SOME ELEMENTARY PROOFS OF PUISEUX'S THEOREMS

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Abstract. This paper presents a short elementary proof of the Newton–Puiseux theorem to the effect that the quotient field of the ring of Puiseux series with complex coefficients is algebraically closed. As a consequence, we deduce the classical Puiseux theorem on parametrization of one-dimensional analytic germs.

We begin with setting up the notation:

\( \mathbb{C}[[z]] \) and \( \mathbb{C}\{z\} \) denote the rings of formal and convergent power series, respectively;
\( \mathbb{C}(z) \) and \( \mathbb{C}\{z\} \) are their quotient fields;
a formal (or convergent) Puiseux series is any series of the form \( f(z^{1/r}) \) with \( f(z) \in \mathbb{C}[[z]] \) (or \( f(z) \in \mathbb{C}\{z\} \)) and \( r \in \mathbb{N} \);
\( \mathbb{C}[[z^*]] \) and \( \mathbb{C}\{z^*\} \) denote the rings of formal and convergent Puiseux series, respectively;
\( \mathbb{C}(z^*) \) and \( \mathbb{C}\{z^*\} \) are their quotient fields.

Any element \( \phi(z) \in \mathbb{C}(z^*) \) can be written as \( \sum_{k=n}^{\infty} a_k \cdot z^{k/r} \) with \( r \in \mathbb{N} \), \( n \in \mathbb{Z} \), \( a_k \in \mathbb{C} \); when \( a_n \neq 0 \), we say that \( \phi(z) \) is of order \( n/r \), \( \text{ord} \phi(z) = n/r \).
The units of the rings \( \mathbb{C}[[z]] \), \( \mathbb{C}\{z\} \), \( \mathbb{C}[[z^*]] \) and \( \mathbb{C}\{z^*\} \) are exactly the elements of order zero.

NEWTON–PUISEUX THEOREM. (see e.g. [4], p. 61) The fields \( \mathbb{C}(z^*) \) and \( \mathbb{C}\{z^*\} \) are algebraically closed.

Proof. It suffices to prove that any monic polynomial
\[ P(z, T) = T^n + a_1(z)T^{n-1} + \cdots + a_n(z) \]
of degree \( n > 1 \) with coefficients in \( \mathbb{C}(z^*) \) (or \( \mathbb{C}\{z^*\} \)) is reducible. Making use of the Tschirnhausen transformation of variables \( T' = T + 1/n \cdot a_1(z) \), we

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can assume that \( a_1(z) \equiv 0 \). Put \( r_k := \text{ord} \ a_k(z) \in \mathbb{Q} \) unless \( a_k(z) \equiv 0 \), and \( r := \min\{r_k/k\} \); obviously, \( r_k/k - r \geq 0 \) and we have equality for at least one \( k \). Take a positive integer \( q \) so large that all the Puiseux series \( a_k(z) \) are of the form \( f_k(z^{1/q}) \) with \( f_k(z) \) in \( \mathbb{C}[[z]] \) (or \( \mathbb{C}\{z\} \)), and let \( r = p/q \) with \( p \in \mathbb{Z} \). After the transformation of variables \( z = w^q \), \( T = U \cdot w^p \), we get
\[
P(z, T) = w^{np} \cdot Q(w, U),
\]
where
\[
Q(w, U) = U^n + b_2(w)U^{n-2} + \cdots + b_n(w)
\]
with \( b_k(w) = a_k(w^q)w^{-kp} \). Since \( \text{ord} \ b_k(z) \in \mathbb{Z} \) and
\[
\text{ord} \ b_k(w) = q \cdot r_k - p \cdot k = qk(r_k/k - r) \geq 0,
\]
\( Q(w, U) \) is a polynomial with coefficients in \( \mathbb{C}[[z]] \) (or \( \mathbb{C}\{z\} \)); furthermore, \( \text{ord} b_k(z) = 0 \) for at least one \( k \), and thus \( b_k(0) \neq 0 \) for every such \( k \). Therefore the complex polynomial
\[
Q(0, U) = U^n + b_2(0)U^{n-2} + \cdots + b_n(0) \neq (U - c)^n
\]
for any \( c \in \mathbb{C} \), and consequently, \( Q(0, U) \) is the product of two relatively prime complex polynomials. Hence and by Hensel's lemma (see e.g. [1], Chap. I, §5.6), \( Q(w, U) \) is the product of two polynomials \( Q_1(w, U) \cdot Q_2(w, U) \) with coefficients in \( \mathbb{C}[[z]] \) (or \( \mathbb{C}\{z\} \)). Then
\[
P(z, T) = z^{nr} \cdot Q_1(z^{1/q}, z^{-r}T) \cdot Q_2(z^{1/q}, z^{-r}T),
\]
and the theorem follows. \( \square \)

In the sequel, \( \epsilon_n \) shall denote an \( n \)-th primitive root of unity.

**Lemma.** If \( f(z) \) is an element of \( \mathbb{C}[[z]] \) (or \( \mathbb{C}\{z\} \)) and \( r \in \mathbb{N} \), then
\[
Q(z, T) := (T - f(z)) \cdot (T - f(\epsilon_r z)) \cdots (T - f(\epsilon_r^{-1} z))
\]
is a monic polynomial in \( T \) with coefficients in \( \mathbb{C}[[z^r]] \) (or \( \mathbb{C}\{z^r \} \).

**Proof.** For a proof, consider the elementary symmetric polynomials
\[
s_j(U_1, \ldots, U_r) \ (j = 1, 2, \ldots, r) \text{ in variables } U_1, \ldots, U_r; \text{ let } S_j : \mathbb{C}[[z]] \rightarrow \mathbb{C}[[z]]
\]
be defined by
\[
S_j(f(z)) := s_j(f(z), f(\epsilon_r z), \ldots, f(\epsilon_r^{-1} z)).
\]
It is to be shown that \( S_j(f(z)) \in \mathbb{C}[[z^r]] \) for all \( f(z) \in \mathbb{C}[[z]] \). Since the mappings \( S_j \) are continuous in the maximal-adic topology of \( \mathbb{C}[[z]] \), it is sufficient to prove the above assertion only for polynomials \( f(z) \in \mathbb{C}[z] \). But this follows from the fact that
\[
\sigma_i : \mathbb{C}(z) \rightarrow \mathbb{C}(z), \quad \sigma_i(z) = \epsilon_i \cdot z \quad (i = 0, 1, \ldots, r - 1)
\]
form the Galois group $G$ of the field $\mathbb{C}(z)$ over $\mathbb{C}(z^r)$. Indeed, if $f(z) \in \mathbb{C}[z]$, then $S_j(f(z))$ is, of course, an invariant of $G$ whence

$$S_j(f(z)) \in \mathbb{C}(z^r) \cap \mathbb{C}[z] = \mathbb{C}[z^r],$$

as desired. \qed

**Proposition.** The rings $\mathbb{C}[[z^*]]$ and $\mathbb{C}\{z^*\}$ are integral over the rings $\mathbb{C}[[z]]$ and $\mathbb{C}\{z\}$, respectively. If a Puiseux series $\phi(z)$ from $\mathbb{C}[[z^*]]$ (or from $\mathbb{C}\{z^*\}$) is a root of an irreducible monic polynomial $P(z, T)$ of degree $n$ with coefficients in $\mathbb{C}[[z]]$ (or $\mathbb{C}\{z\}$), then $\phi(z)$ is of the form $g(z^{1/n})$ where $g(z)$ belongs to $\mathbb{C}[[z]]$ (or $\mathbb{C}\{z\}$). Moreover, the elements conjugate to $\phi(z)$ are exactly $g(\epsilon_i^iz^{1/n})$, $i = 0, 1, \ldots, n - 1$.

**Proof.** The Puiseux series $\phi(z)$ is of the form $f(z^{1/r})$ where $f(z)$ belongs to $\mathbb{C}[[z]]$ (or $\mathbb{C}\{z\}$). It follows immediately from the above lemma that

$$Q(z, T) := \prod_{i=0}^{r-1}(T - f(\epsilon_i^iz^{1/r}))$$

is a monic polynomial in $T$ with coefficients in $\mathbb{C}[[z]]$ (or $\mathbb{C}\{z\}$). Therefore the polynomial $Q(z, T)$ is divisible by $P(z, T)$ whence every root of $P(z, T)$ is of the form $f(\epsilon_i^iz^{1/r})$.

Conversely, each Puiseux series $f(\epsilon_i^iz^{1/r})$ is a root of $P(z, T)$. Indeed, $f(z) = f((z^r)^{1/r})$ is a root of the polynomial $P(z^r, T)$, and thus $f(\epsilon_i^iz)$ is a root of $P((\epsilon_i^iz)^r, T) = P(z^r, T)$. Hence $f(\epsilon_i^iz^{1/r})$ is a root of $P(z, T)$, as asserted.

Summing up, the set $X$ of Puiseux series

$$f(\epsilon_i^iz^{1/r}) \quad (i = 0, 1, \ldots, r - 1)$$

consists of precisely $n$ roots of the polynomial $P(z, T)$. Consider now an action of the group $\mathbb{Z}_r$ on the set $X$ defined by the formula

$$(j \mod r, f(\epsilon_i^iz^{1/r})) \mapsto f(\epsilon_i^{i+j}z^{1/r})).$$

As the set $X$ is the orbit of the element $f(z^{1/r})$, the stabilizer of $f(z^{1/r})$ is a subgroup of $\mathbb{Z}_r$ of index $n$, and thus it is the subgroup $\mathbb{Z}_s \subset \mathbb{Z}_r$ where $r = n \cdot s$. This yields

$$f(\epsilon_i^iz^{1/r}) = f(z^{1/r}) \quad (i = 0, 1, \ldots, s - 1).$$

Hence and by the lemma,

$$s \cdot f(z^{1/s}) = f((z^n)^{1/r}) + f(\epsilon_s(z^n)^{1/r}) + \cdots + f(\epsilon_s^{s-1}(z^n)^{1/r}) =$$

$$= f(z^{1/s}) + f(\epsilon_s z^{1/s}) + \cdots + f(\epsilon_s^{s-1}z^{1/s})$$
belongs to $\mathbb{C}[z]$ (or $\mathbb{C}\{z\}$). Therefore, $f(z^{1/s}) = g(z)$ with $g(z)$ in $\mathbb{C}[z]$ (or $\mathbb{C}\{z\}$). Consequently,

$$\phi(z) = f(z^{1/r}) = f((z^{1/n})^{1/s}) = g(z^{1/n}),$$

and the proof is complete. \qed

We conclude this paper with a corollary concerning parametrization of a one-dimensional analytic germ (cf. [3] or [2], Chap. II, §6).

**Puiseux Theorem.** If $P(z, T) \in \mathbb{C}\{z\}[T]$ is an irreducible monic polynomial in $T$ of degree $n$, then there exists a convergent power series $g(z) \in \mathbb{C}\{z\}$ such that

$$P(z^n, T) = \prod_{i=0}^{n-1} (T - g(\epsilon^n_i z^n)).$$

**Proof.** Indeed, according to the Newton–Puiseux theorem, the polynomial $P(z, T)$ has a root $\phi(z)$ in $\mathbb{C}\{z^n\}$; $\phi(z)$ is, of course, a convergent Puiseux series. Now it follows from the proposition that $\phi(z) = g(z^{1/n})$ for some $g(z) \in \mathbb{C}\{z\}$, and that

$$P(z, T) = \prod_{i=0}^{n-1} (T - g(\epsilon^n_i z^{1/n})).$$

This finishes the proof. \qed

**Remark.** The above assertion can be interpreted geometrically as follows. If an irreducible analytic germ $V$ at $0 \in \mathbb{C}^2$ is determined by the polynomial $P(z, T) \in \mathbb{C}\{z\}[T]$, then

$$(\mathbb{C}, 0) \ni z \mapsto (z^n, g(z)) \in (V, 0)$$

is a parametrization of $V$ near zero.

**References**


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