EMBODIED, SYMBOLIC AND FORMAL ASPECTS OF BASIC LINEAR ALGEBRA CONCEPTS

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Many students find their first experience with linear algebra at university very challenging. They may cope with the procedural aspects of the subject, solving linear systems and manipulating matrices, but struggle to understand the crucial conceptual ideas underpinning them. This makes it very difficult to make progress in more advanced courses. In this research we have sought to apply APOS theory, in the context of Tall’s three worlds of mathematics, to the learning of the linear algebra concepts of linear combination, span, and subspace by a group of second year university students. The results suggest that the students struggled to understand the concepts through mainly process conceptions, but embodied, visual ideas proved valuable for them.

BACKGROUND

The motivation for considering student understanding of linear algebra is well summed up by Carlson (1993, p. 39),

My students first learn how to solve systems of linear equations, and how to calculate products of matrices. These are easy for them. But when we get to subspaces, spanning, and linear independence, my students become confused and disoriented.

Many university teachers will have had a similar experience. Students start well and cope with the procedural aspects of first courses, solving linear systems and manipulating matrices, but struggle to understand some of the crucial conceptual ideas underpinning the material, such as subspace, span, and linear independence, mentioned by Carlson. The action-process-object-schema (APOS) development in learning proposed by Dubinsky and others (Dubinsky & McDonald, 2001) suggests an approach different from the definition-theorem-proof that often characterises university courses. Instead mathematical concepts are described in terms of a genetic decomposition into their constituent actions, process and objects in the order these should be experienced by the learner. For example, there is little point presenting students with the concept of span if they do not understand linear combination, since span is an object constructed from the objects of scalar multiple and linear combination, each of which must be encapsulated from mathematical processes.

Tall and others (Gray & Tall, 1994) have extended these ideas to talk about procepts, the symbolisation of both a process and an object, so that symbols such as $3v, a_1u_1 + a_2u_2 + ... + a_nu_n$ etc. may be viewed from either perspective. In more recent developments of the theory Tall has introduced the idea of three worlds of mathematics, the embodied, symbolic and formal (Tall, 2004). The worlds describe a hierarchy of qualitatively different ways of thinking that individuals develop as new
conceptions are compressed into more thinkable concepts (Tall & Mejia-Ramos, 2006). The embodied world, containing embodied objects (Gray & Tall, 2001), is where we think about the things around us in the physical world, and it “includes not only our mental perceptions of real-world objects, but also our internal conceptions that involve visuo-spatial imagery.” (Tall, 2004, p. 30). The symbolic world is the world of procecepts, where actions, processes and their corresponding objects are realized and symbolized. The formal world of thinking comprises defined objects (Tall, Thomas, Davis, Gray, & Simpson, 2000), presented in terms of their properties, with new properties deduced from objects by formal proof. This theoretical stance implies that students can benefit from constructing embodied notions underpinning concepts by performing actions that have physical manifestations, condensing these to processes and encapsulating these as objects in the embodied world, alongside working in the symbolic world and, finally, the formal world. Many linear algebra concepts have embodied and symbolic representations; in fact several representations (Hillel, 2000). Thus a linear combination of two vectors may be experienced as a triangle of vector lines, symbolized as \( au + bv, \ a(u_1, u_2, u_3) + b(v_1, v_2, v_3) \), or otherwise. In linear algebra few students are given time and opportunities to develop embodied notions of basic ideas that may be considered trivial by the teacher. The research presented here used a framework (Figure 1) based on genetic decompositions of linear combination, span and subspace to investigate student understanding of these concepts and whether embodied constructs are useful.

<table>
<thead>
<tr>
<th>APOS</th>
<th>Embodied World</th>
<th>Algebra</th>
<th>Matrix</th>
<th>Formal World</th>
</tr>
</thead>
</table>
| Action | Can create a new vector \( w \) by, say addition, e.g. \( w = 3u + 5u = 8u \) | Can calculate with linear combinations, e.g. \[
\begin{bmatrix}
2 \\
3 \\
1 \\
\end{bmatrix} +
\begin{bmatrix}
4 \\
-1 \\
1 \\
\end{bmatrix} =
\begin{bmatrix}
16 \\
3 \\
5 \\
\end{bmatrix}
\]
Can determine whether a vector \( w \) is a linear combination of \( u \) and \( v \) using row reduction |
| Process | Can think of linear combinations of vectors e.g. \( w = 3u + 5u \) | Can consider operations on vectors without performing them e.g. \[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
\end{bmatrix} +
\begin{bmatrix}
x_4 \\
x_5 \\
x_6 \\
\end{bmatrix}
\begin{bmatrix}
y_4 \\
y_5 \\
y_6 \\
\end{bmatrix}
\]
| Object | Sees resultant as new vector object and can operate on it | Can operate on a linear combination e.g. \( T(3u + 5v) \) | \( w = c_1v_1 + c_2v_2 + \ldots + c_6v_6 \) |
|        | Can generalise addition of multiples of vectors | Can operate on a linear combination e.g. \( M \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
\end{bmatrix} +
\begin{bmatrix}
x_4 \\
x_5 \\
x_6 \\
\end{bmatrix}
\begin{bmatrix}
y_4 \\
y_5 \\
y_6 \\
\end{bmatrix} \) |

\text{Figure 1.} Part of a framework for linear algebra concepts.

\textbf{METHOD}

This research comprised a case study of a small group of 2\textsuperscript{nd} year 2006 undergraduates at the University of Auckland studying their second general mathematics course: 40% linear algebra and 60% calculus. The students were offered

4-202 PME31—2007
two supplementary linear algebra tutorials at the end of the course, 4 days prior to the examination, taught by the first-named researcher, and attended by ten students. Prior to the first they were given linear algebra questions to assess their existing conceptual thinking (see Figure 2 for questions). The tutorials covered the concepts of linear combination, span, linear independence, subspace, and basis. The aim was to give students an explanation of these topics including elements of embodied, symbolic, and formal worlds. For example, linear combinations were presented by showing embodied, visual aspects of the addition of scalar multiples of directed line segments, along with algebraic and matrix symbolisations. This was generalised to describe the notion of span and the two concepts were linked using a variety of diagrams. In each case the formal definition was given after the symbolic and visual aspects were addressed. Following the course examination three of the ten students returned and did a second, parallel test, although a controlled experiment was not intended, and two of them, students J and Y, were also interviewed. A post-doctoral mathematics student did the final test for comparison purposes.

RESULTS

Linear algebra is a large subject and we identified the sequence of concepts: vector, scalar multiple, linear combination, span, subspace, as the initial focus of attention for the research. Only the last three of these received attention in the course.

1. If \( \mathbf{v} \) is a vector as shown below, then show how to construct the following vectors:

   \[
   3\mathbf{v} ; \; -\frac{1}{2}\mathbf{v} ; \; -\frac{3}{2}\mathbf{v}
   \]

3. Describe the following terms in your own words.
   Linear combination; Span; Linearly independent; Basis; Subspace

4. Consider the following vectors \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \):

   Copy these vectors and show how to construct a diagram to demonstrate the following: \( \mathbf{c} = k\mathbf{a} + m\mathbf{b} \)

5. Which one of the following diagrams represent the linearly dependent vectors? Explain.

10. If \( \mathbf{v}_1, \mathbf{v}_2 \) are non-collinear vectors in \( \mathbb{R}^2 \), explain how the following are related:
    span \{\( \mathbf{v}_1, \mathbf{v}_2 \)\}; a subspace of \( \mathbb{R}^2 \) containing both \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \); the set of all linear combinations of \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \)
    of the form \( a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 \) where \( a_1, a_2 \in \mathbb{R} \).

11. When is it possible to find scalars \( c_1 \) and \( c_2 \) and vectors \( \mathbf{v}_1, \mathbf{v}_2 \) in \( \mathbb{R}^3 \) such that \( c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 \) lies along the \( z \)-axis? Explain.

13. Let \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) be 2 vectors in \( \mathbb{R}^2 \). Consider the linear combinations \( c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 \) and \( c_3 \mathbf{v}_1 + c_4 \mathbf{v}_2 \). How is the linear combination of these, \( E(c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2) + m(c_3 \mathbf{v}_1 + c_4 \mathbf{v}_2) \), related to the original 2 vectors?

Some formatting has been changed.

Figure 2. A selection from the first and second test questions.

PME31—2007
Scalar multiple

Question 1 on the test asked the students, given a vector \( \mathbf{v} \), to show how to construct scalar multiples. On the first test, Figure 3 shows that student F has a wrong embodied conception of scalar multiple, not appreciating that \( k\mathbf{v} \) is parallel to \( \mathbf{v} \) for all \( k \). However, by the second test this has improved and he displays an action conception of scalar multiple in the embodied world. While only two of his vectors are shown here (he also drew 2\( \mathbf{v} \)) he constructed each of them as a separate entity, each with their own distinct action. In contrast student Y has combined all three multiples into the same straight line. This gives evidence of the generalization of the scalar multiple \( k\mathbf{v} \) of the vector \( \mathbf{v} \), and may be described as the embodiment of the process of scalar multiplication. The straight line itself can be seen as the encapsulation of this process into an object-like ‘\( k\mathbf{v} \)’. Each of the students was able to use the symbolic world to represent the vectors as \( 3\mathbf{v}, 1/3\mathbf{v}, \) and \(-3/2\mathbf{v} \), etc.

![Figure 3. Action and process embodied perceptions of scalar multiple.](image)

Linear combination

In question 3 the students were asked to describe in their own words what they thought a linear combination is. Student J is unsure and describes it (first test) as “A vector can be present as a relationship between other 1 or more vectors”, and in the second test the ‘relationship’ is expounded as “One vector is the combination of the others”. In her interview she also seemed confused, and when asked what a linear combination is, she said “Ok, linear combination, for example, ah, eh.. is a hmm, it’s kind of vector equation and I think the linear combination is like that. Is one or two vectors are independent, they are form a plane or space”. When asked for an example she gave two vectors \((1, 0, 1)\) and \((-1, 0, -1)\) with the second a multiple, \(-1\), of the first. Student F also had problems. Unable to answer in the first test he resorts to a procedural, or action, explanation in the second test, “Linear combination is looking for whether the last column is composed by another two columns after reduction raw.” In contrast, Student Y uses the symbolic world for his first test answer, in a structural, proceptual form \(x\mathbf{v} + y\mathbf{u} \) (Figure 4a), and in the second test gives the only glimpse of an embodied view, saying that “several linearly independent vectors combined together form a line, plane”. In his interview he struggled to try and recall both a symbolic form and a learned definition:

Y: I can’t quite remember the definition, I can just remember those forms something like \( \mathbf{b} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 \ldots \) Linear combination is an object class in a space formed by the two vectors and \( x, y \) are scalars, this is my understanding of linear combination.
The post-doctoral respondent was clearly thinking in the formal world and gives a “standard” kind of definition (Figure 4b).

Figure 4. Symbolic and formal descriptions of linear combination.

Asked to construct a diagram to represent a linear combination in question 4, students J and Y had no problem, showing the parallelogram and construction lines (Y’s in Figure 5a). This is at least an action conception in the embodied world, and suggests a process view. Student F on the other hand has some idea of what to do but has apparently focussed on a recalled embodied relationship rather than the construction. In the first test (Figure 5b) he didn’t keep the vectors in the same direction as those given, but this was corrected in the second test (Figure 5c), although his vectors don’t form a parallelogram with c, and it seems this information is not part of his embodied schema for linear combination.

Figure 5. Symbolic and formal descriptions of linear combination.

Question 11 was designed to see whether the students could link a concept across the visual and symbolic representations. A process perspective on linear combination in the symbolic world should enable one to take the generalised symbolic form of linear combination \(c_1v_1 + c_2v_2\) given in this question and reason on its possible embodied implications. This proved too difficult for student F, who we have already seen is mostly at an action-process conception of linear combination. Student J also failed to answer the question, but did try to link the information to the matrix representation that she presumably felt more comfortable with (Figure 6a). Student Y was able to link to the embodied, visual representation (see Figure 6b). In the first test he explained that the ‘space’ formed by \(v_1\) and \(v_2\) has to contain the z-axis.

Figure 6. Symbolic and embodied perspectives for linear combination.

The three cases in the diagrams show an embodied understanding of what is required even though precise ideas of span or subspace are not used, and no line or plane is
mentioned. This approach contrasts with that of the post-doc, who used the higher level concept of span to write that “Non-trivially if \( \text{Span}(v_1, v_2) \) contains the z-axis”. A key identifying characteristic of an object view of a symbolism (procept) is whether one can operate on the object, as symbolised. In question 13 both J and Y tackled \( k(c_1v_1+c_2v_2)+m(c_3v_1+c_4v_2) \) using process knowledge in the symbolic world, multiplying out the brackets and collecting terms (J’s in Figure 7). However, in the post-test, Y stated that the result was “Still a linear combination of \( v_1 \) and \( v_2 \),” seeing the structure of the result of his symbolic manipulations.

\[
k(c_1v_1+c_2v_2)+m(c_3v_1+c_4v_2) = k(c_1v_1+c_2v_2)+m(c_3v_1+c_4v_2) = \text{the linear combination of these has} \\
\text{a relationship like: } kuv_1 + mvd_1 (k, m \in \mathbb{R})
\]

Figure 7. Process working in the symbolic world for linear combination.

**Span and subspace**

Student F struggled to describe the concepts, writing nothing in the first test and for span in the second test he wrote “Span is used to collect vectors that those vectors have taken E-value.” J similarly did not write anything the first time but in the second test she displayed an embodied notion of span, saying “the plane that vectors form.” Y wrote “Span is some vectors form a basis. i.e. \{v, w\},” but then wrote nothing for basis, and this was added to slightly in the post-test with “span: several linearly independent vectors form the basis.” In the interviews J was unclear, saying “Span, ...hmm...I think if there are independent vectors formed a plane, it’s uh... infinite, like if there are 3 or 4 more two vectors form a basis yeah, yeah.” But Y seemed to have a better understanding of span, not based on a definition, but an embodied view:

Y: I forgot the definition but my understanding of span is...it’s like little module of that space, subspace, 3-D like this, and the two vectors form a little plane, and those two vectors in this plane. This plane is a subspace of \( \mathbb{R}^3 \), ...Span is a little module of the subspace. The general form, something like linear combination.

It was clear that the concept of span was not well understood, and only the post-doc was able to link span to linear combination, and to generalise, stating that it “…is the set of all linear combinations of \( v_1 \ldots v_n \)”.

Question 7 was directed at identification of a subspace in \( \mathbb{R}^3 \), although interpretation of the third picture proved problematic. Student F wrote nothing in the first test for this question, but in the second gave the second picture as a subspace since “Those vectors being contained by same plane.” J described both of the planes as subspaces in the first test but changed this in the second to say, correctly, that the second was
not a subspace; no reason was given in the test, but in the interview this was clarified “And this one before the tutorial I think is a subspace after the tutorial I think is not a subspace... because it looks like this plane is not like this one [referred to the first picture] from the origin.” Y chose only the first plane as a subspace in both tests, saying “Those vectors form a plane in a 3-D space. A plane is a subspace of 3-D space.”, and “this is a subspace. It is a plane in a 3-D space.” It is not clear whether this confirms his idea above that any line or plane could be a subspace, since he rejected the second plane without reasons. When asked about the question in the interview his reply showed confusion over vectors not from the origin “Subspace is from the.. It’s formed by those two vectors. And the second graph.. this looks odd, because they are not from the origin. Only the post-doc correctly identified the answers, with reasons in the test: “Subspace, since it is a plane containing the origin” and “Not a subspace – since it does not contain the point (0, 0, 0)”, but it seems that J had the embodied idea after the tutorials. The purpose of question 10 was to ascertain if links between concepts were being made. Students F and J wrote nothing in either test for this, although when pressed in the interview J showed she is moving towards some embodied understanding “Linear.. it means the two vectors are linearly independent, if you look at span like a subspace and the set of all linear combinations lies on the span and also lies on the subspace, or they form”.

Figure 8 shows Y’s answers in both tests. In the first test (top) he still links span to basis, and has an embodied view of both subspace and the set of linear combinations, but in the second test he is much more able to link these concepts together, although not as succinctly as the post-doc who wrote “(a) and (c) are obviously the same since they both describe the plane generated by $v_1, v_2$. “ In his interview Y was able to express the problem he was having with relating the three concepts:

Y: Span and subspace...those two are related together, and if we call subspace $W$...and those vectors are the span, because those vectors formed that subspace...Linear combination, yeah, this question confuses me. How do we distinguish between linear combination and subspace?

\[ (a) \text{Span}(v_1, v_2) \text{ is a basis, } b \text{ is a subspace of } \mathbb{R}^3 \text{ containing both } v_1 \text{ and } v_2 \text{ is a plane, the subspace all the vectors can be expressed in terms of } v_1 \text{ and } v_2, (c) \text{ the form } a_1v_1 + a_2v_2, a, b, c \text{ is the plane} \]

\[(a) \text{ is a small model of } (c), \text{ (b) and (c) maybe the same, or, (c) contains (b).} \]

Figure 8. Y’s improving links between span, subspace and linear combination.

He sees the span of three vectors in $\mathbb{R}^3$ as always forming a subspace, since position vectors always go through the origin, but can’t link to linear combination.
CONCLUSIONS

This research confirms the idea that some students struggle with basic linear algebra concepts such as linear combination, span and subspace. It seems to us, on the basis of limited data, that the use of embodied notions in the tutorials helped. We asked the students if this was the case and how they would explain some of the ideas to others. J was very clear that a visual, embodied approach had greater value for her than beginning with definitions:

J: Basically when I was in lecture I mixed up. All the relationship between definitions of subspace linear comb….But when I came to your tutorial there was some graphs and also very clear explanation that helped me to understand…And if I become a tutor I teach as your way, first I...graph them, not the definition, I think its too difficult to understand, makes them confused.

Y agreed “First give them a picture and start from something in the real life, not from the maths because students are just started studying maths, they couldn’t understand definitions.” Commenting on appreciation of the tutorials F and J wrote “F: It was really helpful to understand the basic of concept. When I began reviewing without tutorials, I could do any question about that part.”, “J: It’s quite helpful. It clarifies my confusion of theories…such as basis, subspace, span-- which I confused through the semester.” We continue to construct the framework, based on APOS theory and the three worlds of thinking, which presents embodied, symbolic and formal experiences that students could have with linear algebra concepts. Further research is under way to examine the value of the framework in learning.

REFERENCES


