PROTOTYPE PHENOMENA AND COMMON COGNITIVE PATHS IN THE UNDERSTANDING OF THE INCLUSION RELATIONS BETWEEN QUADRILATERALS IN JAPAN AND SCOTLAND

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This study explores the status and the process of understanding of ‘the inclusion relations between quadrilaterals’, which are known to be difficult to understand, in terms of the prototype phenomena and the common cognitive paths. As a result of our analysis of data gathered in Japan and Scotland, we found that the students’ understanding was significantly different for each inclusion relation, and that there were strong prototype phenomena related to the shapes of the square and rectangle in Japan, and related to angles in Scotland, the factors which prevent students from fully grasping inclusion relations. We also confirmed the existence of common cognitive paths in Japan and Scotland, and based on these paths discussed a possible route to teach the inclusion relations between quadrilaterals by analogy.

INTRODUCTION

The learning of the inclusion relations between quadrilaterals provides students with an opportunity to develop logical reasoning skills, and is regarded as an introductory process into deductive geometry (Crowley, 1987, van Hiele, 1986, 1999). In terms of van Hiele’s model, at level 3 students are expected to be able to deduce that a rectangle is a special type of parallelogram, based on the definition and the properties of each quadrilateral, while at level 2 they simply recognize the properties of each separate shape (We are using the 1-5 numeration of van Hiele’s model). However, research evidence suggests that the rate of progress from level 2 to level 3 is slow, and that many students remain at level 2 even at the end of secondary schools (e.g., Senk, 1989). Thus, the classification of quadrilaterals by inclusion has been shown to be a difficult task (de Villiers, 1990, 1994).

However, it has also been suggested that some inclusion relations between quadrilaterals are easier to grasp than others (Okazaki, 1995). For example, Japanese 6th grade students are more likely to recognize a rhombus as a special type of parallelogram than to see a square or a rectangle as a parallelogram. In this paper we shall first investigate whether this phenomenon can be recognized at an international level by comparing data from Japan and the UK (Scotland). If indeed evidence of this is found, we shall then explore the common cognitive paths (Vinner and Hershkowitz, 1980) that suggest the process students commonly follow in understanding the links between different shapes, starting from the easier and progressing to more difficult conceptual links. Such information will suggest routes by which we may enable students to understand the inclusion relations between quadrilaterals more effectively.
THEORETICAL BACKGROUNDS

Prototype examples as implicit models in the geometrical thinking

A geometrical figure is a ‘figural concept’ that has aspects, which are both conceptual (ideality, abstractness, generality and perfection), and figural (shape, position, and magnitude) (Fischbein, 1993). However, Fischbein indicated that “the fusion between concept and figure in geometrical reasoning expresses only an ideal, extreme situation usually not reached absolutely because of psychological constraints” and that “the figural structure may dominate the dynamics of reasoning” for many students. This has also been observed in primary trainee teachers. For example, Fujita and Jones (2006) found that among primary trainee teachers in Scotland prototype images in their personal figural concepts have a strong influence over how they define/classify figures.

This tendency to rely on figural aspects is known as the ‘prototype phenomenon’ (Hershkowitz, 1990). The key factor is the prototype example, which is “the subset of examples that is the ‘longest’ list of attributes – all the critical attributes of the concept and those specific (noncritical) attributes that had strong visual characteristics” (ibid., p. 82). Students often see figures in a static way rather than in the dynamic way that would be necessary to understand the inclusion relations of the geometrical figures (de Villiers, 1994). As a result of this static visualisation, some students are likely to implicitly add certain properties such as ‘in parallelograms, the adjacent angles are not equal’ and ‘in parallelograms, the adjacent sides are not equal’ besides the true definition (Okazaki, 1995), which are likely to be a result of the prototypical phenomenon of parallelograms. We assume that figural concepts, including tacit (falsely assumed) properties, act as ‘implicit models’ (Fischbein et al, 1985) in geometrical thinking. This hypothesis will later be used to analyse the difficulties in understanding the classification of quadrilaterals by inclusion relations.

Common cognitive paths

The ‘common cognitive path’ refers literally to a statistical method for identifying a path that many students follow to recognize similar concepts (Vinner and Hershkowitz, 1980). The basic idea is as follows (pp. 182-183):

Denote by a, b, c, d respectively the subgroups of people that answered correctly the items that test aspects A, B, C, D. Suppose, finally, that it was found that a ⊃ b ⊃ c ⊃ d.

We may claim then that A→B→C→D is a common cognitive path for this group (in the sense that nobody in the group can know D without knowing also A, B, C and so on).

This view is of course idealistic. It may be found for example that there are students who answer A incorrectly and B correctly. Thus Vinner and Hershkowitz proposed that the existence of a common cognitive path from A to B be recognized where a significant difference between m(a)/N and m(a and b)/m(b) exists through the chi-square test (Vinner and Hershkowitz, 1980). Several researchers have already found some common cognitive paths. For example, Vinner and Hershkowitz (ibid.) investigated them for obtuse and straight angles, right-angled triangles, and the altitude
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in a triangle. Nakahara (1995) also found for basic quadrilaterals that parallelogram → rhombus → trapezium is a common cognitive path among Japanese primary school children.

METHODOLOGICAL CONSIDERATION

Subjects

We collected data from 234 9th graders from Japanese public junior high school in 1996 and Scottish 111 trainee primary teachers in their first year of university study in 2006 (aged about 15 and 18 respectively). Although there is an age difference between the subjects in the two counties, we consider this comparison to be worthwhile for the following reasons: Under their respective geometry curricula, both of the subject groups have finished studying the classification of, and relations between, quadrilaterals. In the Japanese case, where the students were still in school, no revision of this topic is specified in the curriculum for their remaining years of high school education. The two subject groups were given exactly the same questionnaire in their own languages. If we could find evidence of common mathematical behaviour, we believe this could indicate more general, global findings, which override local factors.

Questionnaire and analysis

A questionnaire, as shown in Table 1 (Okazaki, 1999), was designed based on Nakahara’s study (1995). It consists of five main questions, each with sub-questions, giving in total 40 questions. Questions 1, 2 and 3 ask students to choose images of parallelograms, rectangles and rhombuses from various quadrilaterals. These questions are used to check what mental/personal images of quadrilaterals students have. Question 4 asks whether mathematical statements concerning parallelograms, rectangles and rhombuses are true or false, which is used to judge what implicit properties our students have developed in terms of the inclusion relations. For example, they were requested to judge whether the following statement is true or false, ‘There is a parallelogram which has all its sides equal’. The fifth and last question asks students to judge directly the inclusion relations between rhombuses/parallelograms, rectangles/parallelograms, squares/rhombuses and squares/rectangles. While questions 1-3 reveal what mental/personal images of quadrilaterals students have, this is not enough to judge at what level of van Hiele’s model they may be assessed. As we have discussed, geometrical figures are fundamentally ‘figural concepts’, and the performance of students in questions 4 and 5 will provide us with more information about how students understand the relations between quadrilaterals.

Next, common cognitive paths will be examined based on the students’ overall performance in the questionnaire’s problems related to the inclusion relations between quadrilaterals. To do this, we first identify the students who are considered to have more or less sound understanding of each inclusion relation. As a standard for choosing the students, we adopt more than 70 % correct answers to all of the questions in line with Nakahara’s approach (1995). Next, we produce 2 by 2 cross tables to
examine whether a common cognitive path may exist between each relation by using the chi-square test.

Table 1. Questionnaire.

| Q1. | Q4. Read the following sentences carefully, and put ( / ) for those you think are correct, ( X ) for those that are incorrect, and if you are not sure, put ( ? ) Questions about Parallelograms (a) ( ) The lengths of the opposite sides of parallelograms are equal. (b) ( ) There are no parallelograms which have equal adjacent sides. (c) ( ) The opposite angles of parallelograms are equal. (d) ( ) There are no parallelograms which have equal adjacent angles. (e) ( ) There is a parallelogram which has all its sides equal. (f) ( ) There is a parallelogram which has all equal angles. Questions about Rectangles (a) ( ) The lengths of the opposite sides of rectangles are equal. (b) ( ) There are no rectangles which have equal adjacent sides. (c) ( ) The adjacent angles of rectangles are equal. (d) ( ) The opposite angles of rectangles are equal. (e) ( ) There is a rectangle which has all equal sides. Questions about Rhombuses (a) ( ) The lengths of the opposite sides of rhombuses are equal. (b) ( ) The adjacent sides of rhombuses are equal. (c) ( ) There are no rhombuses which have equal adjacent angles. (d) ( ) The opposite angles of rhombuses are equal. (e) ( ) There is a rhombus which has all equal angles. Q5. Read the following sentences carefully, and put ( / ) for those you think are correct, ( X ) for those which are incorrect, or if you are not sure, put ( ? ). 1. About parallelograms and rhombuses (a) ( ) It is possible to say that parallelograms are special types of rhombuses. (b) ( ) It is possible to say that rhombuses are special types of parallelograms. 2. About parallelograms and rectangles (a) ( ) It is possible to say that parallelograms are special types of rectangles. (b) ( ) It is possible to say that rectangles are special types of parallelograms. 3. About squares and rhombuses (a) ( ) It is possible to say that squares are special types of rhombuses. (b) ( ) It is possible to say that rhombuses are special types of squares. 4. About squares and rectangles (a) ( ) It is possible to say that rectangles are special types of squares. (b) ( ) It is possible to say that squares are special types of rectangles. 

RESULT AND DISCUSSION

Does the prototype phenomenon happen in students’ understanding of inclusion relations?

Table 2 summarises the results (the percentages of correct answers) to the parts of questions (Q) 1-4 which relate to inclusion relations. Through analysis of these results we can observe some interesting mathematical behaviours among our subjects.
Let us examine the answers to Q1~3 (images). Firstly, both Japanese and Scottish students gave very similar responses to Q1. That is, while over 74% students recognized rhombuses as parallelograms (Q1 d and f), many failed to see rectangles as a special type of parallelogram, as their scores dropped by over 15% (Q1c and e).

<table>
<thead>
<tr>
<th></th>
<th>Parallelogram (%)</th>
<th></th>
<th>Rectangle (%)</th>
<th></th>
<th>Rhombus (%)</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Images</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q1 d (Rho.)</td>
<td>78.6</td>
<td>75.7</td>
<td>Q2 c (Squ.)</td>
<td>29.5</td>
<td>45.1</td>
<td>Q3 b (Squ.)</td>
</tr>
<tr>
<td>Q1 f (Rho.)</td>
<td>78.2</td>
<td>74.8</td>
<td>Q2 e (Squ.)</td>
<td>28.6</td>
<td>41.4</td>
<td>Q3 d (Squ.)</td>
</tr>
<tr>
<td>Q1 e (Rec.)</td>
<td>51.3</td>
<td>60.4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q1 e (Rec.)</td>
<td>49.6</td>
<td>58.6</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Properties</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q4PA b (Rho.)</td>
<td>62.8</td>
<td>55.9</td>
<td>Q4RE b (Squ.)</td>
<td>38</td>
<td>55.9</td>
<td>Q4RH e (Squ.)</td>
</tr>
<tr>
<td>Q4PA e (Rho.)</td>
<td>59</td>
<td>49.5</td>
<td>Q4RE e (Squ.)</td>
<td>25.2</td>
<td>38.7</td>
<td>Q4RH e (Squ.)</td>
</tr>
<tr>
<td>Q4PA d (Rec.)</td>
<td>40.6</td>
<td>50.5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q4PA f (Rec.)</td>
<td>41</td>
<td>36.9</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Secondly, we can observe that many students failed to see a square as a special type of a rectangle and a rhombus. However, the two groups’ performance in this question was different; for Japanese students the most difficult was to see a square as a rectangle, and for Scottish trainees the main problem was to recognize a square as a rhombus.

Thirdly, there is a significant difference between the scores for two questions relating to two identical inclusion relations. While the rhombus/parallelogram relation is related to the ‘length of sides’ and corresponds with the square/rectangle relation by analogy, the scores for Q1 d (rhombus/parallelogram) were 78% in Japan and 75% in Scotland while for Q2 c (square/rectangle) they were 30% and 45% respectively. Similarly, the inclusion relations between rectangle/parallelogram and square/rhombus are both related to angles, but we can observe a similar tendency: in particular, the percentage of correct answers by Scottish students dropped by over 25%. We consider these tendencies suggest that both Japanese and Scottish students’ reasoning is not governed conceptually, but rather is influenced by the prototype images of quadrilaterals, i.e. that the prototype phenomenon occurs.

Now, let us examine students’ performance in Q4 (properties). While the scores are slightly worse than for Q1~3, we can observe similar tendencies in the answers. These results suggest that our subjects did not only judge based on their own images, but they at least implicitly create and utilise ‘additional’ properties, such as ‘parallelograms do not have equal adjacent angles’. For the true properties of quadrilaterals such as ‘the opposite angles of parallelograms are equal’, our subjects in both countries showed a good understanding. The score for questions with a ‘True’ correct answer was generally over 85%, with the following exceptions: 74% of Japanese students answered correctly Q4RHb (adjacent sides of rhombuses), and 69%, 77%, 55% and 70% of Scottish trainees answered correctly Q4REc (adjacent angles of rectangles), Q4RHa (opposite sides of rhombuses), Q4RHb (adjacent angles of rhombuses), and
Q4RHd (opposite angles of rhombuses), respectively. As we have seen in Q1~3, Scottish trainees showed particularly weak knowledge in rhombuses.

Finally, let us examine Q5, shown in table 3 below.

Table 3: The correct answers (%) for the direct questions.

<table>
<thead>
<tr>
<th></th>
<th>Rhom/Parall</th>
<th>Rect/Parall</th>
<th>Sq/Rect</th>
<th>Sq/Rhom</th>
</tr>
</thead>
<tbody>
<tr>
<td>Direct (Q5)</td>
<td>J 69%</td>
<td>S 41%</td>
<td>J 50%</td>
<td>S 40%</td>
</tr>
<tr>
<td></td>
<td>J 50%</td>
<td>S 59%</td>
<td>J 37%</td>
<td>S 40%</td>
</tr>
<tr>
<td></td>
<td>J 40%</td>
<td>S 25%</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Data from Japan again showed a similar tendency to Scottish students, and these results are consistent with those from Q1~4. Japanese students’ performance is better than the Scottish group. On the one hand, we speculate that our Scottish trainees might not be familiar with the type of question posed for Q5 and that is why they performed relatively poorly for Q5. On the other hand, this poor performance also suggests that the prototype phenomenon appears more strongly among Scottish trainees than Japanese students.

In summary, considering all answers to Q1~4, we suggest that students’ personal figural concepts of all quadrilaterals are likely to consist of ‘Prototype image + true properties + implicit properties caused by prototype images’.

Common cognitive paths

Table 3 above suggests that an order of difficulty exists within the understanding of the relations between quadrilaterals, e.g. the rhombus/parallelogram relationship might be grasped more easily than the square/rhombus relation, which suggests the existence of common cognitive paths. We first examined the number of nearly achieving subjects by the criteria described in the methodology section (using a standard of more than 70%). However, as we have seen in table 3, Scottish trainees showed particularly low scores for question 5, and hence the number of such subjects significantly dropped. Thus, in this paper we judge evidence of good understanding to be correct answers in 3 out of the first 4 questions (Q1~4, images and properties), as summarised in table 4. We then produce a 2 by 2 table and examine by using the chi-square test whether a common cognitive path may exist between each relation.

Table 4: Nearly achieving subjects on the images and the properties (%).

<table>
<thead>
<tr>
<th></th>
<th>Rhom/Parall</th>
<th>Rect/Parall</th>
<th>Sq/Rect</th>
<th>Sq/Rhom</th>
</tr>
</thead>
<tbody>
<tr>
<td>J 62%</td>
<td>S 52%</td>
<td>J 35%</td>
<td>S 41%</td>
<td>J 19%</td>
</tr>
<tr>
<td>J 19%</td>
<td>S 37%</td>
<td>J 44%</td>
<td>S 17%</td>
<td></td>
</tr>
</tbody>
</table>

As we can see in figures 1 and 2, common cognitive paths are identified among our subjects. The subjects firstly understand the relation between rhombus/parallelogram in both countries. If we look at the paths more simply, the Japanese students’ path is square/rhombus, rectangle/parallelogram and finally square/rectangle, while the Scottish path is rectangle/parallelogram, square/rectangle and square/rhombus.
Implication for teaching of inclusion relations between quadrilaterals

Finally, we shall consider the implications for the learning and teaching of inclusion relations between quadrilaterals. Both Japanese and Scottish students are likely to first of all grasp the rhombus/parallelogram relation. We should make sure that they have fully appreciated this relationship, and then by using this relation as a starting point, consider teaching sequences based on the cognitive paths identified above. We speculate that if we taught them in the opposite order of the common cognitive path, students might not recognize all the relations. We may then consider the pedagogical approach by analogy in which we may use an easier relation in teaching more difficult relations.

For Japanese students, it is obvious that the prototype phenomenon appears strongly in squares and rectangles, and such prototype images and implicit properties are obstacles for the correct understanding of the rectangle/parallelogram and square/rectangle relations. The curriculum design in Japan might influence these tendencies: in Japan children learn these quadrilaterals first in the 2nd grade in primary schools, and in so doing they also informally learn them as ‘regular quadrangle’ and ‘oblong’. To tackle this problem, considering the common cognitive paths of Japanese students, it is suggested that, as a teaching strategy we use rhombus/parallelogram relation as analogy for square/rectangles (sides) and square/rhombus for rectangle/parallelogram (angles).

For Scottish trainees, while they have relatively flexible images of parallelograms, the strongest prototype phenomenon appears in squares. Also, considering their answers regarding rectangle/parallelogram relation (41%), it seems that the common cognitive barrier for them is ‘the size of angles’, i.e. for both cases it is very difficult for them to recognize that parallelograms or rhombuses can have all equal angles. In fact, a similar behaviour is observed in our pilot study, i.e. trainees in Scotland tend to define squares and rectangles by mentioning only ‘sides’ but not ‘angles’ (Fujita and Jones, 2006). Thus, for Scottish trainees, it might be effective to use the rhombus/parallelogram as analogy for square/rectangle, i.e. we could agitate their prototype images and implicit properties by asking them ‘Is it possible to have a parallelogram which has all equal angles if it is possible to have a parallelogram whose ‘sides’ are all equal?’
As a concluding remark, in this paper we have identified prototype phenomenon and common cognitive paths by quantitative approaches, and our next task is to examine qualitatively teaching sequences and approaches suggested by the identified cognitive paths (e.g. by conducting clinical interview etc.).

References


