I^{n-1}(x,f) = \Big(\sqrt[n-1]{K_I(x,f)}\Big)^{(n-1)^2} \tag{50}$$

which holds when $J(x,f)>0$.

4. Mollifying the distributional Jacobian

We shall consider a mapping $f\in W^{1,n-1}_{\mathrm{loc}}(\Omega,\mathbf{R}^n)$, $n>2$, whose cofactor matrix lies in $L^q_{\mathrm{loc}}(\Omega,\mathbf{R}^n)$ with

$$q = \frac{n^2-n}{n^2-n-1} < \frac{n}{n-1}.$$

By Sobolev's embedding we know that $|f|\in L^p_{\mathrm{loc}}(\Omega)$, $p=n^2-n$. Hence $|f|\,|D^\sharp f|\in L^1_{\mathrm{loc}}(\Omega)$. The distributional Jacobian, denoted by $\mathscr{J}_f$, is a Schwartz distribution acting on test functions $\varphi\in C_0^\infty(\Omega)$ according to the rule

$$\mathscr{J}_f[\varphi] = -\int_\Omega f^i\, df^1\wedge\cdots\wedge df^{i-1}\wedge d\varphi\wedge df^{i+1}\wedge\cdots\wedge df^n. \tag{51}$$

We recall that this integral does not depend on the choice of the index $i = 1,2,\dots,n$, see [3, Section 2].

Let $\Phi_t(x) = t^{-n}\Phi(t^{-1}x)$, $t > 0$, be a standard approximation of unity; that is, $\Phi \in C_0^\infty(\mathbf{B})$, is non-negative and has integral 1. The convolution

$$(52)\quad (\mathscr{J}_f * \Phi_t)(a) = -\int_\Omega f^i(x)\, df^1\wedge\cdots\wedge df^{i-1}\wedge d\Phi_t(a-x)\wedge df^{i+1}\wedge\cdots\wedge df^n$$

is a smooth function defined on the set $\Omega_t = \{a\in\Omega : \operatorname{dist}(a,\partial\Omega) > t\}$. Now the following lemma provides us with a beneficial link between the distribution $\mathscr{J}_f$ and the point-wise Jacobian $J(a,f)$.

Lemma 4.1. *For almost every $a\in\Omega$ we have*

$$J(a,f) = \lim_{t\to 0}(\mathscr{J}_f * \Phi_t)(a). \tag{53}$$

Proof. The main idea of this proof cames from [16], although it differs in a number of details.

Let us disclose in advance that the points $a\in\Omega$ for which we achieve equation (53) are determined by the properties

$$\lim_{t\to 0}\left(\fint_{B(a,t)} |D^\sharp f(a) - D^\sharp f(x)|^q\, dx\right)^{1/q} = 0 \tag{54}$$

and

$$\lim_{t\to 0}\frac{1}{t}\left(\fint_{B(a,t)} |f(x) - f(a) - Df(a)(x-a)|^p\, dx\right)^{1/p} = 0. \tag{55}$$

The first requirement is fulfilled at the Lebesgue points of $D^\sharp f$. The second requirement is guaranteed at almost every point $a\in\Omega$ since $f\in W^{1,n-1}_{\mathrm{loc}}(\Omega)$ and $p = n^2 - n$, see [1]. We now split the integral at (52) as

$$(\mathscr{J}_f * \Phi_t)(a) = I_1 + I_2,$$

where

$$I_1 = -\int_\Omega \bigl[f^1(x) - f^1(a) - \langle\nabla f^1(a), x-a\rangle\bigr]\, d\Phi_t(a-x)\wedge df^2\wedge\cdots\wedge df^n$$

and

$$I_2 = -\int_\Omega \bigl[f^1(a) + \langle\nabla f^1(a), x-a\rangle\bigr]\, d\Phi_t(a-x)\wedge df^2\wedge\cdots\wedge df^n.$$

The first integral, I_1, can be estimated by Hölder's inequality and the fact that $|d\Phi_t(a-x)| \le C(n)t^{-n-1}\chi_{B(a,t)}(x)$:

$$
\begin{aligned}
|I_1| \le \frac{C(n)}{t}\Big(&⨍_{B(a,t)} |f^1(x)-f^1(a)-\langle \nabla f^1(a), x-a\rangle|^p\,dx\Big)^{1/p} \\
&\times\Big(⨍_{B(x,t)} |D^\sharp f(x)|^q\,dx\Big)^{1/q} \to 0
\end{aligned} \tag{56}
$$

by the requirements (55) and (54), where $p = n^2 - n$ and

$$q = \frac{n^2-n}{n^2-n-1} < \frac{n}{n-1}.$$

Concerning the second term, we are allowed to integrate by parts to obtain

$$
\begin{aligned}
|I_2| &= \int_\Omega \Phi_t(a-x)\,df^1(a)\wedge df^2(x)\wedge\cdots\wedge df^n(x) \\
&= J(a,f) + \int_\Omega \Phi_t(a-x)\,df^1(a)\big[df^2\wedge\ldots\wedge df^n - df^2(a)\wedge\ldots\wedge df^n(a)\big]
\end{aligned}
$$

where the latter integral converges to zero, as it is bounded by

$$
\begin{aligned}
\int_\Omega &|\Phi_t(a-x)|\,|df^1(a)|\,|D^\sharp f(x) - D^\sharp f(a)|\,dx \\
&\le C(n)|df^1(a)| ⨍_{B(a,t)} |D^\sharp f(x) - D^\sharp f(a)|\,dx \\
&\le C(n)|df^1(a)|\Big(⨍_{B(a,t)} |D^\sharp f(x) - D^\sharp f(a)|^q\,dx\Big)^{1/q} \to 0.
\end{aligned} \tag{57}
$$

The proof is complete.

5. Isoperimetric inequality

In this section we formulate perhaps the most general isoperimetric inequality based on our result in [3]. We shall not give all details but only confine ourselves to verifying the rather involved hypotheses (6.1) in [15].

Proposition 5.1. *Let $f \in W^{1,n-1}_{\mathrm{loc}}(\Omega,\mathbf{R}^n)$, $n > 2$, be an orientation preserving mapping, i.e. $J(x,f) \ge 0$ almost everywhere in Ω, whose cofactors satisfy*

$$|D^\sharp f|^{n/(n-1)} \in L^P(\Omega). \tag{58}$$

Here the Orlicz function P satisfies the divergence condition

$$\int_1^\infty \frac{P(s)\,ds}{s^2} = \infty \tag{59}$$

and the technical condition

$$\left[t^{-1}P(t)\right]' \le 0 \le \left[t^{-s}P(t)\right]', \qquad s = \frac{n^2-2n+1}{n^2-n-1}. \tag{60}$$

Then for $x \in \Omega$, we have

$$\int_{B(x,r)} J(x,f)\,dx \le C(n)\left(\int_{\partial B(x,r)} |D^\sharp f(x)|\,dx\right)^{n/(n-1)} \tag{61}$$

for almost every $0 < r < \operatorname{dist}(x, \partial\Omega)$.

Proof. As we are going to appeal to Lemma 6.1 in [15], we first observe that $f \in W^{1,1}_{\rm loc}(\Omega, \mathbf{R}^n)$, its cofactor matrix $D^\sharp f \in L^1_{\rm loc}(\Omega, \mathbf{R}^{n\times n})$ and by [3, Theorem 1.3] the Jacobian $\det Df \in L^1_{\rm loc}(\Omega)$. Inequality (61) is immediate from Lemma 6.1 in [15] once we verify the equation (6.1) in [15]. The reader will easily recognize that equation (6.1) is a result of integration by parts, which we have proven in [3, Theorem 1.3]. Specifically, let $v = (v^1, \dots, v^n)$ be a vector field of class $C^1_0(\mathbf{R}^n, \mathbf{R}^n)$. We consider the mapping

$$F_i = \left(f^1, \dots, f^{i-1}, \lambda f^i + v^i(f), f^{i+1}, \dots, f^n\right) \in W^{1,n-1}_{\rm loc}(\Omega, \mathbf{R}^n)$$

where λ is a sufficiently large positive parameter to be selected later. The point is that F_i is still orientation preserving and $|D^\sharp F_i|^{n/(n-1)} \in L^P_{\rm loc}(\Omega)$. To see this, we compute

$$\begin{aligned} J(x,F_i)\,dx &= df^1 \wedge \dots \wedge df^{i-1} \wedge \left(\lambda df^i + \sum_{\nu=1}^n \frac{\partial v^i}{\partial y_\nu}\, df^\nu\right) \wedge df^{i+1} \wedge \dots \wedge df^n \\ &= \left(\lambda + \frac{\partial v^i}{\partial y_i}\right) J(x,f)\,dx \end{aligned} \tag{62}$$

showing that $J(x,F_i) \ge 0$. Concerning cofactors of F_i, we only need to observe that

$$|D^\sharp F_i| \le C(\lambda + \|Dv\|_\infty)|D^\sharp f|. \tag{63}$$

Thus $|D^\sharp F_i|^{n/(n-1)} \in L^P_{\rm loc}(\Omega)$. With this condition in hand we can now apply Theorem 1.3 in [3] to conclude that

$$\int_\Omega \psi(x) J(x,F_i)\,dx = -\int_\Omega df^1 \wedge \dots \wedge df^{i-1} \wedge \left[\lambda f^i + v^i(f)\right] d\psi \wedge df^{i+1} \wedge \dots df^n \tag{64}$$

for every $\psi \in C_0^1(\Omega)$. This yields

$$\begin{aligned}\lambda \int_\Omega \psi(x) J(x,f)\,dx + \int_\Omega \psi(x)\frac{\partial v^i}{\partial y_i}(f)J(x,f)\,dx \\ = -\lambda \int_\Omega df^1\wedge\cdots\wedge df^{i-1}\wedge d\psi\wedge df^{i+1}\wedge\cdots\wedge df^n \\ = -\int_\Omega v^i(f)\,df^1\wedge\cdots\wedge df^{i-1}\wedge d\psi\wedge df^{i+1}\wedge\cdots\wedge df^n.\end{aligned}$$

The integrals with factor λ in front cancel out, again by Theorem 1.3 in [3]. What remains upon cancellation is the identity

$$\int_\Omega v^i(f)\,df^1\wedge\cdots\wedge df^{i-1}\wedge d\psi\wedge df^{i+1}\wedge\cdots\wedge df^n + \psi(x)\frac{\partial v^i}{\partial y_i}(f)J(x,f)\,dx = 0.$$

Finally, we sum it up with respect to all indices $i = 1,2,\dots,n$ and arrive at the identity (6.1) in [15]. All the hypotheses of Lemma 6.1 in [15] are therefore fulfilled, completing the proof of inequality (61).

6. Higher integrability

We combine the isoperimetric inequality and maximal theorem in [5] to prove the following higher integrability result.

Theorem 6.1. *Under the hypothesis of Proposition* 5.1, *the Jacobian determinant belongs to the Orlicz space* $L^\Psi_{\mathrm{loc}}(\Omega)$, *where*

$$\Psi(t) = P(t) + t\int_0^t \frac{P(s)}{s^2}\,ds. \tag{65}$$

Precisely, for each pair of concentric cubes $Q \subset nQ \subset \Omega$ *we have*

$$\|\det Df\|_{L^\Psi(Q)} \le C_P(n)\big\||D^\sharp f|^{n/(n-1)}\big\|_{L^P(nQ)}. \tag{66}$$

Here, as usual, we assume that the integral converges near zero. The estimate at (66) gives us an advantage over the higher integrability result in [3] because is explicit and depends exclusively on the cofactors of Df.

Proof. We recall that for $u \in L^1_{\mathrm{loc}}(\Omega)$ the maximal function Mu is defined by

$$Mu(x) = M_\Omega u(x) = \sup\Big\{ ⨍_Q |u(z)|\,dz : x \in Q \subset \Omega\Big\}. \tag{67}$$

Fix concentric cubes $Q \subset nQ \subset \Omega$. Let $Q' = Q'(a,r)$ be an arbitrary cube in Q. Certainly $nQ' \subset nQ$. By the isoperimetric inequality (61) we have

$$\left(\int_{B(a,s)} J(x,f)\,dx\right)^{(n-1)/n} \le C(n)\int_{\partial B(a,s)} |D^\sharp f(x)|\,dx$$

for almost every $0 < s < \operatorname{dist}(x,\partial\Omega)$. Next, we integrate this estimate from $\sqrt{n}\,r$ to nr:

$$\int_{\sqrt{n}\,r}^{nr}\left(\int_{B(a,s)} J(x,f)\,dx\right)^{(n-1)/n} ds \le C(n)\int_{nQ'} |D^\sharp f(x)|\,dx. \tag{68}$$

Since the Jacobian is non-negative almost everywhere this yields

$$(n-\sqrt{n}\,)r\left(\int_{Q'} J(x,f)\,dx\right)^{(n-1)/n} \le C(n)\int_{nQ'} |D^\sharp f(x)|\,dx. \tag{69}$$

Divide both sides by r^{n-1}, to obtain

$$\fint_{Q'} J(x,f)\,dx \le C(n)\left(\fint_{nQ'} |D^\sharp f(x)|\,dx\right)^{n/(n-1)}. \tag{70}$$

Finally, we take the supremum over all cubes $Q' \subset Q$ containing a given point $x \in Q$ and arrive at the point-wise estimate

$$M_Q(\det Df) \le C(n)\big(M_{nQ}|D^\sharp f|\big)^{n/(n-1)} \tag{71}$$

of the maximal functions. In order to make use of the maximal inequality we introduce $\widetilde{P}(t) = P\big(t^{n/(n-1)}\big)$ and notice that $t^{-p}\widetilde{P}(t)$ is increasing for $p = (n^2-n)/(n^2-n-1) > 1$, by condition (60). We conclude from [5, Lemma 5.2] that

$$\begin{aligned} \|\det Df\|_{L^\Psi(Q)} &\le C(n)\|M_Q(\det Df)\|_{L^P(Q)} \\ &\le C(n)\|M_{nQ}(D^\sharp f)\|^{n/(n-1)}_{L^{\tilde P}(nQ)}. \end{aligned} \tag{72}$$

As the maximal operator $M_{nQ}\colon L^{\tilde P}(nQ) \to L^{\tilde P}(nQ)$ is bounded, by [5, Lemma 5.1], we see that

$$\|M_{nQ}(D^\sharp f)\|_{L^{\tilde P}(nQ)} \le C(n)\|D^\sharp f\|_{L^{\tilde P}(nQ)} \tag{73}$$

which together with previous estimate yields

$$\|\det Df\|_{L^\Psi(Q)} \le C(n)\||D^\sharp f|^{n/(n-1)}\|_{L^P(nQ)} \tag{74}$$

as desired.

Theorem 6.2. *Suppose that* $f \in W^{1,n-1}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$, $n > 2$, $J(\,\cdot\,, f) \in L^1(\Omega)$, *and* f *satisfies the inequality*

$$|\textstyle\bigwedge^l f(x)|^n \le \mathscr{K}_l(x)\big[J(x,f)\big]^l \qquad \text{a.e.} \tag{75}$$

with

$$\sqrt[l]{\mathscr{K}_l(x,f)} \in \operatorname{Exp}\mathscr{A}(\Omega) \tag{76}$$

for some $l = 1, \dots, n-1$, *where* $\mathscr{A}$ *is an Orlicz function satisfying the divergence condition* (8) *and the technical condition* (9). *Let* $\mathscr{L}(t) = \int_0^t s^{-2}\mathscr{A}(s)\,ds$. *Then for each compact set* $\Omega' \subset \Omega$

$$\int_{\Omega'} J(x,f)\mathscr{L}\big[J(x,f)\big]\,dx \le C(n,\mathscr{A},\Omega')\big\|\sqrt[l]{\mathscr{K}_l(x,f)}\big\|_{\operatorname{Exp}\mathscr{A}(\Omega)} \int_\Omega J(x,f)\,dx. \tag{77}$$

Proof. The point of special note is that (75) implies the same condition for

$$\sqrt[n-1]{K_I(x,f)} \in \operatorname{Exp}\mathscr{A}(\Omega) \tag{78}$$

and hence

$$|D^\sharp f|^{n/(n-1)} \le \sqrt[n-1]{K_I(x,f)}\, J(x,f). \tag{79}$$

By Hölder's inequality at (40) and (49) we find that

$$\|\,|D^\sharp f|^{n/(n-1)}\|_{L^P(\Omega)} \le C(\mathscr{A})\big\|\sqrt[l]{\mathscr{K}_l}\,\big\|_{\operatorname{Exp}\mathscr{A}(\Omega)} \int_\Omega J(x,f)\,dx. \tag{80}$$

Using (32) we see that the Orlicz function P satisfies (59). By (29) and Lemma 2.3 we have that also the technical condition (60) is fulfilled and so by Theorem 6.1 we conclude

$$\int_{\Omega'} J(x,f)L\big[J(x,f)\big]\,dx \le C(n,\mathscr{A},\Omega')\big\|\sqrt[l]{\mathscr{K}_l(x,f)}\,\big\|_{\operatorname{Exp}\mathscr{A}(\Omega)} \int_\Omega J(x,f)\,dx \tag{81}$$

where

$$L(t) = \frac{P(t)}{t} + \int_0^t \frac{P(s)}{s^2}\,ds. \tag{82}$$

This estimate is stronger than our higher integrability statement in this theorem, since

$$\mathscr{L}(t) = \int_0^t s^{-2}\mathscr{A}(s)\,ds \le C \cdot L(t), \qquad t \ge 1,$$

by the analysis throughout (27)–(34).

7. Weak compactness

Let $f \in W^{1,n-1}(\Omega, \mathbf{R}^n)$. In Theorem 1.1 we have assumed that

$$\big|\bigwedge^l f(x)\big|^{n/l} \le \sqrt[l]{\mathscr{K}_l(x)}\, J(x,f) \tag{83}$$

with

$$\int_\Omega e^{\mathscr{A}(\sqrt[l]{\mathscr{K}_l})}\, dx \le B < \infty \tag{84}$$

where as always

$$\int_1^\infty \frac{\mathscr{A}(s)\, ds}{s^2} = \infty. \tag{85}$$

We also recall the assumptions (10) and (9):

$$\textit{the function } t \to e^{\mathscr{A}(\sqrt[l]{t})} \textit{ is convex for } t \ge 1 \tag{86}$$

and

$$\lim_{t\to\infty} t\mathscr{A}'(t) = \infty. \tag{87}$$

These additional assumptions on $\mathscr{A}$ cause practically very little loss of generality. Note that (86) holds for the function $\mathscr{A}(s) = \lambda s$, whenever $\lambda > 0$ and $\lambda \ge l-1$.

Proof of Theorem 1.1. We notice that the inequality (83) implies

$$|D^\sharp f(x)|^{n/(n-1)} \le \sqrt[l]{\mathscr{K}_l(x)}\, J(x,f); \tag{88}$$

see our estimates at (49). Fix $s \in [1, n/(n-1))$. Hölder's inequality and the assumption (87) yield

$$\|D^\sharp f\|_s^{n/(n-1)} \le \big\|\sqrt[l]{\mathscr{K}_l}\, J\big\|_{(n-1)s/n} \le C_s(n,\mathscr{A},B)\int_\Omega J(x,f)\, dx. \tag{89}$$

Because of the uniform bound at (89) we see that the set of cofactor matrices is bounded in $L^q(\Omega,\mathbf{R}^n)$ for $q \in (n^2-n/(n^2-n-1), n/(n-1))$. Using this for a weakly in $W^{1,n-1}(\Omega,\mathbf{R}^n)$ (and so strongly for a subsequence in $L^s(\Omega,\mathbf{R}^n)$ for all $s \in [1, n^2-n)$) converging sequence $\{f_\nu\}$ of mappings in $\mathscr{F}$ we have that the distributional Jacobians $\mathscr{J}_{f_\nu}$ converge to $\mathscr{J}_f$ in $\mathscr{D}'(\Omega)$ i.e.

$$\mathscr{J}_f[\varphi] = \lim_{\nu\to\infty} \mathscr{J}_{f_\nu}[\varphi] = \lim_{\nu\to\infty}\int_\Omega \varphi(x) J(x,f_\nu)\, dx, \tag{90}$$

for $\varphi \in C_0^\infty(\Omega)$. Here we have passed to a subsequence. Each mapping in the sequence $\{f_\nu\}$ obeys the rule of integration by parts i.e.

$$\int_\Omega \eta(x) J(x,f)\,dx = \mathscr{J}_f[\eta]$$

for every $\eta \in C_0^\infty(\Omega)$ and $i = 1,2,\dots,n$. Integration by parts follows from Theorem 1.3 in [3]. Here we used equations (32), (29) and Lemma 2.3. Therefore, if we take a non-negative test function φ, say $\varphi(x) = \Phi_t(a-x)$ from Section 5, $\mathscr{J}_f[\varphi] \ge 0$. We see that $|D^\sharp f| \in L^q(\Omega)$, where $q \in (n^2-n/(n^2-n-1), n/(n-1))$, by (89), and so the assumptions of Lemma 4.1 are fulfilled and thus we have

$$J(a,f) = \lim_{t\to 0} \mathscr{J}_f[\varphi] \ge 0$$

for almost every $a \in \Omega$. Therefore, f is an orientation preserving map i.e. $J(x,f) \ge 0$ almost everywhere in Ω.

Next we want to show that (90) remains valid for any bounded function $\varphi \in L^\infty(\Omega)$ with compact support (see [8, Theorem 8.4.2]). Of course we need only consider test functions φ satisfying the bound

$$|\varphi(x)| \le \chi_Q(x) \tag{91}$$

where χ_Q is the characteristic function of a cube Q and $2nQ \subset \Omega$. Using (32) we see that the Orlicz function P satisfies (59). By (29) and Lemma 2.3 we have that also the technical condition (60) is fulfilled and so by Theorem 6.1 we conclude

$$\|\det Df_\nu\|_{L^\Psi(2Q)} \le C_P(n) \big\| |D^\sharp f_\nu|^{n/(n-1)} \big\|_{L^P(2nQ)} \le M \tag{92}$$

with M independent of ν and Ψ given by (65), by Proposition 2.2. This translates into the integral estimate

$$\int_{2Q} \Psi\left(\frac{J(x,f_\nu)}{M}\right) dx \le 1 \tag{93}$$

by definition (18). By (65), we have

$$t \le \Psi(t) \left(\int_0^t \frac{P(s)}{s^2}\,ds \right)^{-1}. \tag{94}$$

We mollify φ by convolution with Φ_t, where t is chosen to be so small that $\varphi_t \in C_0^\infty(2Q)$. For given $k \ge 1$ we have by (94) that

$$\begin{aligned} \int_{2Q} |\varphi_t(x) - \varphi(x)| J(x,f_\nu) &\le \int_{\{x\in 2Q: J(x,f_\nu)\le Mk\}} |\varphi_t(x)-\varphi(x)| J(x,f_\nu)\,dx \\ &\quad + 2\int_{\{x\in 2Q: J(x,f_\nu)\ge Mk\}} J(x,f_\nu)\,dx \\ &\le Mk\|\varphi_t - \varphi\|_{L^1(2Q)} + 2M \left(\int_1^k \frac{P(s)}{s^2} \right)^{-1}. \end{aligned} \tag{95}$$

Combining (95) and (90) we conclude with

$$\lim_{\nu\to\infty}\int_\Omega J(x,f_\nu)\chi_Q(x)\,dx = \int_\Omega J(x,f)\chi_Q(x)\,dx \tag{96}$$

as desired.

Next we will prove the critical lower semicontinuity property

$$\int_\Omega \eta(x)e^{\mathscr{A}(\sqrt[l]{\mathscr{K}_l(x,f)})}\,dx \le \liminf_{\nu\to\infty}\int_\Omega \eta(x)e^{\mathscr{A}(\sqrt[l]{\mathscr{K}_l(x,f_\nu)})}\,dx \tag{97}$$

for each non-negative test function $\eta\in L^\infty(\Omega)$ with compact support (see [8, Theorem 8.10.1]). Fix $\varepsilon>0$ and write

$$\mathscr{K}_l^\varepsilon(x,f) = \frac{\left|\bigwedge^l f(x)\right|^n}{\big(\varepsilon+J(x,f)\big)^l}.$$

Since the function $g(x,y)=x^n y^{-l}$ is convex on $(0,\infty)\times(0,\infty)$ whenever $n\ge l+1\ge 1$ [8, Lemma 8.8.2] and $t\to e^{\mathscr{A}(\sqrt[l]{t})}$ is increasing and convex, for all $t\ge 1$ we have

$$\begin{aligned} e^{\mathscr{A}(\sqrt[l]{\mathscr{K}_l^\varepsilon(x,f_\nu)})} - e^{\mathscr{A}(\sqrt[l]{\mathscr{K}_l^\varepsilon(x,f)})} &\ge \frac{1}{l}\frac{\sqrt[l]{\mathscr{K}_l^\varepsilon(x,f)}}{\mathscr{K}_l^\varepsilon(x,f)} e^{\mathscr{A}(\sqrt[l]{\mathscr{K}_l^\varepsilon(x,f)})}\mathscr{A}'\Big(\sqrt[l]{\mathscr{K}_l^\varepsilon(x,f)}\Big)\\ &\quad\times\bigg[-l\bigg(\frac{\left|\bigwedge^l f\right|^n}{\big(\varepsilon+J(x,f)\big)^{l+1}}\bigg)\big(J(x,f_\nu)-J(x,f)\big)\\ &\quad + n\bigg(\frac{\left|\bigwedge^l f\right|^{n-1}}{(\varepsilon+J(x,f))^l}\bigg)\times\Big(\big|\textstyle\bigwedge^l f_\nu\big|-\big|\bigwedge^l f\big|\Big)\bigg]. \end{aligned} \tag{98}$$

Set

$$\begin{aligned} E_k = \bigg\{&\bigg|\frac{\sqrt[l]{\mathscr{K}_l^\varepsilon(x,f)}}{\mathscr{K}_l^\varepsilon(x,f)}e^{\mathscr{A}(\sqrt[l]{\mathscr{K}_l^\varepsilon(x,f)})}\mathscr{A}'\Big(\sqrt[l]{\mathscr{K}_l^\varepsilon(x,f)}\Big)\bigg(\frac{\left|\bigwedge^l f\right|^n}{(\varepsilon+J(x,f))^{l+1}}\bigg)\bigg|\le k\\ &\text{and}\\ &\bigg|\frac{\sqrt[l]{\mathscr{K}_l^\varepsilon(x,f)}}{\mathscr{K}_l^\varepsilon(x,f)}e^{\mathscr{A}(\sqrt[l]{\mathscr{K}_l^\varepsilon(x,f)})}\mathscr{A}'\Big(\sqrt[l]{\mathscr{K}_l^\varepsilon(x,f)}\Big)\bigg(\frac{\left|\bigwedge^l f\right|^{n-1}}{(\varepsilon+J(x,f))^{l}}\bigg)\bigg|\le k\bigg\} \end{aligned}$$

for all $k\in\mathbf{N}$. Fix $k\in\mathbf{N}$.

We now consider what happens in (98) when we first multiply (98) by a non-negative $\eta\in L^\infty_{\mathrm{loc}}(\Omega)$ and χ_{E_k}, then integrate over Ω and let $\nu\to\infty$. The first term in the right-hand side converges to 0, because $J(\,\cdot\,,f_\nu)$ converges to $J(\,\cdot\,,f)$

weakly in $L^1_{\mathrm{loc}}(\Omega)$, by (96). We fix unit matrix fields $\xi=\xi(x)$ and $\zeta=\zeta(x)$, each valued in the space $\binom{n}{l}\times\binom{n}{l}$-matrices such that

$$|\textstyle\bigwedge^l f(x)| = \left\langle \textstyle\bigwedge^l f(x)\xi,\zeta\right\rangle.$$

Thus

$$|\textstyle\bigwedge^l f_\nu(x)| - |\textstyle\bigwedge^l f(x)| \ge \left\langle \textstyle\bigwedge^l f_\nu(x) - \textstyle\bigwedge^l f(x), \xi\otimes\zeta\right\rangle. \tag{99}$$

Using the assumption that $\{f_\nu\}$ converges weakly in $W^{1,s}(\Omega,\mathbf{R}^n)$ where $s>l$ we have that $\bigwedge^l f_\nu$ converges to $\bigwedge^l f$ weakly in $L^1_{\mathrm{loc}}(\Omega,\mathbf{R}^{n\times n})$ and $|\eta(x)|\xi\otimes\zeta$ belongs to the space $L^\infty_{\mathrm{loc}}(\Omega)$ we have

$$\begin{aligned}\int_\Omega \eta(x)\chi_{E_k}(x)e^{\mathscr{A}(\sqrt[l]{\mathscr{K}_l^\varepsilon(x,f)})}\,dx &\le \lim_{\nu\to\infty}\inf\int_\Omega \eta(x)\chi_{E_k}(x)e^{\mathscr{A}(\sqrt[l]{\mathscr{K}_l^\varepsilon(x,f_\nu)})}\,dx\\ &\le \lim_{\nu\to\infty}\inf\int_\Omega \eta(x)e^{\mathscr{A}(\sqrt[l]{\mathscr{K}_l(x,f_\nu)})}\,dx.\end{aligned} \tag{100}$$

Letting $\varepsilon\to 0$ and $k\to\infty$ we have (97). Combining this and (96) we complete the proof of Theorem 1.1.

Proof of Theorem 1.2. Theorem 1.2 is proven in the case $n=2$ in [8] and so we will assume that $n\ge 3$. The uniform bound at (i) is rather simple. To see this we notice that

$$K_O(x) \le \left[\sqrt[l]{\mathscr{K}_l(x)}\,\right]^{(n-1)^2} \tag{101}$$

see our estimates at (49) and (50); notice that $Df(x)=0$ a.e. in the set $\{x\in\Omega : J(x,f)=0\}$. Hölder's inequality yields

$$\begin{aligned}\|Df\|^n_{L^p(\Omega)} &\le \|K_O J\|_{p/n} \le \|K_O\|_{p/(n-p)}\|J\|_1\\ &\le \big\|\sqrt[l]{\mathscr{K}_l}\big\|^{(n-1)^2}_{p(n-1)^2/(n-p)}\int_\Omega J(x,f)\,dx \le C_p(n,B)\int_\Omega J(x,f)\,dx,\end{aligned} \tag{102}$$

the factor in front of the volume integral being finite due to much stronger assumption at (15). This follows directly from the assumption (9). We now proceed to Assertion (ii). Let $f_\nu\colon\Omega\to\mathbf{R}^n$ be a sequence of mappings in the family $\mathscr{F}$, converging weakly in $W^{1,p}(\Omega,\mathbf{R}^n)$ to a mapping f. We want to show that $f\in\mathscr{F}$. The conditions (14) and (15) for the limit map f follow from Theorem 1.1 and so it suffices to show that f has a finite outer distortion. The proof is very similar to that for inequality (15): Since the function $g(x,y)=x^ny^{-1}$ is convex on $(0,\infty)\times(0,\infty)$ [8, Lemma 8.8.2] we have for each $\varepsilon>0$

$$\begin{aligned}\frac{|Df_\nu(x)|^n}{\varepsilon+J(x,f_\nu)} - \frac{|Df(x)|^n}{\varepsilon+J(x,f)} &\ge -\left(\frac{|Df|^n}{\big(\varepsilon+J(x,f)\big)^2}\right)\big(J(x,f_\nu)-J(x,f)\big)\\ &\quad + n\left(\frac{|Df|^{n-1}}{\varepsilon+J(x,f)}\right)(|Df_\nu|-|Df|).\end{aligned}$$

Because of the uniform bound at (102) the sequence $\{f_\nu\}$ actually converges weakly in $W^{1,s}(\Omega,\mathbf{R}^n)$ for every $1 \le s < n$ and by the analysis as in (91)–(96) we have that $J(\,\cdot\,,f_\nu)$ converges to $J(\,\cdot\,,f)$ weakly in $L^1_{\rm loc}(\Omega)$. Furthermore observing that Df_ν converges weakly to Df in $L^1_{\rm loc}(\Omega,\mathbf{R}^{n\times n})$, we have (see (100))

$$\int_\Omega \eta(x)\frac{|Df(x)|^n}{\varepsilon+J(x,f)}\,dx \le \liminf_{\nu\to\infty}\int_\Omega \eta(x)\frac{|Df_\nu(x)|^n}{\varepsilon+J(x,f_\nu)}\,dx$$

for every $\eta\in L^\infty_{\rm loc}(\Omega)$. It then follows that

$$|Df(x)|^n \le M(x)J(x,f) \qquad \text{a.e.}$$

for some measurable function $1\le M(x)<\infty$ and this completes the proof of Theorem 1.2.

Proof of Theorem 1.3. We begin the construction of the example of Theorem 1.3 by recalling the existence of suitable Lipschitz functions.

Lemma 7.1. *Let $\varepsilon>0$. Then there exists a Lipschitz function $u\colon Q_1=[-1,1]^n\to[-1,1]$, smooth outside a closed set of measure zero, such that $u=0$ on the boundary of the unit cube Q_1, the alternative*

$$|\nabla u(x)-e_1|<\varepsilon \qquad \text{or} \qquad |\nabla u(x)+e_1|<\varepsilon \tag{103}$$

holds for a.e. $x\in Q_1$, where $e_1=(1,0,\dots,0)$ as usually.

Proof. See the work of Fonseca, Müller and Šverak [17, Lemma 5.1] and references therein.

The idea of the following lemma comes from [13].

Lemma 7.2. *Suppose that $\mathscr{B}$ is a strictly increasing non-negative function and*

$$\int_1^\infty \frac{\mathscr{B}(s)}{s^2}<\infty.$$

Given $0\le t\le t$, let $Q_t=[-t,t]\times[-1,1]^{n-1}$. There is a continuous mapping $f_t\colon Q_t\to\mathbf{R}^n$ such that $f_t(x)=(0,x_2,\dots,x_n)$ on the boundary of the set Q_t,

(1) *$f\in W^{1,p}(Q_t,\mathbf{R}^n)$ and $\int_{Q_t}|Df|^p\le Ct$, for all $p\in[1,n)$,*

(2) *$\int_{Q_t}J(x,f_t)\le Ct$,*

(3) *f_t has finite distortion in int(Q_t), with $\int_{Q_t}e^{\mathscr{B}\left(K_O(x,f_t)\right)}\,dx\le Ct$,*
and

(4) *$|f_t(x)-f_t(y)|\le Ct$ if $|x-y|\le t$.*

Proof. Given $\varepsilon > 0$, using Lemma 7.1 we construct a mapping

$$g(x_1, \dots, x_n) = \bigl(u(x_1, \dots, x_n), x_2, \dots, x_n\bigr)$$

for all $x = (x_1, \dots, x_n) \in Q_t$ so that

$$g(Q_t) \subset Q_t \tag{104}$$

$$g(x) = (0, x_2, \dots, x_n) \qquad \text{on } \partial Q_t$$

and for a.e. $x \in Q_t$ we have

$$|\nabla u(x) - e_1| < \varepsilon \qquad \text{or} \qquad |\nabla u(x) + e_1| < \varepsilon.$$

Set

$$Q_t^+ = \{x \in Q_t : J(x, g) > 0\}, \qquad Q_t^- = \{x \in Q_t : J(x, g) < 0\}.$$

By choosing a suitably small $\varepsilon > 0$ we may assume that a.e.

$$|Dg(x)| \le 2 \tag{105}$$

and either

$$\tfrac{1}{2} \le J(x, g) \le \tfrac{3}{2} \tag{106}$$

or

$$-\tfrac{3}{2} \le J(x, g) \le -\tfrac{1}{2}. \tag{107}$$

We define

$$f_t(x) = g(x), \qquad x \notin Q_t^-.$$

On Q_t^-, which we may assume to be open, we will precompose g with suitable mappings with negative Jacobians so as to obtain a mapping with a positive Jacobian. For this we recall the mapping constructed in [12, Theorem 1.2(b)]: Under the assumptions of Lemma 7.2 there is a continuous mapping $h \in W^{1,p}(Q_1, [-M, M]^n)$ for some $M > 1$ and all $p \in [1, n)$ so that there does not exist a set $A \subset Q_1$ of measure zero such that $|h(A)| > 0$,

$$\begin{aligned} h(x) &= x && \text{on } \partial Q_1, \\ J(x, h) &< 0 && \text{a.e in } Q_1, \\ \int_{Q_1} |J(x, h)| \, dx &< \infty \end{aligned}$$

and h satisfies

$$|Dh(x)|^n \le K(x)|J(x,h)| \quad \text{a.e.}$$

with

$$\int_{Q_1} e^{\mathscr{B}(K(x))}\,dx < \infty.$$

Here we used the following fact: If $\mathscr{B}$ is an increasing non-negative function such that

$$\int_1^\infty \frac{\mathscr{B}(s)}{s^2}\,ds < \infty, \qquad \text{then} \qquad \int_1^\infty \frac{\mathscr{B}'(s)}{s}\,ds < \infty.$$

This follows from integration by parts and from a minor estimate.

Now we decompose Q_t^- into pairwise disjoint cubes Q so that $M\overline{\mathrm{Q}} \subset Q_t^-$ for each cube in this decomposition $\mathscr{W}$. Using the above mapping h we find, for each Q, a mapping

$$h_Q\colon \overline{\mathrm{Q}} \to M\overline{\mathrm{Q}} \tag{108}$$

such that $h_Q(x) = x$ on ∂Q, $J(x,h_Q) < 0$ a.e., there does not exist a set $A \subset \overline{\mathrm{Q}}$ of measure zero such that $|h_Q(A)| > 0$,

$$\int_Q |J(x,h_Q)|\,dx \le \int_{Q_1} |J(x,h)|\,dx \cdot |Q|,$$
$$\int_Q |Dh_Q(x)|^p\,dx \le \int_{Q_1} |Dh(x)|^p\,dx \cdot |Q|$$

for all $p \in [1,n)$ and h_Q satisfies $|Dh_Q(x)|^n \le K_Q(x)|J(x,h_Q)|$ a.e. in Q with

$$\int_Q e^{\mathscr{B}(K_Q(x))}\,dx \le \int_{Q_1} e^{\mathscr{B}(K(x))}\,dx \cdot |Q|. \tag{109}$$

We define

$$f_t(x) = u\bigl(h_Q(x)\bigr), \qquad x \in \overline{\mathrm{Q}}$$

when $Q \in \mathscr{W}$. To see that f_t has the desired properties, notice that the outer distortion of f_t on Q is, by (105) and (107), no more that 2^{n+1} times the "outer distortion" of h_Q on Q. The desired integrability condition on the outer distortion of f_t then follows using the fact that there does not exist a set $A \subset \overline{\mathrm{Q}}$ of measure zero such that $|h_Q(A)| > 0$ and applying (109) in Q_t^- and (105), (106) in $Q_t \setminus Q_t^-$. The construction shows that f_t is continuous. Finally condition (4) follows from (104) and (108).

Proof of Theorem 1.3. Consider a fixed Cantor-set construction on $[-1,1]$. At stage j of the construction, $[-1,1]$ is divided into a finite number of intervals, some of which are being removed at this stage and some of which were removed in a previous step. Let $[a,b]$ be an interval in the subdivision of $[-1,1]$. We define

$$F_j(x) = f_{(b-a)/2}\big(x_1 - \tfrac{1}{2}(a+b), x_2, \dots, x_n\big)$$

for $x \in [a,b] \times [-1,1]^{n-1}$, where $f_{(b-a)/2}$ is the map from Lemma 7.2. Because of the boundary values of the maps f_t in Lemma 7.2, this procedure gives a consistent definition of F_j in all of Q_1 and it follows (using (1)) that

$$F_j \in W^{1,p}(Q_1, \mathbf{R}^n)$$

for all $p \in [1,n)$. From (1)–(3) we further conclude that F_j has finite distortion and that

$$\int_{Q_1} |DF_j|^p + J(x, F_j) + e^{\mathscr{B}(K_O(x,F_J))} \le C.$$

Notice also that F_k coincides with F_j for $k \ge j$ on each of the intervals that was removed either earlier or at stage j. Select now the Cantor-construction so that the resulting Cantor-set E_1 has positive length. As mentioned above, $F_k(x) = F_j(x)$ for all sufficiently large j and k when x is not in $E_1 \times [-1,1]^{n-1}$. Using (4) and the boundary values of the maps f_t on Q_t, it follows that the limit

$$\lim_{j\to\infty} F_j(x) = (0, x_2, \dots, x_n)$$

exists for $x \in E_1 \times [-1,1]^{n-1}$. We conclude that $F_j(x) \to F(x)$ for all $x \in Q_1$, where $F \in W^{1,p}(Q_1, \mathbf{R}^n)$, for all $p \in [1,n)$, is continuous and satisfies

$$F(x) = (0, x_2, \dots, x_n)$$

for all $x \in E_1 \times [-1,1]^{n-1}$. Set $E = E_1 \times (-1,1)^n$. It is a straightforward computation to find that $\big|\bigwedge^l F(x)\big| = 1$ and $|J(x,f)| = 0$ for a.e. $x \in E$.

8. Uniform convergence

Theorems 1.1 and 1.2 deal with weak convergence. Our final result shows that, for mappings of finite distortion, also locally uniform convergence can be used. For related results in the homeomorphic case see [2], [18], and [19].

Theorem 8.1. *Assume that an Orlicz function* $\mathscr{A}$ *satisfies* (8), (9) *and* (10) $(l=1)$, $n \ge 2$, *and let* $A, B > 0$. *Let* $\mathscr{F}$ *be the family of mappings* $f\colon \Omega \to \mathbf{R}^n$ *of finite distortion such that*

$$\int_\Omega J(x,f)\,dx \le A \tag{110}$$

and

$$\int_\Omega e^{\mathscr{A}\left(K_O(x)\right)}\,dx \le B. \tag{111}$$

Fix some $x_0 \in \Omega$ and define $\widetilde{\mathscr{F}} = \{g : g(x) = f(x) - f(x_0) \text{ and } f \in \mathscr{F}\}$. Then each sequence of mappings in $\widetilde{\mathscr{F}}$ contains a locally uniformly converging subsequence, and the limit of any such a sequence belongs to $\widetilde{\mathscr{F}}$.

Proof. Notice first that, by Theorem 1.2, $\mathscr{F}$ is bounded in each $W^{1,p}(\Omega', \mathbf{R}^n)$, $1 \le p < n$, when Ω' is compactly contained in Ω. Moreover, by Hölder's inequality at (40) we find that

$$\big\| |Df|^n \big\|_{L^P(\Omega)} \le C(\mathscr{A}) \|K_O\|_{\mathrm{Exp}\,\mathscr{A}(\Omega)} \int_\Omega J(x,f)\,dx. \tag{112}$$

Define $\Phi(t) = P(t^n)$, for all $t \ge 0$. Then it follows from (32) and Lemma 2.4 that $\int_1^\infty \Phi(t)/t^{n+1} = \infty$ and the function $\tau \to \Phi\big(\sqrt[p]{\tau}\big)$ is convex for some $p > n-1$. From [7, Theorem 1.6] we can conclude that each $f \in \mathscr{F}$ is continuous (the continuity means the existence of a continuous representative) with the uniform bound

$$|f(x) - f(y)| \le C(n,R) \|Df\|_{L^\Phi(B)} \omega_\Phi\left(\frac{|x-y|}{R}\right),$$

where $x, y \in B(a,R) \subset B(a,2R) \subset \Omega$ and $\omega = \omega_\Phi(t)$, $0 < t \le 1$, is determined uniquely from the equation $1 = \int_t^1 \Phi(\Phi/s)\,ds$, provided that f is weakly monotone, see [7]. This additional assumption is automatically guaranteed because we can integrate by parts against the Jacobian (see [3, Theorem 1.3]), and thus the argument used in [7, Section 4] based on Stokes' theorem applies also in our setting. Combining Theorem 1.2 with Ascoli's theorem and the equicontinuity property of the family $\mathscr{F}$ the claim follows.

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