GENERALIZED $k$-MITTAG LEFFLER FUNCTION AND ITS COMPOSITION WITH PATHWAY INTEGRAL OPERATORS

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Abstract. Our purpose in this paper is to consider a more generalized form of the Mittag Leffler function. For this extended Mittag Leffler function, we obtain some composition formulas with pathway fractional integral operators. We also point out some important special cases of the main results.

1. Introduction

Mittag Leffler functions are important in studying solutions of fractional differential equations, and they are associated with a wide range of problems in many areas of mathematics and physics. These considerations have led various workers in the field of special functions for exploring the possible extensions and applications for the Mittag Leffler function. A useful generalization of the Mittag Leffler called as $k$-Mittag Leffler function has been introduced and studied in [1]. Here we aim at introducing a more generalized $k$-Mittag Leffler function and also presenting certain image formulas under pathway fractional integral formulas for the newly defined function.

Throughout this paper, let $\mathbb{C}, \mathbb{R}, \mathbb{N}$ be the sets of complex numbers, real numbers, positive integers respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Let $\alpha, \beta, \gamma \in \mathbb{C}, k \in \mathbb{R}, \{\Re(\alpha), \Re(\beta), \Re(\gamma)\} > 0$ and $q \in (0, 1) \cup \mathbb{N}$, then the generalized $k$-Mittag Leffler function is defined by Gehlot [1] as:

\begin{equation}
E_{\gamma,q}^{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k}}{\Gamma_{k}(\alpha n + \beta)} z^n,
\end{equation}

where $(\gamma)_{nq,k}$ is $k$-Pochhammer symbol given by

\begin{equation}
(x)_{n,k} = x(x+k)(x+2k)\ldots(x+(n-1)k) \quad (x \in \mathbb{C}, k \in \mathbb{R}, n \in \mathbb{N}).
\end{equation}

The integral form of the generalized $k$-Gamma function is given by

\begin{equation}
\Gamma_k(z) = \int_0^{\infty} e^{-\frac{x}{t}, t^{-1}} dt,
\end{equation}

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where $k \in \mathbb{R}$, $z \in \mathbb{C}$, $\Re(z) > 0$ and

\begin{align*}
\Gamma_k(x + k) &= x\Gamma_k(x), \\
\Gamma_k(\gamma) &= (k)^{\gamma - 1} \Gamma(\frac{\gamma}{k}).
\end{align*}

(1.4)

In this paper, we introduce a more generalized $k$-Mittag Leffler function as under:

\begin{equation}
E^{\gamma,q}_{k,\alpha,\beta,\delta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)^{nq,k}}{\Gamma_k(\alpha n + \beta)} \frac{z^n}{(\delta)^n},
\end{equation}

(1.5)

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, $k \in \mathbb{R}$, $\{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta)\} > 0$ and $q \in (0,1) \cup \mathbb{N}$.

Particular cases:

(i) For $\delta = 1$, we get the generalized $k$-Mittag Leffler function (1.1) (see [1]).

(ii) Again, if $\delta = q = 1$ then (1.5) reduces to $E^{\gamma}_{k,\alpha,\beta}(z)$ (see [3]).

Recently, Nair [9] introduce a pathway fractional integral operator by using the pathway idea of Mathai [8] and developed further by Mathai and Haubold [10], [11], and it is defined as follows:

Let $f(x) \in L(a, b)$, $\eta \in \mathbb{C}$, $\Re(\eta) > 0$, $a > 0$ and the pathway parameter $\alpha < 1$ as (cf. [12]), then

\begin{equation}
(P^{(\eta, \alpha)}_{0+} f)(x) = x^\eta \int_0^1 \left[ 1 - \left( \frac{a(1-\alpha)t}{x} \right)^{\frac{\alpha}{1-\alpha}} \right]^{\frac{\alpha}{1-\alpha}} f(t) \, dt.
\end{equation}

(1.6)

For a real scalar $\alpha$, the pathway model for scalar random variables is represented by the following probability density function (p.d.f.):

\begin{equation}
f(x) = c |x|^{\gamma - 1} \left[ 1 - a(1-\alpha) |x|^\beta \right]^{\frac{\beta}{1-\alpha}},
\end{equation}

(1.7)

provided that $-\infty < x < \infty$, $\delta > 0$, $\beta \geq 0$, $\left[ 1 - a(1-\alpha) |x|^\beta \right] > 0$, and $\gamma > 0$ where $c$ is the normalizing constant and $\alpha$ is called the pathway parameter.

Further, for real $\alpha$, the normalizing constant as follows:

\begin{align*}
c &= \frac{1}{2} \frac{\delta [a(1-\alpha)]^{\gamma}}{\Gamma(\frac{\beta}{1-\alpha} + 1)}, \text{ for } \alpha < 1 \\
&= \frac{1}{2} \frac{\delta [a(1-\alpha)]^{\gamma}}{\Gamma(\frac{\beta}{1-\alpha} + 1)}, \text{ for } \frac{1}{1-\alpha} - \frac{\gamma}{\delta} > 0, \alpha > 1 \\
&= \frac{1}{2} \frac{(a\beta)^{\gamma}}{\Gamma(\frac{\beta}{1-\alpha})}, \alpha \to 1.
\end{align*}

Note that for $\alpha < 1$ it is a finite range density with $\left[ 1 - a(1-\alpha) |x|^\beta \right] > 0$ and (1.7) remains in the extended generalized type-1 beta family. The pathway density in (1.7), for $\alpha < 1$, includes the extended type-1 beta density, the triangular density, the uniform density and many other p.d.f.
For instance, $\alpha > 1$, writing $(1 - \alpha) = - \frac{\alpha - 1}{2}$ in (1.6) gives

$$P_{0+}^{(\eta, \alpha)} f(x) = x^\eta \int_0^1 \left[ 1 + \frac{a (\alpha - 1) t}{x} \right]^{\frac{-\eta}{\alpha - 1}} f(t) dt,$$

and

$$f(x) = c |x|^{-1} \left[ 1 + a (\alpha - 1) |x|^{\delta} \right]^{\frac{-\eta}{\alpha - 1}},$$

provided that $-\infty < x < \infty, \delta > 0, \beta \geq 0, \text{ and } \alpha > 1$ which is the extended generalized type-2 beta model for real $x$. It includes the type-2 beta density, the $F$ density, the Student-$t$ density, the Cauchy density and many more.

Moreover, when $\alpha \to 1_-$, the operator (1.6) reduces to the Laplace integral transform, and when $\alpha = 0, a = 1$ and $\eta$ replacing by $\eta - 1$, the operator (1.6) reduces to the Riemann-Liouville fractional integral operator. For more details on the pathway model and its particular cases, the reader is referred to the recent papers of Mathai and Haubold [10], [11] and Nair [9].

It is observed that the pathway fractional integral operator (1.6), can lead to other interesting examples of fractional calculus operators, related to some probability density functions and applications in statistics. This has led various workers in the field of fractional calculus for exploring the possible extensions of the known results. For example, the composition of the integral transform operator (1.6) with the product of generalized Bessel function of the first kind is given in [12]. Recently Nisar et. al studied the pathway fractional integral operator associated with Struve function of first kind [17]. The results provided in [12] are extensions of the results given by Agarwal and Purohit [16] and Nair [9]. The purpose of this work is to investigate the composition formula of integral transform operator due to Nair, with the more generalized $k$-Mittag Leffler function introduced in presiding section.

2. Pathway Fractional Integration of Generalized $k$-Mittag Leffler Function.

In this section, we derive the pathway integral representation of generalized $k$-Mittag-Leffler function, which is defined in (1.5). The results given in this section are based on the preliminary assertions giving by composition formula of pathway fractional integral (1.6) with a power function. The results are given in the following theorems.

**Theorem 1.** Let $\rho, \beta, \gamma, \delta, \xi \in \mathbb{C}, k \in \mathbb{R}, \{\Re(\rho), \Re(\beta), \Re(\gamma), \Re(\delta), \Re(\xi)\} > 0, \eta > 0, \Re \left(1 + \frac{\xi}{1 - \alpha}\right) > 0, \alpha < 1, k, w \in \mathbb{R}$ and $q \in (0, 1) \cup \mathbb{N}$. Then the following formula holds true:

$$P_{0+}^{(\xi, \alpha)} \left[t^{\xi-1} E_{k, \rho, \beta, \delta}^{\gamma, q} \left(w t^\xi \right)\right] (x) = x^{\xi+\frac{\eta}{\alpha}} k^{1+\frac{\xi}{1-\alpha}} \left[\frac{1+\frac{\xi}{1-\alpha}}{\eta (1-\alpha)}\right]^\frac{\eta}{\alpha} \times E_{k, \rho, \beta+k(1+\frac{\xi}{1-\alpha}), \delta}^{\gamma, q} \left[w \left(\frac{x}{a (1-\alpha)}\right)^\frac{\eta}{\alpha}\right].$$

(2.1)
Proof. By applying (1.5) and (1.6), we have

\[
P_{\alpha}^{(1,1)} \left[ t^{\alpha-1} E_{k,\alpha,\beta} \left( \frac{wt^\alpha}{x} \right) \right] = \int_0^x \frac{1 - \frac{a(1-\alpha)t}{\nu}}{t^{\alpha-1}} \times E_{k,\alpha,\beta} \left( \frac{wt^\alpha}{x} \right) dt.
\]

For convenience, we denote the right hand integral of the above term by \( I \), then

\[
I = x^\xi \sum_{n=0}^\infty \frac{1}{\Gamma_k (\nu n + \beta)} \frac{x^n}{(\nu \nu)^n} \left[ 1 - \frac{a(1-\alpha)t}{\nu} \right] \frac{t^\alpha}{x^\alpha} \frac{dt}{dt}. dt.
\]

Now, by evaluating the inner integral using beta function formula, we get

\[
I = x^\xi \sum_{n=0}^\infty \frac{1}{\Gamma_k (\nu n + \beta)} \frac{x^n}{(\nu \nu)^n} \left[ 1 - \frac{a(1-\alpha)t}{\nu} \right] \frac{t^\alpha}{x^\alpha} \frac{dt}{dt}. dt.
\]

Using (1.4.), we obtain

\[
I = x^\xi \sum_{n=0}^\infty \frac{1}{\Gamma_k (\nu n + \beta)} \frac{x^n}{(\nu \nu)^n} \left[ 1 - \frac{a(1-\alpha)t}{\nu} \right] \frac{t^\alpha}{x^\alpha} \frac{dt}{dt}. dt.
\]

Again, on applying (1.4.), we get

\[
I = x^\xi \sum_{n=0}^\infty \frac{1}{\Gamma_k (\nu n + \beta)} \frac{x^n}{(\nu \nu)^n} \left[ 1 - \frac{a(1-\alpha)t}{\nu} \right] \frac{t^\alpha}{x^\alpha} \frac{dt}{dt}. dt.
\]

which completes the proof of Theorem 1. \( \square \)
Corollary 1. If we put $\delta = 1$, Theorem 1 reduces to the result given in [2]:

$$P^{(\xi, \alpha)}_{0+} \left[ t^{\frac{x}{\alpha}} E^{\gamma, q}_{k, \rho, \beta} \left( wt^\frac{x}{\alpha} \right) \right] (x) = x^{\xi + \frac{x}{\alpha} k (1 + \frac{x}{\alpha})} \frac{\Gamma \left( 1 + \frac{x}{\alpha} \right)}{\left[ a (1 - \alpha) \right]^\frac{x}{\alpha}}$$

\[ \times E^{\gamma, q}_{k, \alpha, \beta + k (1 + \frac{x}{\alpha})} \left[ w \left( \frac{x}{a (1 - \alpha)} \right) \right]. \]

Corollary 2. If we put $\delta = q = 1$, Theorem 1 reduces to the result of [3]:

$$P^{(\xi, \alpha)}_{0+} \left[ t^{\frac{x}{\alpha}} E^{\gamma, q}_{k, \rho, \beta} \left( wt^\frac{x}{\alpha} \right) \right] (x) = x^{\xi + \frac{x}{\alpha} k (1 + \frac{x}{\alpha})} \frac{\Gamma \left( 1 + \frac{x}{\alpha} \right)}{\left[ a (1 - \alpha) \right]^\frac{x}{\alpha}}$$

\[ \times E^{\gamma, q}_{k, \alpha, \beta + k (1 + \frac{x}{\alpha})} \left[ w \left( \frac{x}{a (1 - \alpha)} \right) \right]. \]

Corollary 3. If $\delta = q = 1$ and $k = 1$, we obtain the result of Nair [9]:

$$P^{(\xi, \alpha)}_{0+} \left[ t^{\frac{x}{\alpha}} E^{\gamma, q}_{k, \rho, \beta} \left( wt^\frac{x}{\alpha} \right) \right] (x) = x^{\xi + \frac{x}{\alpha} k (1 + \frac{x}{\alpha})} \frac{\Gamma \left( 1 + \frac{x}{\alpha} \right)}{\left[ a (1 - \alpha) \right]^\frac{x}{\alpha}}$$

\[ \times E^{\gamma, q}_{k, \alpha, \beta + k (1 + \frac{x}{\alpha})} \left[ w \left( \frac{x}{a (1 - \alpha)} \right) \right]. \]

Theorem 2. Let $\rho, \beta, \gamma, \delta, \xi \in \mathbb{C}$, $k \in \mathbb{R}$, $\{\Re (\rho), \Re (\beta), \Re (\gamma), \Re (\delta), \Re (\xi)\} > 0$, $q > 0$, $\Re \left( 1 + \frac{x}{\alpha} \right) > 0$, $\alpha < 1$, $k, w \in \mathbb{R}$ and $q \in (0, 1) \cup \mathbb{N}$. Then the pathway fractional integral representation of (1.5) is given by

$$P^{(\xi, \alpha)}_{0+} \left[ t^{\frac{x}{\alpha}} E^{\gamma, q}_{k, \rho, \beta} \left( wt^\frac{x}{\alpha} \right) \right] (x) = x^{\xi + \frac{x}{\alpha} k (1 - \frac{x}{\alpha})} \frac{\Gamma \left( 1 - \frac{x}{\alpha} \right)}{\left[ -a (1 - \alpha) \right]^\frac{x}{\alpha}}$$

\[ \times E^{\gamma, q}_{k, \alpha, \beta + k (1 - \frac{x}{\alpha})} \delta \left[ w \left( \frac{x}{-a (1 - \alpha)} \right) \right]. \]

Proof. By applying (1.8) and (1.9), we have

$$P^{(\xi, \alpha)}_{0+} \left[ t^{\frac{x}{\alpha}} E^{\gamma, q}_{k, \alpha, \beta} \left( wt^\frac{x}{\alpha} \right) \right] (x) = x^{\xi} \int_{0}^{\infty} t^{\frac{x}{\alpha} - 1} \left[ 1 + \frac{a (\alpha - 1) t}{x} \right] \frac{x}{\alpha} \right]$$

\[ \times E^{\gamma, q}_{k, \alpha, \beta} \left( wt^\frac{x}{\alpha} \right) dt. \]

For convenience, we denote the right hand integral of the above term by $I_2$, then

$$I_2 = x^{\xi} \int_{0}^{\infty} t^{\frac{x}{\alpha} - 1} \left[ 1 + \frac{a (\alpha - 1) t}{x} \right] \frac{x}{\alpha} \right]$$

\[ \times \sum_{n=0}^{\infty} \frac{\Gamma_k (\rho n + \beta)}{(\delta)_n} \frac{(\gamma)_{\alpha, k}}{\Gamma_k (\rho n + \beta)} \left( wt^\frac{x}{\alpha} \right) n. \]
Now, on interchanging the order of integration and summation, we have

\[
I_2 = x^\xi \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k}}{\Gamma_k(\rho n + \beta)} \frac{w^n}{(\delta)_n} \times \int_0^{\infty} \left[ 1 + \frac{a(\alpha - 1) t}{x} \right]^{-\frac{\xi}{\alpha}} t^{\alpha-1} \frac{dt}{a(\alpha - 1)}.
\]

By putting \(a(\alpha - 1) = u\) and evaluating the inner integral by beta function using (1.4), we get

\[
I_2 = x^{\xi + \frac{\beta}{\alpha}} \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k}}{\Gamma_k\left(\frac{\xi}{\alpha}n + \frac{\beta}{\alpha}\right) + 1 - \frac{\xi}{\alpha-1}} \left[ \frac{w}{-(\alpha - a)} \right]^{\frac{\beta}{\alpha}}.
\]

Again, on applying (1.4), we arrive at the following desired result:

\[
I_2 = x^{\xi + \frac{\beta}{\alpha}} k^{1 - \frac{\xi}{\alpha - 1}} E_{\kappa,\alpha,\beta + k(1 - \frac{\xi}{\alpha - 1}),\delta}^{\gamma,q} \left[ w \left( \frac{x}{-(\alpha - a)} \right) \right].
\]

**Corollary 4.** Let \(\delta = 1\), then Theorem 2 reduces to the result given in [2]:

\[
P_{\alpha,(1)} \left[ t^{\frac{\xi}{\alpha} - 1} E_{\kappa,\rho,\beta}^{\gamma,q} \left( wt^{\frac{\beta}{\alpha}} \right) \right](x)
\]

\[
= x^{\xi + \frac{\beta}{\alpha}} \left[ \frac{w}{-(\alpha - a)} \right]^{\frac{\beta}{\alpha}} E_{\kappa,\alpha,\beta + k(1 - \frac{\xi}{\alpha - 1}),\delta}^{\gamma,q} \left[ w \left( \frac{x}{-(\alpha - a)} \right) \right].
\]

**Corollary 5.** If \(\delta = 1\) and \(k = q = 1\), we get the well known results of [9]:

\[
P_{\alpha,(1)} \left[ t^{\frac{\xi}{\alpha} - 1} E_{1,\rho,\beta}^{\gamma,1} \left( wt^\rho \right) \right](x)
\]

\[
= x^{\xi + \beta} \left[ \frac{w}{-(\alpha - a)} \right]^{\frac{\beta}{\alpha}} E_{1,\alpha,\beta + (1 - \frac{\xi}{\alpha - 1}),\delta}^{\gamma} \left[ w \left( \frac{x}{-(\alpha - a)} \right) \right]^\delta.
\]

3. **Conclusion**

In this paper, we have introduced a more generalized Mittag Leffler function. For this extended Mittag Leffler function, we have presented two pathway fractional integral formulas (PFIF). The result obtained in the present paper provides an
extension of the known results, as mentioned earlier. We conclude our paper with the remark that, the function introduced and results deduced above are significant and can lead to yield numerous other integral formulas involving various Mittag Leffler type functions.

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**References**


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