On the Faddeev-Hopf model and a conjecture by Ward

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Abstract
We discuss the variational calculus for the Faddeev-Hopf model on a general Riemannian manifold, with general Kähler target space, in the strong coupling limit. This model has interesting similarities with pure Yang-Mills theory, namely conformal invariance in dimension 4 and an infinite dimensional symmetry group. We calculate the first and second variation formulae for this functional and discuss some solutions and their stability properties. In particular, it is proved that all immersive solutions are stable. From an explicit description of the spectral behaviour of the Hopf map $S^3 \to S^2$, we are able to prove a conjecture of Ward concerning the stability of this map in the full Faddeev-Hopf model.

Keywords: harmonic maps, the Hopf map, stability.

1 Introduction

Theoretical physics has long been a rich source of geometrically interesting and natural variational problems. The Yang-Mills equations, of deep significance for the differential topology of 4-manifolds and the Yang-Mills-Higgs equations, which have led to interesting results in hyperkähler geometry, both originated in elementary particle physics. Harmonic map theory, while not originating in theoretical physics, has found many applications in high energy and condensed matter physics, with physicists frequently independently rediscovering fundamental results, and sometimes contributing genuinely new insights.

The purpose of this talk is to present a study of a variational problem arising in the so-called Faddeev-Hopf (or Faddeev-Skyrme) model [3], originally proposed as a model of quark confinement (among other phenomena) in high energy physics. This is joint work with Martin Speight, and full details and proofs can be found in [6].

Let $M$ be a Riemannian manifold and $N$ a Kähler manifold, the target space, with Kähler form $\omega$. The full Faddeev-Hopf model has a single field $\phi : M \to N$, and is given by

$$E_\alpha(\phi) = \frac{1}{2} \int_M \left( |d\phi|^2 + \alpha |\phi^* \omega|^2 \right), \quad (1)$$

$\alpha \geq 0$ being a coupling constant. The model of original interest has $M = \mathbb{R}^3$, $N = S^2$. The weak coupling limit of this model, $\alpha = 0$, has of course been intensively studied: it is the harmonic map problem. This is conformally invariant if $M$ has dimension 2. By contrast, we shall study the strong coupling limit, $\alpha \to \infty$, or more precisely, the variational problem for the energy functional

$$E(\phi) = \lim_{\alpha \to \infty} \alpha^{-1} E_\alpha(\phi) = \frac{1}{2} \int_M |\phi^* \omega|^2, \quad (2)$$

henceforth referred to as the pure Faddeev-Hopf model.

The pure model has been studied in the specific case $M = \mathbb{R} \times S^3$ (with a Lorentzian metric) and $N = S^2, \mathbb{C}$ or the hyperbolic plane by de Carli and Ferreira [2]. It has some similarities with pure Yang-Mills theory. It is invariant under an infinite dimensional group of symmetries, the group of symplectic diffeomorphisms of $N$, rather than Yang-Mills theory is invariant under gauge transformations. It is also conformally invariant if $M$ has dimension 4.
As is well known, the Hopf map $\pi : S^3 \to S^2 \cong \mathbb{C}P^1$ is a harmonic map and, as such, is known to be unstable. In [8], Ward proved the following result.

**Theorem 1** [8] The Hopf map is critical for the full Faddeev-Hopf model for any $\alpha \geq 0$, and is unstable for this model if $\alpha < 1$.

Furthermore, Ward conjectured that the Hopf map is stable for the full Faddeev-Hopf model if $\alpha \geq 1$.

We here discuss the variation formulæ for the pure Faddeev-Hopf model, and give examples of solutions to the Euler-Lagrange equations. We will also see that all immersive solutions are stable, and that there are no non-vacuum (i.e., $E > 0$) immersive solutions in the case $M = S^4$. Finally, we will discuss a proof of Ward’s conjecture mentioned above.

## 2 The Variation Formulae

In this section we calculate the first and second variation formulæ for the pure Faddeev-Hopf model (2). We assume throughout that $(M^m, g)$ is a compact, oriented Riemannian manifold of dimension $m$, and $(N^n, h, J)$ a Kähler manifold of real dimension $n$ and Kähler form $\omega$. For any vector bundle $F$ over $M$, we denote by $\Gamma(F)$ the space of sections of $F$.

### 2.1 The first variation formula

Let us derive the first variation formula and the Euler-Lagrange equation for $E(\phi)$.

**Proposition 1** For a smooth variation $\phi_t : M \to N$ of $\phi$ with variational vector field $X \in \Gamma(\phi^{-1}TN)$, we have

$$\frac{d}{dt} E(\phi_t) |_{t=0} = \int_M \omega(X, d\phi(\delta \phi^* \omega)) \ast 1.$$ 

**Corollary 1** The map $\phi : M \to N$ is a critical point for the energy if and only if

$$\delta \phi^* \omega \in \ker d\phi.$$

**Example 1** Assume that $M = N$ and $\phi : N \to N$ is the identity map. Then $\phi^* \omega = \omega$, and $\delta \omega = 0$. Hence $\phi$ is a critical point for the functional.

**Remark 1** Clearly $\phi$ is a vacuum, i.e., $E(\phi) = 0$, if and only if $\phi^* \omega = 0$ everywhere, that is, if $\phi$ is isotropic. The set of vacuum solutions to this model is unusually rich. For example, the map

$$\phi : S^4 \to \mathbb{C}P^4,$$

defined as the 2-fold covering by $S^4$ of $\mathbb{R}P^4$, followed by the natural embedding of $\mathbb{R}P^4$ to $\mathbb{C}P^4$, is clearly isotropic, hence a vacuum.

**Remark 2** Assume that $\phi$ is smooth and is immersive on a dense set, that is, for all $x$ in a dense subset of $M$, the differential

$$d\phi : T_x M \to T_{\phi(x)} N$$

is injective. By the corollary, if $\phi$ is a critical point of the functional, then the 1-form $\delta \phi^* \omega$ vanishes almost everywhere, and hence vanishes everywhere by continuity. Hence the 2-form $\phi^* \omega$ is co-closed. Since it is obviously closed, we see that an almost everywhere immersive map is a critical point of the functional if and only if $\phi^* \omega$ is a harmonic 2-form.

In particular, when $H^2(M, \mathbb{R}) = 0$, the only immersive critical points defined on $M$ are isotropic immersions, i.e., maps for which $\phi^* \omega = 0$. As previously remarked, such a map has $E(\phi) = 0$, and hence is a vacuum solution.
Of primary physical interest, given their physical interpretation as instantons, are smooth non-isotropic critical points on \( M = S^4 \) which minimize \( E(\phi) \) within their homotopy class. In particular, one would like a smooth non-isotropic minimizer in the nontrivial class of \( \pi_4(S^2) \), a pure Faddeev-Hopf instanton. In fact, it remains an open question whether smooth non-isotropic critical points exist on \( S^4 \) at all, for any choice of target space. The best we have managed to find is the following example.

**Example 2** The twice punctured sphere \( S^4 \) is conformally equivalent to the cylinder \( \mathbb{R} \times SU(2) \) with the product metric. As \( SU(2) \) acts on \( \mathbb{R} \times SU(2) \) with cohomogeneity one, and also in an obvious way on \( CP^2 \), one can look for equivariant solutions \( \mathbb{R} \times SU(2) \rightarrow CP^2 \) satisfying some appropriate boundary conditions at \( t \rightarrow \pm \infty \).

This approach gives the solution
\[
\phi(t,X) = \left( \begin{array}{cc}
X & 0 \\
0 & 1
\end{array} \right) \begin{bmatrix} 1, 0, t/|t| \sqrt{e^{|t|} - 1} \end{bmatrix} \quad ((t,X \in \mathbb{R} \times SU(2)).
\]

This extends to a continuous map \( S^4 \rightarrow CP^2 \) of finite total energy; however, it fails to be smooth along the equator and at the poles of \( S^4 \).

### 2.2 Critical Submersions

In light of Corollary 1 and Remark 2, it is natural to seek critical maps in the case where the dimension of \( M \) exceeds that of \( N \). In particular, there exists a large number of interesting critical submersions. We begin with a simple example.

**Example 3** Assume that \( (P,k) \) and \( (N,h) \) are two compact Riemannian manifolds and that \( f : P \rightarrow \mathbb{R} \) is a positive, smooth function. The warped product of \( (P,k) \) and \( (N,h) \) by \( f \) is the manifold \( P \times N \) with the Riemannian metric
\[
g = k + f^2 h.
\]

Assume further that \( (N,h,\omega) \) is Kähler. Then the projection map \( \phi : P \times N \rightarrow N \) onto the second coordinate is critical.

To see this, let \( *_P \) and \( *_N \) be the Hodge star operators on \( P \) and \( N \), respectively, so that the volume form on \( P \) is \( *_P 1 \). Then, as is easily seen,
\[
*_P \omega = \frac{f^{(n-4)}}{(n/2 - 1)!} *_P 1 \wedge \omega^{n/2 - 1}.
\]

It follows from this that \( \delta \phi^* \omega = 0 \).

To proceed further in our analysis of critical submersions, let us denote by \( \nabla \) both the Levi-Civita connection on \( TM \) and on \( TN \). Recall that the connection on \( TN \) induces a connection \( \nabla^0 \) on \( \phi^{-1}TN \). This connection, together with the Levi-Civita connection on \( TM \), induces a connection on \( \text{Hom}(TM,\phi^{-1}TN) \), which we also denote by \( \nabla \). The **second fundamental form** of \( \phi \) is the covariant derivative of \( d\phi \):
\[
\nabla d\phi(X,Y) = \nabla_X^0 d\phi(Y) - d\phi(\nabla_X Y) \quad (X,Y \in \Gamma(TM)).
\]

The map \( \phi \) is said to be **totally geodesic** if its second fundamental form vanishes. The **tension field** of \( \phi \) is the trace of the second fundamental form:
\[
\tau(\phi) = \text{trace } \nabla d\phi = \sum_{j=1}^m \nabla d\phi(e_i,e_i).
\]

The map \( \phi \) is said to be a **harmonic map** if its tension field vanishes.

Choose a local orthonormal frame \( \{e_i\}_{i=1}^m \) on \( M \). It is then an easy calculation to see that
\[
d\phi(\delta \phi^* \omega) = \sum_{j=1}^m \left( \sum_{i=1}^m \omega(\nabla d\phi(e_i,e_j), d\phi(e_i)) - \omega(\tau(\phi), d\phi(e_j)) \right) d\phi(e_j).
\]

By our calculation, \( \phi \) is a critical point if and only if this expression vanishes.
Example 4 Assume that \( \phi \) is a Riemannian submersion; thus, at each point \( x \in M \), the differential
\[
d\phi_x : T_x M \to T_{\phi(x)} N
\]
maps the space \((\ker d\phi_x)^\perp \subset T_x M\) isometrically onto \( T_{\phi(x)} N \).
Locally, we can choose an orthonormal frame \( e_1, \ldots, e_m \) for \( T M \) with the property that \( e_1, \ldots, e_{m-n} \) is a local frame for \( \ker d\phi \) and \( e_{m-n+1}, \ldots, e_m \) is a local frame for \((\ker d\phi)^\perp \). Then
\[
d\phi(\delta \phi^* \omega) = - \sum_{j=m-n+1}^m \omega(\tau(\phi), d\phi(e_j)) d\phi(e_j).
\]
Thus, \( \phi \) is critical if and only if \( \phi \) is harmonic. In fact, using the same local frame for \( T M \) gives
\[
\tau(\phi) = - d\phi(\sum_{j=1}^{m-n} \nabla e_j e_j) = - d\phi(H),
\]
where \( H \) is the mean curvature vector of the fibres of \( \phi \). We conclude that a Riemannian submersion is a critical point if and only if it has minimal fibres, and thus is a harmonic morphism, see [1]. Note that such a map, being harmonic, is automatically a critical point of the full Faddeev-Hopf functional for every value of the coupling constant. A similar calculation shows that this result remains true for the wider class of horizontally homothetic maps, see [1].

For example, the natural projection
\[
\phi : S^{2n+1} \to \mathbb{C}P^n, \quad \phi(z) = [z] \quad (z \in S^{2n+1} \subset \mathbb{C}^{n+1})
\]
is a Riemannian submersion with minimal, even totally geodesic, fibres.

2.3 The Second Variation and Stability

In this section we calculate the second variation of the energy functional. Assume that \( \phi : M \to N \) is a critical point of the functional. We define the Hessian of \( E \) at \( \phi \) as
\[
H_{\phi}(X,Y) = \left. \frac{\partial^2}{\partial s \partial t} \right|_{s=t=0} E(\phi_{s,t});
\]
here \( \phi_{s,t} \) is a 2-parameter variation of \( \phi \) with
\[
X = \partial_t \phi_{s,t} \big|_{s=t=0} \quad \text{and} \quad Y = \partial_s \phi_{s,t} \big|_{s=t=0}.
\]
Clearly \( H_{\phi} \) is a symmetric, bi-linear form on \( \Gamma(\phi^{-1} TN) \). The map \( \phi \) is said to be stable if
\[
H_{\phi}(X,X) \geq 0 \quad (X \in \Gamma(\phi^{-1} TN));
\]
the index of \( \phi \) is the dimension of the largest subspace on which \( H_{\phi} \) is negative.

Any map which minimizes the energy within its homotopy class is obviously a stable critical point. In some situations it is easy to give lower bounds for the energy.

Example 5 Let \( M = M^2 \) be a surface. Then
\[
E(\phi) \geq \frac{1}{2\text{Vol}(M)} \left( \int_M \phi^* \omega \right)^2.
\]
Note that the right-hand side is a homotopy invariant. If \( \phi \) attains this lower bound then \( \phi \) is either isotropic (so \( E(\phi) = 0 \)) or has no critical points.

Assume for example that \( \phi : M^2 \to S^2 \) is non-isotropic and attains the bound. Then it is easy to see that \( M \cong S^2 \) and \( \phi^* \omega = \pm \omega \).
Example 6 When \( \dim M = 4 \) then
\[
E(\phi) \geq \frac{1}{2} \int_M \phi^* (\omega \wedge \omega)
\]
with equality if and only if \( \phi^* \omega \) is (anti-)self-dual. Again we note that the right-hand side is a homotopy invariant.

Let us now find an explicit formula for the Hessian of a critical point. Note that the metric \( h \) on \( T N \) induces a metric, also denoted by \( h \), on \( \phi^{-1} T N \).

Proposition 2 Assume that \( \phi \) is a critical point. Then the Hessian of \( \phi \) is given by
\[
H_{\phi}(X, Y) = \int_M h(X, L_{\phi} Y) \ast 1 \quad (X, Y \in \Gamma(\phi^{-1} T N)),
\]
where
\[
L_{\phi} Y = -J(\nabla_{\phi} Y + d\phi(\delta\delta^* Y \omega)) \quad \text{and} \quad Z_{\phi} = \delta\delta^* \omega.
\]

Remark 3 Recall that \( Z_{\phi} \) is the vector field on \( M \) which must lie pointwise in \( \ker d\phi \) given that \( \phi \) is critical, by Corollary 1.

Corollary 2 Assume that \( \phi : M \to N \) is a critical point. Then
\[
H_{\phi}(Y, Y) = \int_M \omega(Y, \nabla_{Z_{\phi}} Y) \ast 1 + \|d\phi^* Y \omega\|_{L^2}^2 \quad (Y \in \Gamma(\phi^{-1} T N)).
\]
In particular, \( \phi \) is stable if \( Z_{\phi} \) vanishes.

Corollary 3 Any critical immersion is stable.

Example 7 The identity map \( \text{Id} \) of any compact Kähler manifold is stable by the above corollary. In the case \( \dim N = 2 \) or \( 4 \), we have the stronger information that \( \text{Id} : N \to N \) globally minimizes \( E \) within its homotopy class. Note that
\[
L_{\text{Id}} = J^{-1} \delta^{-1} \delta d J,
\]
so that \( Y \) is in the kernel of \( L_{\text{Id}} \) if and only if \( d \delta J Y = 0 \), i.e., \( Y \) is a symplectic vector field. In particular, the kernel of \( L_{\text{Id}} \) is infinite dimensional.

Example 8 In Example 3 we proved that the projection of a warped product
\[
\phi : P \times_f N \to N
\]
is critical when \( N \) is a Kähler manifold. This is not an immersion; however, we saw that \( Z_{\phi} = 0 \), so such a map is stable nonetheless.

3 The Hopf Map

In this section we prove that the Hopf map \( \pi : S^3 \to S^2 \) is stable and calculate the spectrum of its Hessian. We then apply this to prove the conjecture of Ward regarding the full Faddeev-Hopf model mentioned in the introduction.

We begin by introducing some Lie group and Lie algebra technicalities regarding symmetric and Hermitian symmetric spaces.

Assume that \( G \) is a compact, connected, simple Lie group and that \( K \) is a compact subgroup of \( G \) such that \( G/K \) is an irreducible Hermitian symmetric space of compact type. On the Lie algebra level we have the standard orthogonal decomposition

\[
g = \mathfrak{k} + \mathfrak{p},
\]
where \( \mathfrak{t} \) is the Lie algebra of \( K \) and \( \mathfrak{p} \) an \( \text{Ad}_K \)-invariant subspace with the property that \([\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{t}\). It is well known that the Hermitian structure on \( G/K \) is induced by the adjoint action of an element in the centre of \( \mathfrak{t} \); as customary, we denote this element by \( J \).

We provide \( G \) with the Riemannian metric induced by the negative of the Killing form (or a suitable multiple thereof), and give \( G/K \) the metric which turns the homogeneous projection

\[
\phi : G \to G/K, \quad g \mapsto g \cdot o
\]

into a Riemannian submersion; here \( o \) denotes the identity coset in \( G/K \). The fibres of \( \phi \) are clearly minimal, even totally geodesic; according to Example 4, \( \phi \) is a critical point of the functional, and indeed for the full Faddeev-Hopf model. For simplicity, we denote by \( \langle \cdot, \cdot \rangle \) the negative of the Killing form on \( g \).

The pullback bundle \( \phi^{-1}TG/K \) is isomorphic to the trivial bundle \( G \times \mathfrak{p} \) by the map

\[
G \times \mathfrak{p} \ni (g, X) \mapsto \frac{d}{dt}|_{t=0} \phi(g) \exp tX \cdot o \in TG \setminus o \in T \phi_o(G)/K;
\]

the metric on \( \phi^{-1}TG/K \) corresponds under this isomorphism to the metric on \( G \times \mathfrak{p} \) induced by the restriction of \( \langle \cdot, \cdot \rangle \) to \( \mathfrak{p} \). Similarly, we can identify \( TG \) with the trivial bundle \( G \times \mathfrak{g} \) by left translation, and this gives the following commutative diagram:

\[
\begin{array}{ccc}
G \times \mathfrak{g} & \xrightarrow{\phi^{-1}} & TG \\
\downarrow & & \downarrow d\phi \\
G \times \mathfrak{p} & \xrightarrow{\phi} & \phi^{-1}TG/K
\end{array}
\]

The map on the left, which we thus identify with \( d\phi \), is induced by orthogonal projection

\[
\mathfrak{g} = \mathfrak{t} + \mathfrak{p} \to \mathfrak{p}.
\]

With this identification in mind, we think of sections of \( TG \) as functions on \( G \) with values in \( \mathfrak{g} \) and sections of \( \phi^{-1}TG/K \) as functions on \( G \) with values in \( \mathfrak{p} \):

\[
\Gamma(TG) \cong C^\infty(G, \mathfrak{g}), \quad \Gamma(\phi^{-1}TG/K) \cong C^\infty(G, \mathfrak{p}).
\]

The sections of \( TG \) are of course also derivations: for any vector space \( V \) and smooth function \( f : G \to V \), an element \( X \in C^\infty(G, \mathfrak{g}) \) acts on \( f \) as

\[
X(f)(g) = df(X)(g) = \left. \frac{d}{dt} \right|_{t=0} f(g \exp(tX(g))) \quad (g \in G).
\]

With these identifications, we can view \( L_\phi \) as a differential operator on \( C^\infty(G, \mathfrak{p}) \).

**Proposition 3** Choose an orthonormal basis \( \{e_k\}_{k=1}^m \) for \( \mathfrak{g} \) such that \( e_1, \ldots, e_{m-n} \) is a basis for \( \mathfrak{t} \) and \( e_{m-n+1}, \ldots, e_m \) a basis for \( \mathfrak{p} \). For the homogeneous projection \( \phi \), the second variation takes the form

\[
L_\phi Y = -J \left( -\frac{\lambda}{2} J(Y) - J \sum_{k=1}^m e_k e_k(Y) + \frac{3}{2} \sum_{a=1}^n \langle e_a(Y), e_a \rangle + \sum_{r,s=m-n+1}^m \omega(e_r e_s, e_r) - \frac{1}{2} [e_r, e_s] e_r \right).
\]

Here \( \lambda \) is the eigenvalue of the Casimir operator associated with the adjoint representation of \( \mathfrak{g} \):\[
-\sum_{k=1}^m [e_k, [e_k, X]] = \lambda X \quad (X \in \mathfrak{g}).
\]

The Hopf map is by definition the map

\[
\pi : S^3 \subset \mathbb{C}^2 \to \mathbb{C}P^1, \quad \pi(z_1, z_2) = [z_1, z_2].
\]
It is easy to see that the Hopf map is a Riemannian submersion with totally geodesic fibres. In particular, it is a harmonic map and so a critical point to the full Faddeev-Hopf model. In fact, by the identifications
\[ S^3 \cong SU(2), \quad CP^1 \cong SU(2)/S(U(1) \times U(1)), \]
we get the alternative definition of the Hopf map as the homogeneous projection
\[ \pi : SU(2) \rightarrow SU(2)/S(U(1) \times U(1)). \]
The Lie algebra \( su(2) \) of \( SU(2) \) has an orthonormal basis
\[ \vartheta_1 = \frac{i}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \vartheta_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \vartheta_3 = \frac{i}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]
and \( p \) is the span of \( \vartheta_1 \) and \( \vartheta_2 \). This gives us the formula
\[ L_\pi = C \cdot \text{Id} + \begin{pmatrix} \vartheta_2^2 & -\vartheta_3 - \vartheta_1 \vartheta_2 \\ \vartheta_3 - \vartheta_2 \vartheta_1 & \vartheta_1^2 \end{pmatrix} \]
where \( C = - (\vartheta_1^2 + \vartheta_2^2 + \vartheta_3^2) \) is the Casimir operator.

Using the Peter-Weyl theorem to study the action of \( L_\pi \) on \( C^\infty(SU(2), p) \), we can deduce the following result.

**Theorem 2** The Hopf map is stable for the pure Faddeev-Hopf model; the Hessian has eigenvalues
\[ \frac{1}{4}(n^2 + 2n) \text{ and } \frac{1}{4}(n - 2k)^2, \quad k = 0, \ldots, n, \quad n = 1, 2, \ldots. \] Each eigenspace is of infinite dimension.

Let us finally return to the full Faddeev-Hopf model
\[ E_\alpha(\phi) = \frac{1}{2} \int_{SU(2)} (|d\phi|^2 + \alpha |\phi^* \omega|^2) \ast 1. \]

Urakawa had shown that the Hessian, or Jacobi operator, of the Hopf map for the usual energy functional is given by
\[ J_\pi = C \cdot \text{Id} - 2 \begin{pmatrix} 0 & \vartheta_3 \\ -\vartheta_3 & 0 \end{pmatrix}. \]

Combining this with (3), we can get precise information on the stability of the Hopf map for the full Faddeev-Hopf model.

**Theorem 3** The Hopf map is an unstable critical point of the Faddeev-Hopf energy functional if \( \alpha < 1 \) and a stable critical point if \( \alpha \geq 1 \).

In particular, this proves Ward’s conjecture mentioned in the introduction.

**References**


