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ON THE Γ-CONVERGENCE OF LAPLACE–BELTRAMI OPERATORS IN THE PLANE

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Abstract. We show here that if f_h is a sequence of mappings of finite distortion K_h, uniformly bounded in some exponential norm, weakly converging to f in $W^{1,2}(\Omega)$, $\Omega \subset \mathbf{R}^2$, then the matrices $A(x, f_h)$ in the Beltrami operators associated to each f_h, Γ-converge, in the sense of De Giorgi, to the matrix $A(x,f)$ in the Beltrami operator associated to f.

1. Introduction

For Ω an open subset of $\mathbf{R}^2$ we shall study mappings $f = (f^1, f^2)\colon \Omega \to \mathbf{R}^2$ in the Sobolev space $W^{1,2}(\Omega, \mathbf{R}^2)$. We say that f has *finite distortion* if

$$|Df(x)|^2 \le \mathscr{K}(x) J(x,f) \qquad \text{a.e.} \tag{1.1}$$

Here $|Df(x)|$ stands for the Hilbert–Schmidt norm of the differential matrix $Df(x) \in \mathbf{R}^{2\times 2}$ and $J(x,f) = \det Df(x)$. That is,

$$|Df(x)|^2 = \sum_{i,j=1}^{2} \left| \frac{\partial f^i}{\partial x_j} \right|^2 \qquad \text{and} \qquad J(x,f) = \frac{\partial f^1}{\partial x_1}\frac{\partial f^2}{\partial x_2} - \frac{\partial f^1}{\partial x_2}\frac{\partial f^2}{\partial x_1}.$$

The function $\mathscr{K} = \mathscr{K}(x)$ is assumed to be measurable with values in the interval $[2, \infty)$. It will be advantageous to write $\mathscr{K}$ as

$$\mathscr{K}(x) = K(x) + \frac{1}{K(x)} \qquad \text{where} \quad 1 \le K(x) < \infty. \tag{1.2}$$

We refer to the smallest such $K(x)$ for which (1.1) holds as the *distortion function* of f. If $K(x)$ is bounded by a constant, say $1 \le K(x) \le K$ a.e., then we say that f is K-quasiregular. An important quantity associated to a mapping with finite distortion is the so called *distortion tensor* $G(\,\cdot\,, f)\colon \Omega \to \mathbf{R}^{2\times 2}$, defined by

$$G(x,f) = \begin{cases} \dfrac{D^t f(x) Df(x)}{J(x,f)} & \text{if } J(x,f) \ne 0, \\ I & \text{if } J(x,f) = 0, \end{cases} \tag{1.3}$$

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where $D^t f(x)$ stands for the transposed differential.

The distortion inequality (1.1) reads as

$$\frac{|\xi|^2}{K(x)} \le \langle G(x,f)\xi,\xi\rangle \le K(x)|\xi|^2 \tag{1.4}$$

and we have $\det G(x,f)=1$ a.e.

The symmetric matrix function $G(\,\cdot\,,f)$ can be viewed as a Riemannian metric on Ω, the pullback of the Euclidean structure via the mapping f. It is obvious that f is conformal with respect to this new metric. This raises an important question: how does $G(\,\cdot\,,f)$ change with f? We are particularly concerned with the continuity property of the map $f\to G(\,\cdot\,,f)$, since many constructions in quasiconformal geometry and elliptic PDE's rely on limiting processes. The natural convergence of the mapping $f_h\colon\Omega\to\mathbf{R}^2$ with finite distortion is that of the weak topology in $W^{1,2}(\Omega,\mathbf{R}^2)$. This, however, does not guarantee convergence of the matrices $G(x,f_h)$ to $G(x,f)$ in any familiar sense (compare with Example 6.1 here and also [LV]). Note that the condition $\det G(x,f_h)=1$ is not necessarily preserved under the weak convergence of $G(x,f)$.

S. Spagnolo [S2] first realized that the proper way to overcome this difficulty is by considering the Γ-convergence of the inverse matrices

$$A(x,f)=G(x,f)^{-1}.$$

This matrix clearly verifies the bounds at (1.4) as well. See Section 3 for the definition of Γ-convergence.

Spagnolo's result dealt with the special case of K-quasiregular mappings in which $A(x,f)$ were bounded and uniformly elliptic matrices. In that case Γ-convergence is equivalent to the L^2-convergence of solutions of the Dirichlet problem. More precisely, given a sequence $\{A_h\}$ of 2×2 matrices satisfying

$$\frac{|\xi|^2}{K} \le \langle A_h(x)\xi,\xi\rangle \le K|\xi|^2, \qquad K\ge 1,$$

we consider the elliptic operators on a bounded open set $\Omega\subset\mathbf{R}^2$

$$\mathscr{L}_h=\operatorname{div}[A_h(x)\nabla]\colon W_0^{1,2}(\Omega)\to W^{-1,2}(\Omega).$$

They are certainly invertible. Following [S1], we say that $\{A_h\}$ Γ-converges to A if for every $\varphi\in W^{-1,2}(\Omega)$, $\mathscr{L}_h^{-1}(\varphi)\rightharpoonup\mathscr{L}^{-1}(\varphi)$ in $L^2(\Omega)$, where $\mathscr{L}=\operatorname{div}[A(x)\nabla]$. Later these results were generalized to the n-dimensional case by [DD].

In the present paper we extend Spagnolo's result to sequences of mappings with pointwise unbounded distortion. Our only assumption will be that the distortion functions stay bounded in the EXP_α class for a certain $\alpha>1$, see Section 2, for the definitions.

The main result is as follows (see Section 5):

Theorem. *Let f_h converge weakly in $W^{1,2}(\Omega, \mathbf{R}^2)$ to a mapping f, and suppose that their distortion functions K_h converge to K weakly in $L^1(\Omega)$ and satisfy*

$$\int_\Omega \exp\left(\frac{K_h(x)}{\lambda}\right)^\alpha dx \le c$$

for some $\alpha > 1$, $\lambda > 0$ and $c > 0$. Then f has distortion K and

$$A(x, f_h) \xrightarrow{\Gamma_\alpha} A(x, f).$$

For the notion of Γ_α-convergence, we refer to the definition in Section 3.

In Section 6 we will relate our results to some known convergence theorems for quasiregular mappings [GMRV], [IK], [Bo].

2. Some Orlicz spaces

Let Ω be a bounded open set in $\mathbf{R}^n$. An Orlicz function is a nonnegative continuosly increasing function $P\colon \mathbf{R}_+ \to \mathbf{R}_+$, verifying $P(0) = 0$ and $P(\infty) = \infty$. The Orlicz space $L^P(\Omega)$ consists of all measurable functions $\varphi\colon \Omega \to \mathbf{R}$ such that

$$\int_\Omega P(\lambda^{-1}|\varphi|) < \infty$$

for some $\lambda = \lambda(\varphi) > 0$ (see [RR]).

For $\alpha > 1$, we denote by $\mathrm{EXP}_\alpha(\Omega)$ the Orlicz space with the defining function $P(t) = \exp(t^\alpha) - 1$. It consists of all measurable functions φ on Ω such that

$$\|\varphi\|_{\mathrm{EXP}_\alpha(\Omega)} = \inf\left\{\lambda > 0 : \fint_\Omega \exp\left(\frac{|\varphi(x)|}{\lambda}\right)^\alpha dx \le 2\right\} < \infty.$$

Here

$$\fint_\Omega \psi = \frac{1}{|\Omega|}\int_\Omega \psi = \psi_\Omega,$$

and $\|\varphi\|_{\mathrm{EXP}_\alpha(\Omega)}$ provides a norm of φ. Another space of interest to us will be the Zygmund space $L^p \log^\beta L(\Omega)$, with $p \ge 1$ and $\beta \ge 0$, with the defining function $P(t) = t^p \log^\beta(e + t)$. It consists of all measurable functions φ on Ω such that

$$\fint_\Omega |\varphi|^p \log^\beta\left(e + \frac{|\varphi|}{|\varphi|_\Omega}\right) dx < \infty.$$

Observe that both are Banach spaces and $\mathrm{EXP}_\alpha(\Omega)$ is the dual to $L^1 \log^\beta L$, when $\beta = 1/\alpha$.

The Luxemburg norm of a function $\varphi \in L^p \log^\beta L(\Omega)$ is given by

$$\|\varphi\|_{L^p \log^\beta L(\Omega)} = \inf\left\{\lambda > 0 : \fint_\Omega \left(\frac{|\varphi|}{\lambda}\right)^p \log^\beta\left(e + \frac{|\varphi|}{\lambda}\right) dx \le 1\right\}.$$

Proposition 2.1 (Generalized Hölder inequality). *Let* $\alpha \geq 1$. *Let* $K(x) \in \mathrm{EXP}_\alpha(\Omega)$, $\varphi \in L^2 \log^{1/\alpha} L$, *and* $\psi \in L^2 \log^{1/\alpha} L$. *Then*

$$\left| \int_\Omega K(x)\varphi(x)\psi(x) \right| dx \leq c\|K\|_{\mathrm{EXP}_\alpha} \|\varphi\|_{L^2 \log^{1/\alpha} L} \|\psi\|_{L^2 \log^{1/\alpha} L}.$$

For $P(t) = t^2 \log^\beta(e+t)$ we denote by $W^{1,P}(\Omega)$ the Orlicz–Sobolev space of functions $\varphi \in L^2 \log^\beta L$ whose gradient belongs to the Zygmund space $L^2 \log^\beta L$. We supply this space with the norm

$$\|\varphi\|_{W^{1,P}(\Omega)} = \|\varphi\|_{L^2 \log^\beta L(\Omega)} + \|\nabla\varphi\|_{L^2 \log^\beta L(\Omega)}. \tag{2.1}$$

3. The Γ-convergence

We denote by $\mathbf{R}_+^{2\times 2}$ the set of symmetric 2×2 matrices A, such that $\langle A\xi, \xi\rangle \geq 0$ for all $\xi \in \mathbf{R}^2$. Consider measurable functions $A\colon \Omega \to \mathbf{R}_+^{2\times 2}$ on $\Omega \subset \mathbf{R}^2$ satisfying

$$\frac{|\xi|^2}{K(x)} \leq \langle A(x)\xi, \xi\rangle \leq K(x)|\xi|^2 \tag{3.1}$$

for some $1 \leq K(x) < \infty$ a.e. The smallest $K(x)$, for which the above holds, denoted by $K_A(x)$, is called the distortion function of A.

The present paper is concerned with mappings whose distortion belongs to the exponential class $\mathrm{EXP}_\alpha(\Omega)$, $1 < \alpha \leq \infty$. For the purpose of this work, we adopt the following variant of De Giorgi's notion of Γ-convergence ([DF]).

Definition 3.1. Let A and A_h $(h = 1, 2, \dots)$ be matrix functions whose distortions K_A and K_{A_h} are uniformly bounded in the norm of $\mathrm{EXP}_\alpha(\Omega)$. We say that $\{A_h\}$ Γ_α-converges to A if the following two conditions are verified:

(1) The inequality

$$\int_\Omega \langle A(x)\nabla u, \nabla u\rangle\, dx \leq \liminf_{h\to\infty} \int_\Omega \langle A_h(x)\nabla u_h, \nabla u_h\rangle\, dx \tag{3.2}$$

holds whenever $|\nabla u_h|, |\nabla u| \in L^2 \log^{1/\alpha} L(\Omega)$ and $u_h \to u$ in $L^2 \log^{1/\alpha} L$.

(2) For every $v \in L^2 \log^{1/\alpha} L(\Omega)$ with $|\nabla v| \in L^2 \log^{1/\alpha}(\Omega)$ there exists a sequence $v_h \in L^2 \log^{1/\alpha} L(\Omega)$ with $|\nabla v_h| \in L^2 \log^{1/\alpha} L$ such that $v_h \to v$ in $L^2 \log^{1/\alpha} L(\Omega)$ and

$$\int_\Omega \langle A(x)\nabla v, \nabla v\rangle = \lim_h \int_\Omega \langle A_h \nabla v_h, \nabla v_h\rangle. \tag{3.3}$$

Remark. The assumption that K_A and K_{A_h} belong to $\mathrm{EXP}_\alpha(\Omega)$ is needed to guarantee that the above integrals are finite. This follows from the inequality

$$\begin{aligned} \int_\Omega \langle A(x)\nabla u, \nabla u\rangle\, dx &\leq \int_\Omega K_A(x)|\nabla u|^2\, dx \\ &\leq c\|K_A\|_{\mathrm{EXP}_\alpha(\Omega)} \|\nabla u\|^2_{L^2 \log^{1/\alpha} L(\Omega)}. \end{aligned} \tag{3.4}$$

If one merely assumes that K_A and $K_{A_h} \in L^1$ then one must be confined to Lipschitz functions. In this case we speak of Γ-convergence. *We say that a sequence A_h of matrix functions $A_h \in L^1(\Omega, \mathbf{R}_+^{2\times 2})$ Γ-converges to A if*:

(1) *Inequality* (3.2) *holds whenever $u, u_h \in \mathrm{Lip}\,(\Omega)$ and $u_h \to u$ in $L^2(\Omega)$*;

(2) *For every $v \in \mathrm{Lip}\,(\Omega)$ one can find a sequence $v_h \in \mathrm{Lip}\,(\Omega)$ converging to v in $L^2(\Omega)$ satisfying* (3.3).

Actually, by the general properties of Γ-convergence, conditions (1) and (2) remain true if we replace Ω by any of its open subsets.

We report here the fundamental compactness result concerning Γ-convergence [MS].

Theorem 3.1. *Let A_h be a sequence of symmetric 2×2 matrices satisfying*

$$0 \le \langle A_h(x)\xi, \xi\rangle \le K_h(x)|\xi|^2 \quad \text{for a.e. } x \in \Omega \text{ and } \xi \in \mathbf{R}^2.$$

Assume that $K_h \rightharpoonup K$ weakly in $L^1(\Omega)$. Then there exists a subsequence A_{h_r} Γ-converging to a symmetric matrix A. Moreover, this matrix A also satisfies

$$0 \le \langle A(x)\xi, \xi\rangle \le K(x)|\xi|^2.$$

In this connection it is appropriate to mention another important notion of convergence of matrix functions $A_h\colon \Omega \to \mathbf{R}_+^{2\times 2}$, the so-called G-convergence. For simplicity we confine ourselves to bounded domains and to sequences such that

$$1 \le K_{A_h}(x) \le K \qquad \text{a.e.} \tag{3.5}$$

for $h = 1, 2, \dots$, and

$$1 \le K_A(x) \le K \qquad \text{a.e.}$$

We recall from the introduction the elliptic operators and their inverse

$$\mathscr{L}_h = \operatorname{div}[A_h(x)\nabla]\colon W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega), \quad \mathscr{L}_h^{-1}\colon W^{-1,2}(\Omega) \to W_0^{1,2}(\Omega),$$
$$\mathscr{L} = \operatorname{div}[A(x)\nabla]\colon W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega), \quad \mathscr{L}^{-1} : W^{-1,2}(\Omega) \to W_0^{1,2}(\Omega).$$

Following Spagnolo [S1], $\{A_h\}$ G-converges to A if $\mathscr{L}_h^{-1}(\varphi) \rightharpoonup \mathscr{L}^{-1}(\varphi)$ weakly in $W_0^{1,2}(\Omega)$, for every $\varphi \in W^{-1,2}(\Omega)$. We emphasize that under condition (3.5) all the above notions of convergence are equivalent, though we shall not pursue this matter here, see [MS].

4. Mappings of finite distortion and the Laplace–Beltrami operators

Let Ω be a bounded open set in $\mathbf{R}^2$ and $f=(f^1,f^2)\in W^{1,2}(\Omega,\mathbf{R}^2)$ be a mapping of finite distortion $K\colon\Omega\to[1,\infty)$, i.e. satisfying, for a.e. $x\in\Omega$,

$$|Df(x)|^2\le[K(x)+K^{-1}(x)]J(x,f), \tag{4.1}$$

where $J(x,f)$ is the Jacobian determinant of f. The distortion tensor $G(x,f)$ of f at x is defined in (1.3). It is easy to check that G is a symmetric matrix with $\det G(x,f)=1$ and that (1.4) is equivalent to (4.1). In fact, for any 2×2-matrix F with $\det F>0$, we can consider

$$G=\frac{F^tF}{\det F}.$$

Then, obviously

$$\det G=1.$$

Moreover, recalling the Hilbert–Schmidt norm of F,

$$|F|^2=\operatorname{tr} F^tF$$

the distortion inequality

$$|F|^2\le\left(K+\frac{1}{K}\right)\det F$$

is equivalent to

$$\operatorname{tr} G\le K+\frac{1}{K}.$$

Let λ and $1/\lambda$ be the eigenvalues of G. Then the last inequality means that

$$\lambda+\frac{1}{\lambda}\le K+\frac{1}{K};$$

hence $1/K\le\lambda\le K$.

Now we consider the inverse matrix

$$A(x,f)=G(x,f)^{-1}$$

which obviously satisfies the ellipticity condition

$$\frac{|\xi|^2}{K(x)}\le\langle A(x,f)\xi,\xi\rangle\le K(x)|\xi|^2.$$

Connections between mappings of finite distortion and PDEs are established via the Laplace–Beltrami operator $\mathscr{L}=\operatorname{div}[A(x,f)\nabla]$. Note that the components f^i $(i=1,2)$ solve the equations

$$\begin{cases}\mathscr{L}[f^i]=0,\\ \langle A(x,f)\nabla f^i,\nabla f^j\rangle=\delta_{ij}J(x,f),\end{cases} \tag{4.2}$$

see for example [BI] and [HKM]. Planar mappings with unbounded distortion have been recently studied by [D], [IŠ] and most recently by [MM], [BJ], [RSY], [IS]. In particular in [MM] the following higher integrability result, which will be useful to us, was established.

Theorem 4.1. *If* $f \in W^{1,2}(\Omega)$ *satisfies* (4.1) *with* $K \in \mathrm{EXP}_\alpha(\Omega)$, *for certain* $\alpha > 1$, *then* $|Df|$ *belongs to* $L^2 \log^{1/\alpha} L(\Omega_1)$ *for any* $\Omega_1 \subset\subset \Omega$ *and the following inequality holds*:

$$\|Df\|_{L^2 \log^{1/\alpha} L(\Omega_1)} \le c(\Omega_1) \|K\|_{\mathrm{EXP}_\alpha(\Omega)} \|Df\|_{L^2(\Omega)}. \tag{4.3}$$

This is true in all dimensions, provided the exponent 2 is replaced by the dimension n.

In view of Hadamard's inequality

$$\langle A(x,f)\nabla f^i, \nabla f^i\rangle = J(x,f) \le \tfrac{1}{2}|Df(x)|^2,$$

we deduce by (4.3)

$$\|\langle A(x,f)\nabla f^i, \nabla f^i\rangle\|_{L^1 \log^{1/\alpha} L(\Omega_1)} \le c(\Omega_1)\|K\|_{\mathrm{EXP}_\alpha(\Omega)} \int_\Omega |Df|^2\,dx. \tag{4.4}$$

We show here that the limit mapping f of a weakly convergent sequence of mappings f_h with finite distortion also has finite distortion. Our arguments are based on the weak continuity of the Jacobian determinant [R], [Mü] and the concept of polyconvexity. General n-dimensional results of this type have been recently obtained by F.W. Gehring and T. Iwaniec in [GI]. They adopted slightly different definition of the distortion, which for $n = 2$ reduces to

$$|Df(x)|^2 \le 2K(x)J(x,f).$$

Theorem 4.2. *Let* $f_h \colon \Omega \to \mathbf{R}^2$ *be mappings of finite distortion* $K_h(x)$:

$$|Df_h(x)|^2 \le \left[K_h(x) + \frac{1}{K_h(x)}\right] J(x, f_h). \tag{4.5}$$

Assume that K_h *are integrable and converge weakly to* K *in* $L^1(\Omega)$, *while* $f_h \rightharpoonup f$ *weakly in* $W^{1,2}(\Omega, \mathbf{R}^2)$. *Then the above inequality remains valid for the limit map.*

Proof. Let us first introduce some useful notation. Set $F = (B, E)$ where the vectors B, E are defined by

$$E = \nabla f^1, \qquad B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \nabla f^2$$

and let

$$F^+ = \tfrac{1}{2}(E + B), \qquad F^- = \tfrac{1}{2}(E - B).$$

It is obvious that

$$\begin{aligned} J(x,f) = \langle B, E\rangle = |F^+|^2 - |F^-|^2 := J(F), \\ |F|^2 = 2(|F^+|^2 + |F^-|^2). \end{aligned}$$

Hence the distortion inequality

$$|F|^2 \leq \left(K + \frac{1}{K}\right) J(F)$$

is easily seen to be equivalent to

$$|F^-| \leq \frac{K-1}{K+1}|F^+|.$$

This, in turn, is equivalent to

$$\|F\|^2 \leq K J(F), \tag{4.6}$$

where we have used another norm of F defined by $\|F\| = |F^+| + |F^-|$.

Now, assume that $F_h \rightharpoonup F$ weakly in L^2 and

$$\frac{\|F_h\|^2}{J(F_h)} \leq K_h$$

with $K_h \rightharpoonup K$ weakly in L^1. The desired conclusion

$$\frac{\|F\|^2}{J(F)} \leq K \tag{4.7}$$

follows by applying the inequality

$$\frac{\|F\|^2}{J(F)} \leq \frac{\|F_h\|^2}{J(F_h)} + \frac{2\|F\|}{J(F)}(\|F\| - \|F_h\|) - \frac{\|F\|^2}{J(F)^2}[J(F) - J(F_h)]. \tag{4.8}$$

The latter is immediate from the convexity of the function $(x,y) \to x^2/y$. The well-known weak continuity property of the Jacobians [R], together with the lower semicontinuity of the norm $\|\cdot\|$, imply (4.7). Here, for simplicity, we have assumed $J(F) > 0$ and $J(F_h) > 0$. To get rid of this redundant assumption one must replace $J(F)$ by the expression $J(F) + \varepsilon\|F\|$, $J(F_h)$ and then pass to the limit as $\varepsilon \to 0$.

5. The convergence theorem

In this section we consider a sequence $f_h = (f_h^1, f_h^2) \in W^{1,2}(\Omega, \mathbf{R}^2)$ of non-constant mappings with distortion $1 \leq K_h(x) < \infty$, that is

$$|Df_h(x)|^2 \leq [K_h(x) + K_h^{-1}(x)] J(x, f_h). \tag{5.1}$$

Our basic assumptions are:

(i) *There exists* $\alpha > 1$ *and* $c_0 > 0$ *such that*

$$\|K_h\|_{\mathrm{EXP}_\alpha(\Omega)} \leq c_0 \qquad \textit{for } h = 1, 2, \dots .$$

(ii) $K_h \rightharpoonup K$ *weakly in* $L^1(\Omega)$.

(iii) $f_h \rightharpoonup f = (f^1, f^2)$ *weakly in* $W^{1,2}(\Omega, \mathbf{R}^2)$.

By virtue of Theorem 3.1 there exists a subsequence $A_r(x) = A(x, f_{h_r})$, $r = 1, 2, \dots$, such that

$$A(x, f_{h_r}) \xrightarrow{\Gamma} A(x) \tag{5.2}$$

where $A(x)$ is a symmetric matrix field satisfying

$$0 \le \langle A(x)\xi, \xi\rangle \le K(x)|\xi|^2. \tag{5.3}$$

Our aim here is to prove that $A(x)$ can be identified with $A(x, f)$, which is the inverse of the distortion tensor of f:

$$A(x, f) = [D^t f(x) Df(x)]^{-1} J(x, f). \tag{5.4}$$

As a byproduct of our proof, we improve the lower bound at (5.3)

$$K^{-1}(x)|\xi|^2 \le \langle A(x)\xi, \xi\rangle$$

and show that actually the entire sequence $\{A(x, f_h)\}$ Γ-converges to $A(x, f)$.

Theorem 5.1. *Under the above assumptions*

$$\int_{\Omega_1} \langle A(x)\nabla f^i, \nabla f^i\rangle\, dx = \lim_{r\to\infty} \int_{\Omega_1} \langle A(x, f_{h_r})\nabla f^i_{h_r}, \nabla f^i_{h_r}\rangle\, dx \tag{5.5}$$

on compact subdomains $\Omega_1 \subset \Omega$, for $i = 1, 2$.

Proof. In fact, we have

$$\begin{aligned} \int_\Omega \langle A(x, f_h)\nabla u, \nabla u\rangle &\le \int_\Omega K_h|\nabla u|^2\, dx \le c\|K_h\|_{\mathrm{EXP}_\alpha(\Omega)} \|\nabla u\|^2_{L^2 \log^{1/\alpha} L(\Omega)} \\ &\le cc_0 \|u\|^2_{W^{1, L^2 \log^{1/\alpha} L(\Omega)}}. \end{aligned} \tag{5.6}$$

It then follows that the functionals $\left(\int_\Omega \langle A(x, f_h)\nabla u, \nabla u\rangle\, dx\right)^{1/2}$ are equilipschitz in $W^{1,P}(\Omega)$ with $P(t) = t^2 \log^{1/\alpha}(e + t)$, a legitimate reason for passing from Γ-convergence to the stronger one

$$A(x, f_{h_r}) \xrightarrow{\Gamma_\alpha} A(x); \tag{5.7}$$

see [MS] for details.

For $i = 1, 2$ fixed, set for simplicity $u_r = f^i_{h_r}$ and $u = f^i$. Note that $u_r \to u$ in $L^2 \log^{1/\alpha} L(\Omega_1)$. Let now (v_r) be a sequence in $W^{1,P}(\Omega_1)$ such that $v_r \to u$ in $L^2 \log^{1/\alpha} L(\Omega_1)$ and

$$\lim_{r\to\infty} \int_{\Omega_1} \langle A(x, f_{h_r})\nabla v_r, \nabla v_r\rangle\, dx = \int_{\Omega_1} \langle A(x)\nabla u, \nabla u\rangle\, dx.$$

Let Ω' be an arbitrary compact subdomain of Ω_1 and $\varphi \in C_0^\infty(\Omega_1)$ be such that $0 \le \varphi \le 1$, $\varphi \equiv 1$ in Ω'; then for every $t \in]0,1[$

$$\begin{aligned}
&\int_{\Omega_1} \langle A(x, f_{h_r})\nabla u_r, \nabla u_r\rangle \, dx \\
&\quad \le \int_{\Omega_1} \langle A(x, f_{h_r})\nabla(\varphi v_r + (1-\varphi)u_r), \nabla(\varphi v_r + (1-\varphi)u_r)\rangle \, dx \\
&\quad = \int_{\Omega_1} \Big\langle A(x, f_{h_r})\Big\{\frac{t}{t}(\nabla\varphi)(v_r - u_r) + \frac{1-t}{1-t}(\varphi\nabla v_r + (1-\varphi)\nabla u_r)\Big\}, \\
&\qquad\qquad \Big\{\frac{t}{t}(\nabla\varphi)(v_r - u_r) + \frac{1-t}{1-t}(\varphi\nabla v_r + (1-\varphi)\nabla u_r)\Big\}\Big\rangle \, dx \\
&\quad \le t\int_{\Omega_1} \Big\langle A(x, f_{h_r})\Big\{\frac{1}{t}(\nabla\varphi)(v_r - u_r)\Big\}, \Big\{\frac{1}{t}(\nabla\varphi)(v_r - u_r)\Big\}\Big\rangle \, dx \\
&\qquad + (1-t)\int_{\Omega_1} \Big\langle A(x, f_{h_r})\Big\{\frac{1}{1-t}(\varphi\nabla v_r + (1-\varphi)\nabla u_r)\Big\}, \\
&\qquad\qquad \Big\{\frac{1}{1-t}(\varphi\nabla v_r + (1-\varphi)\nabla u_r)\Big\}\Big\rangle \\
&\quad \le \frac{1}{t}\int_{\Omega_1} K|D\varphi|^2|v_r - u_r|^2 \, dx + \frac{1}{1-t}\int_{\Omega_1} \langle A(x, f_{h_r})\nabla v_r, \nabla v_r\rangle\varphi \, dx \\
&\qquad + \frac{1}{1-t}\int_{\Omega_1} \langle A(x, f_{h_r})\nabla u_r, \nabla u_r\rangle(1-\varphi) \, dx.
\end{aligned}$$

This yields

$$\begin{aligned}
(1-t)\int_{\Omega_1} \langle A(x, f_{h_r})\nabla u_r, \nabla u_r\rangle \, dx \le{}& \frac{1-t}{t} c\|v_r - u_r\|^2_{L^2 \log^{1/\alpha} L} \cdot \|D\varphi\|^2_{L^\infty(\Omega_1)} \\
&+ \int_{\Omega_1} \langle A(x, f_{h_r})\nabla v_r, \nabla v_r\rangle\varphi \, dx \\
&+ \int_{\Omega_1} \langle A(x, f_{h_r})\nabla u_r, \nabla u_r\rangle(1-\varphi) \, dx.
\end{aligned}$$

The final estimate reads as

$$\begin{aligned}
\int_{\Omega_1} \langle A(x, f_{h_r})\nabla v_r, \nabla v_r\rangle\varphi \, dx \ge{}& \int_{\Omega_1} \langle A(x, f_{h_r})\nabla u_r, \nabla u_r\rangle(1 - t - 1 + \varphi) \, dx \\
&- \frac{1-t}{t} c\|D\varphi\|^2_{L^\infty(\Omega_1)} \cdot \|v_r - u_r\|^2_{L^2 \log^{1/\alpha} L}.
\end{aligned}$$

Now, passing to the limit as $r \to \infty$, we obtain

$$\int_{\Omega_1} \langle A(x)\nabla u, \nabla u\rangle \, dx \ge \limsup_{r\to\infty} \int_{\Omega_1} \langle A(x, f_{h_r})\nabla u_r, \nabla u_r\rangle(\varphi - t) \, dx.$$

We let the parameter t go to zero

$$\int_{\Omega_1} \langle A(x)\nabla u, \nabla u\rangle \geq \limsup_{r\to\infty} \int_{\Omega_1} \langle A(x, f_{h_r})\nabla u_r, \nabla u_r\rangle \varphi$$
$$\geq \liminf_{r\to\infty} \int_{\Omega'} \langle A(x, f_{h_r})\nabla u_r, \nabla u_r\rangle \geq \int_{\Omega'} \langle A(x)\nabla u, \nabla u\rangle.$$

Since Ω' was arbitrary, we get (5.5). □

Now we are in a position to rigorously state and prove our main result.

Theorem 5.2. *Under the conditions* (i), (ii), *and* (iii), *the limit mapping* f *is either constant or, if not, has finite distortion* $K(x)$ *and*

$$A(x, f_h) \xrightarrow{\Gamma_\alpha} A(x, f). \tag{5.8}$$

Proof. That f has finite distortion $K(x)$ was already established in Section 4. Since we wish to identify the Γ_α-limit of $A(x, f_h)$, we can assume that in (5.2) and (5.5) the convergence of the entire sequence holds.

As in the proof of Theorem 5.1, set $u_h = f_h^i$, $u = f^i$, for $i = 1, 2$ and $A_h(x) = A(x, f_h)$.

For the compact subdomain $\Omega_1 \subset \Omega$ consider step functions

$$\varphi = \sum_{j=1}^{\nu} \lambda_j \chi_{B_j}, \qquad \lambda_j \geq 0, \tag{5.9}$$

where B_j are pairwise disjoint open subsets of Ω_1 such that $|\Omega_1 \setminus \bigcup_{j=1}^{\nu} B_j| = 0$.

From (5.5) it follows that

$$\liminf_{h\to\infty} \int_{\Omega_1} \langle A_h(x)\nabla u_h, \nabla u_h\rangle \varphi \, dx \geq \int_{\Omega_1} \langle A(x)\nabla u, \nabla u\rangle \varphi \, dx. \tag{5.10}$$

Moreover, by an approximation, this also holds if φ is a nonnegative continuous function on $\overline{\Omega}_1$.

Let us now prove more, namely, that (5.10) holds as equality for every continuous function φ in $\overline{\Omega}_1$, not necessarily nonnegative.

Applying (4.4), we infer that the sequence $J(x, f_h) = \langle A_h(x)\nabla u_h, \nabla u_h\rangle$ admits a subsequence weakly converging in $L^1(\Omega_1)$ to a function $E(x)$. Thus

$$\lim_{r\to\infty} \int_{\Omega_1} \langle A_{h_r}(x)\nabla u_{h_r}(x), \nabla u_{h_r}(x)\rangle \varphi(x) \, dx = \int_{\Omega_1} E(x)\varphi(x) \, dx \tag{5.11}$$

for any $\varphi \in C^0(\overline{\Omega}_1)$. By (5.10) it follows

$$\int_{\Omega_1} \langle A(x)\nabla u, \nabla u\rangle \varphi(x) \, dx \leq \int_{\Omega_1} E(x)\varphi(x) \, dx. \tag{5.12}$$

Let S be a measurable subset of Ω_1 and let $(\varphi_k) \subset C^0(\overline{\Omega}_1)$ be such that $\varphi_k(x) \to \chi_s(x)$ a.e. in Ω_1. Then from the previous relation and the Lebesgue theorem it follows that

$$\int_S \langle A(x)\nabla u, \nabla u\rangle \leq \int_S E(x)\,dx. \tag{5.13}$$

On the other hand we deduce from (5.11) and Theorem 5.1 that

$$\int_{\Omega_1} \langle A(x)\nabla u, \nabla u\rangle\,dx = \int_{\Omega_1} E(x)\,dx. \tag{5.14}$$

Hence

$$E(x) = \langle A(x)\nabla u, \nabla u\rangle \qquad \text{a.e. in } \Omega_1.$$

Therefore, we have for the whole sequence

$$\lim_{h\to\infty} \int_{\Omega_1} \langle A(x, f_h)\nabla u_h, \nabla u_h\rangle \varphi\,dx = \int_{\Omega_1} \langle A(x)\nabla u, \nabla u\rangle \varphi\,dx \tag{5.15}$$

for every $\varphi \in C^0(\overline{\Omega}_1)$.

Now we recall from (4.2) that

$$\langle A(x, f_h)\nabla {f_h}^i(x), \nabla {f_h}^j(x)\rangle = J(x, f_h)\delta_{ij} \qquad \text{a.e. on } \Omega,\ i, j = 1, 2. \tag{5.16}$$

By the symmetry of the matrix $A(x, f_h)$, (5.15), (5.16) and the weak continuity property of Jacobian ([R]) we have

$$\begin{aligned} \int_{\Omega_1} \langle A(x)\nabla f^i, \nabla f^j\rangle \varphi\,dx &= \lim_{h\to\infty} \int_{\Omega_1} \langle A(x, f_h)\nabla {f_h}^i, \nabla {f_h}^j\rangle \varphi\,dx \\ &= \lim_{h\to\infty} \int_{\Omega_1} J(x, f_h)\delta_{ij}\varphi\,dx = \int_{\Omega_1} J(x, f)\delta_{ij}\varphi\,dx, \end{aligned} \tag{5.17}$$

where $\varphi \in C_0^\infty(\Omega_1)$, $i, j = 1, 2$. Since φ was arbitrary, it follows that

$$\langle A(x)\nabla f^i(x), \nabla f^j(x)\rangle = J(x, f)\delta_{ij} \qquad \text{a.e. in } \Omega_1,\ i, j = 1, 2, \tag{5.18}$$

and consequently, as $J(x, f)$ is a.e. positive,

$$A(x) = J(x, f)[Df(x)^t \cdot Df(x)]^{-1} \qquad \text{a.e. in } \Omega_1. \tag{5.19}$$

Since Ω_1 was arbitrary, (5.18) holds a.e. in Ω. Hence (5.8) holds . □

6. The Bers–Bojarski theorem

For the sake of brevity we will now confine ourselves to the particular case $K(x) = K \geq 1$ and relate our results to some classical convergence theorems for quasiregular mappings.

Let $G(x, f)$ be defined as in (1.3). No natural continuity result can be traced for the map

$$f \to G(x, f) \tag{6.1}$$

of the type obtained in the present paper for the map

$$f \to A(x, f)$$

unless we consider a convergence $f_h \to f$ stronger than weak-$W^{1,2}$; see also [LV], [D].

Example 6.1. Let $\psi_h\colon \mathbf{R} \to \mathbf{R}$ be a sequence of bounded measurable functions such that $0 < K^{-1} \leq \psi_h(t) \leq K$ and

$$\psi_h \rightharpoonup 1, \qquad \frac{1}{\psi_h} \rightharpoonup \frac{1}{c} \quad (c \neq 1),$$

in $\sigma(L^\infty, L^1)$; for example, let us choose

$$\psi_h(t) = 1 + \delta \frac{\sin ht}{|\sin ht|} \qquad (0 < \delta < 1).$$

Then, the sequence of K-quasiregular mappings

$$f_h(x_1, x_2) = \left(\int_0^{x_1} \psi_h(t)\, dt, x_2 \right)$$

converges locally uniformly to the identity mapping $f(x_1, x_2) = (x_1, x_2)$.

It is immediate that the distortion tensor of f_h is

$$G(x, f_h) = \begin{pmatrix} \psi_h(x_1) & 0 \\ 0 & \bigl(\psi_h(x_1)\bigr)^{-1} \end{pmatrix}$$

and the distortion tensor of the limit f is

$$G(x, f) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The sequence $G(x, f_h)$ does not converge weakly nor does it Γ-converge to the identity matrix $G(x, f)$. Actually

$$G(x, f_h) \rightharpoonup \begin{pmatrix} 1 & 0 \\ 0 & c^{-1} \end{pmatrix} \qquad \text{weakly in } L^1(\Omega, \mathbf{R}^{2\times 2}).$$

Moreover it can be proved that

$$G(x, f_h) \xrightarrow{\Gamma} \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}.$$

Thus, of the two matrices $A(x, f)$, $G(x, f)$ only the first one exhibits a suitable continuity behaviour as a function of f.

In the following we deduce by our results a well-known theorem of Bers–Bojarski for planar K-quasiregular mappings whose n-dimensional version has been recently proved in [GMRV] (see also [IK]). The result states that if $f_h\colon \Omega \subset \mathbf{R}^2 \to \mathbf{R}^2$ verify a.e. in Ω ($K \geq 1$)

$$|Df_h(x)|^2 \leq \left(K + \frac{1}{K}\right) J(x, f_h);$$

if $f_h \to f$ locally uniformly and the distortion tensors $G(x, f_h)$ defined as in (1.3) converge a.e. to $G_0(x)$ then $G_0(x) = G(x, f)$. Namely we have the following

Theorem 6.1. *Let f_h be a sequence of mappings of finite distortion $K \geq 1$ on Ω such that*

(i) *$f_h \rightharpoonup f$ in $W^{1,2}(\Omega)$,*

(ii) *$G(x, f_h) \to G_0(x)$ a.e. in Ω.*

Then

$$G_0(x) = G(x, f) \qquad \text{a.e. in } \Omega.$$

We start with

Lemma 6.1. *Let A_h be a sequence of symmetric 2×2 matrices satisfying*

$$\frac{|\xi|^2}{K} \leq \langle A_h(x)\xi, \xi\rangle \leq K|\xi|^2 \qquad \text{for a.e. } x \in \Omega.$$

If

$$A_h^{-1} \to A_0^{-1} \quad \text{in } L^1(\Omega, \mathbf{R}^{2\times 2})$$

and

$$A_h \xrightarrow{\Gamma} A \tag{6.2}$$

then

$$A = A_0.$$

Proof. It is easy to check that

$$A_h - A_0 = A_h(A_0^{-1} - A_h^{-1})A_0.$$

So by our assumptions we deduce

$$A_h \to A_0 \qquad \text{in } L^1(\Omega, \mathbf{R}^{2\times 2}).$$

Since it is well known that strong L^1 convergence of coefficients matrices imply Γ-convergence [S1], we get

$$A_h \xrightarrow{\Gamma} A_0$$

and therefore, by (6.2)

$$A = A_0.$$

Proof of Theorem 6.1. Theorem 5.2 implies that $A(x, f_h) \xrightarrow{\Gamma} A(x, f)$. By (ii) and Vitali's theorem we deduce

$$G(x, f_h) = A(x, f_h)^{-1} \xrightarrow{L^1} G_0(x) = A_0^{-1}(x)$$

so Lemma 6.1 implies $A(x, f) = A_0(x) = G_0^{-1}(x)$ and this means $A^{-1}(x, f) = G_0(x)$, that is $G(x, f) = G_0(x)$.

Actually, L^1-convergence of the coefficient matrix A_h to A implies strong convergence in $W^{1,2}_{\mathrm{loc}}$ of local solutions u_h of the equation

$$\operatorname{div} A_h(x)\nabla u_h = 0$$

to local solutions u of

$$\operatorname{div} A(x)\nabla u = 0$$

(see [S1, Theorem 5]). So, in particular, under our assumptions we deduce $f_h^i \to f^i$ in $W^{1,2}_{\mathrm{loc}}$, for $i = 1, 2$, due to the fact that $\operatorname{div} A_h(x, f_h)\nabla f_h^i = 0$.

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