ABDUCTION IN PATTERN GENERALIZATION

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In this paper we explain generalization of patterns in algebra in terms of a combined abduction-induction process. We theorize and provide evidence of the role abduction plays in pattern formation and generalization and distinguish it from induction.

INTRODUCTION

The focus of this theoretical paper is to extrapolate the role of abduction in generalization with examples drawn from our recent research on patterning in algebra among sixth-grade students (Becker & Rivera, 2006a, 2006b). We utilize a question-and-answer format in order to surface important issues that pertain to abduction as it is explored in the context of elementary algebraic thinking and learning.

WHAT IS ABDUCTION?

Peirce introduced the notion of abduction in the 19th century in relation to induction and deduction. For him, an inferential act takes at least three forms and the choice of which form to pursue ultimately depends on the available knowledge base. He argues that while deductive inferences will always be valid, however, their validity rests on having a complete knowledge base. Further, he points out that the fundamental task of abduction and induction involves the production of generalizations from an always-already incomplete knowledge base. Hence, they are both deductively invalid.

Figure 1 provides an illustrative summary of the three inferential modes. Following Peirce, Deutscher (2002) foregrounds how inferences are either deductive or ampliative. While deductive inferences always yield valid and necessary conclusions, ampliative inferences tend to produce generous and, consequently, fallible, conclusions that have not necessarily been drawn from the premises. For example, James’s general formula \( C = 2n - 1 \) and general description (“doubling a row and minus using a chip”) and Jane’s diagrammatic description (Figure 3) for the circle pattern in Figure 2 assume the additional information that it is increasing. Thus, it is possible for learners to perceive the same sequence in different ways.

IS ABDUCTION THE SAME AS OR DIFFERENT FROM INDUCTION?

Deutscher (2002) distinguishes abduction and induction in terms of conceptual leap and generalization, respectively. That is, while induction involves generalizing an attribute or a relationship from at least two particular instances to a presumed entire class of objects with some additional assumptions, abduction necessitates a conceptual leap from the given instances to an explanatory hypothesis. Further, while induction constructs obvious generalizations, abduction produces an entirely different level of abstraction (p. 471). For Abe (2003), Peircean abduction is another form of discovery or suggestive reasoning that “discovers new events” (p. 234) and yields

Reasoning Types

Nature of Knowledge

Deduction
Valid and necessary inference; conclusions are not generalizations; could only be performed with a complete knowledge base

Abduction
Generates a viable inference from an incomplete knowledge base; inference is ampliative

Induction
Conclusions are generalizations and ampliative; relies on an incomplete knowledge base

Predict

Inference

Confirm

Inference

Figure 1: Taxonomy of the inferential trivium

Figure 2: Dot Pattern

Figure 3: Jane’s Diagrammatic Description of the Succeeding Steps in the Dot Pattern

Figure 4: Pattern Generalization Scheme
explanations rather than predictions because they are not directly knowable. It is similar to induction insofar as both are concerned with discovery. However, it is distinguished from induction in that the latter “discovers tendencies that are not new events” (p. 234). Induction tests an abducted hypothesis through extensive experimentation and increased success on trials means increased confidence in the hypothesis. For example, James’s inductive success in calculating several far generalization tasks for the circle pattern in Figure 2 (such as obtaining the total number of circles in steps 10 and 100) has increased the confidence he has in his abducted formula. Seeing consistency in the calculated values, the formula, and the figures, the combined abduction-induction process enabled him to finally state a generalization. Figure 4 illustrates how the combined process materializes in a generalization activity from the beginning phase of noticing a regularity $R$ in a few specific cases to the establishment of a general form $F$ as a result of confirming it in several extensions of the pattern and then finally to the statement of a generalization.

**IF AN ABDUCTION IS INFERRED FOR A PATTERN, IS IT THE BEST?**

Nothing so far has been said about how to decide which abduction makes the most sense. In fact, what we can assume to be a consequence of theory or concept generation in abduction is that the process cannot be taken lightly in the form of “happy guesses.” Thus, it makes sense to add an evaluation component to abduction by ascertaining if it is the inference or reasoning that yields the best explanation. J. Josephson and S. Josephson (1994) summarize this broader version of abduction in the following manner:

- $D$ is a collection of data (facts, observations, givens).  
- $H$ explains $D$ (would, if true, explain $D$).  
- No other hypothesis can explain $D$ as well as $H$ does.  
- Therefore, $H$ is probably true.

(Josephson & Josephson, 1994, p. 5)

Thus, while Peirce’s version recommends steps (1), (2), and (4), J. Josephson and S. Josephson point out the necessity of step (3). Further, J. Josephson (1996) lists the following “normative considerations” in assessing the “strength of an abductive conclusion:” (1) How good $H$ is by itself, independently of considering the alternatives; (2) How decisively $H$ surpasses the alternatives, and; (3) How thorough the search was for alternative explanations (p. 1). This reconceptualized version of abduction enables us to further distinguish between an abductive reasoning process and an abductive justification, with the former focusing on satisfaction and the latter confidence in accepting a stated abduction. Further, J. Josephson (1996) argues that while generalization assists in explaining a perceived characteristic of or a commonality among a given sequence, “it does not explain the instances themselves” (p. 2). The warrant in an explanation lies in its capacity to “give causes” and it
certainly does not make sense to think that a generalization can provide an explanation that causes the instances. For example, while the general form \( C = 2n - 1 \) explains the relationship between elements in the class \{1, 3, 5, 7, \ldots\} in Figure 2, it does not cause them. That is, the nature of “explanation” in this type of generalization is determined not by an observed fact but by the observed “frequency of [a] characteristic” in both the small and extended samples (p. 3). Thus, what needs to be explained or be given a “causal story” deals with the nature in which frequencies in a class are produced and justified. In particular, in a pattern sequence that consists of figural cues, a generalization may be explained in terms of how it is reflected in the cues themselves that produce them. For example, Dung articulated his explanation of the general formula \( C = 2n - 1 \) in Figure 2 under item E in Figure 5. His explanation justifies why his calculated frequencies were the way they were, including the inductive projection (i.e., observations \( \rightarrow \) All A’s are B’s \( \rightarrow \) The next A will be a B) which he employed in dealing with all far generalization tasks.

Figure 5: Dung’s explanation of the generalization \( C = 2n - 1 \) in Figure 2

HOW CAN TEACHERS ASSIST THEIR STUDENTS TO ASSESS THE REASONABLENESS OF AN ABDUCTION?

How do we justify the logic of reason behind abductive inferences involving patterns especially if we consider the fact that there might be several available plausible alternatives to choose from? Resolving this issue will in some way address the practical concern of mathematics teachers who need to assist their students to develop reasoned judgments about ampliative inferences made in relation to a generalizing task, including ways to evaluate and reconcile students’ generalizations with the intended ones. Psillos (1996) advances the following conditions that an ampliative inference must fulfill: (1) It must be non-monotonic, i.e., it must allow that a certain conclusion be defeated by the inclusion of extra information in the premises; (2) It must deal with the “cut-off point” problem, i.e., it must show how and why generalizations from samples to populations as a whole are warranted; (3) It must allow for vertical extrapolation, i.e., it must support conclusions that involve reference to types of entity (or, more generally, vocabulary) that are not already referred to in the premises, e.g., positing scientific unobservables, and; (4) It must accommodate the eliminative dimension of ampliative reasoning, i.e., the fact that in typical cases of ampliative reasoning, more than one hypotheses consistent with the
premises are considered and attempt is being made to find grounds to eliminate (ideally) all but one of them (pp. 1-2). If we assume that any abductive generalization made about a pattern sequence of objects should meet the extended requirement of being an inference to the best explanation, then we fulfill the above conditions in the following manner:

1. Non-monotonicity is satisfied. An abductive generalization of a pattern that offers the best explanation can still be shown false if additional or different assumptions are made which would then necessitate developing a different generalization. For example, the best general formula for the pattern sequence in Figure 2 is \( C = 2n - 1 \) if we agree with James’s assumptions. However, if we add the premise that the pattern is, say, oscillating after every four terms given the available cues as the original premises, James’s rule would no longer hold to be true.

2. The cut-off point problem is solved. Mere abduction develops generalizations out of a few instances and inducing a general form out of repeated abduction of the same form for several more instances might still not provide the best explanation. However, an abduced generalization that offers the best explanation provides the cut-off point in that it can explain why the stated generalization that depends only on a few instances (sample) actually holds for the entire class (population). For example, James and Dung provided the best visual-based explanations that warrant the form \( C = 2n - 1 \) for the sequence in Figure 2. There were other students who provided abductions that were not warranted such as Cherrie who hypothesized that since step 10 has 19 circles (after listing the number of circles per step from step 1 to 9), then step 20 has 29 circles, step 30 has 39 circles, and so on based on a numerical relationship that she perceived among the digits in both dependent and independent terms.

3. Vertical extrapolation is achieved. An abductive generalization of a pattern that provides the best explanation oftentimes draws on the deep structure of the available and unavailable cues. For example, Demetrio’s additive generalization “just add two for every figure” for the sequence in Figure 2 is a superficial observation and could not be easily employed when confronted with a far generalization task. The ones offered by James and Dung relied on an hypothesis that was based on an unobservable perceptual knowledge which enabled them to see a relationship between two sets of circles.

4. The eliminative dimension is accommodated. An abductive generalization of a pattern that offers the best explanation has been chosen from several plausible ones and judged most tenable on the basis that it provides a maximal understanding of the pattern beyond what is superficially evident. For example, the strength and unifying power of Dung’s abductive generalization eliminated Demetrio’s version despite the fact that both students saw an additive relationship among the cues.

Teachers who are aware of the above conditions in relation to the formation of a generalization about a pattern sequence of objects will be capable of exercising...
judgment about which abduction will offer the best explanation; it will also enable them to “separate good from bad potential explanations” (Psillos, 1996, p. 6). Further, students will not be misled into thinking that anything goes in abducting a generalization. The requirement of non-monotonicity foregrounds the necessity of stating assumptions about a pattern undergoing generalization; it assists in confronting biases and in resolving situations of conflict between several viable claims of generalization for the same pattern. The requirement of a cut-off point surfaces the need to provide a justification for a global-type of generalization (versus a local one) that holds in both specific cases and the entire class of cases. The requirement of vertical extrapolation focuses on providing a generalization that can be explained in a deeper way by using perceptual knowledge or other relevant mathematical idea or concept that bears on the class. Finally, the requirement of eliminative dimension makes it possible to consider several possible generalizations for a pattern, however, it also necessitates making a judgment about which one(s) will make the most sense.

WHAT ARE SOME IMPLICATIONS FOR THEORY AND RESEARCH?

Abduction plays a significant role in the logic of discovering and establishing a generalization of a pattern sequence. Expressing generalities about patterns cannot simply be reduced to, and equated with, training for it is abductive which can be approached in several different ways (leading to different hypotheses) and is always-already complicated by the fact that it is mediated by signs. Following Peirce, induction takes the position of verifying an already established generalization which has been initially drawn and captured through abduction. Psillos’s (1996) four general conditions above can assist teachers and learners to develop an abductive generalization that provides the best explanation. Hence, it goes without saying that not all abductions are equally valuable and tenable despite the fact that all are equally viable. A more pressing issue deals with how to decide the goodness of an abduction in relation to pattern construction and generalization. Aside from Psillos, we acquire three more conditions from Peirce, as follows: (1) A good abductive generalization made about a pattern should be able to explain the facts, i.e., there is a reliable and justifiable causal story behind why the known, including and especially the unknown, instances are the way they are; (2) The generalization should not surprise us, i.e., we expect that it will hold in the largest domain possible. We do not want to frustrate ourselves with a generalization that seems to always fail in situations when new cases are introduced for verification; (3) The generalization should stand experimental verification, i.e., in Psillos’s (1996) terms, it is non-monotonic with a well-justified cut-off point and has been vertically extrapolated (Peirce, 1958, vol. 5, par. 197).

In the context of a pattern sequence that consists of figural cues, Radford (2006) distinguishes between naïve induction and generalization as follows: naïve induction is when a student primarily employs a numerical heuristic such as trial and error and exhibits improbable reasoning in order to guess a formula for the pattern; generalization is when a student searches for a commonality among the available figural cues in the
pattern, notices one or several common features or relationships and then establishes a generalization in the sense that a general property has been noticed in the particular. The theoretical distinction has allowed Radford to define algebraic generalization more precisely as follows: “Generalizing a pattern algebraically rests on the capability of grasping a commonality noticed on some elements of a sequence S, being aware that this commonality applies to all the terms of S [i.e., the formation of a genus] and being able to use it to provide a direct expression [i.e., elaborated in the form of a schema] of whatever terms of S” (p. 5). Two issues are worth clarifying from the point of view of abduction as it has been explored in this paper. First, Radford’s distinction between naïve induction and algebraic generalization can actually be collapsed under abductive reasoning since both exhibit probable reasoning in an apparent, purposeful search to discover and establish a generalization from an incomplete knowledge base S. The statement of a rule in Radford’s naïve induction, even if it has been obtained “by accident,” can be shown to involve all the three elements which for him constitutes an algebraic generalization: there is a commonality that is noticed from one term to the next; an assumption has been imposed in which the rule applies to all the terms, and; there is the presence of a direct expression despite the use of a cumbersome method such as trial and error. If a sequence S consists of only numerical cues, most students will have no other recourse to generalizing except through trial and error, systematically acquired or otherwise (Becker & Rivera, 2005). A more theoretically tenable and useful distinction in relation to pattern generalization involves acts of abduction and induction. That is, does an act constitute developing and discovering a perceived commonality (abduction), or is it verifying the commonality (induction) leading to a generalization? Second, Radford is right in claiming that the construction of a direct expression depends on the algebraic capacity of a student making the generalization. That is, the different layers of expressing generality (factual, contextual, and symbolic) rely at the very least on the student’s facility and fluency in representing with variables. However, what is not clearly articulated in his characterization of algebraic generalization deals with how to assess if it is the best generalization possible for S. Generalizing a pattern algebraically rests on all three elements that Radford stated as important and necessary with the additional condition that it addresses the criteria that have been identified as important by Peirce, Josephson, and Psillos above. Other useful criteria perhaps need to be further explored.

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REFERENCES


