THE POTENTIAL OF PATTERNING ACTIVITIES TO GENERALIZATION

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This study presents partial results from the project of Ma (2002). It was conducted to obtain the deeper appreciation and knowledge of children approaches to quadratic sequence though patterning activities. The participants were 40 elementary school students in Taiwan. The conclusions drawn from this study were: (a) it was not to guarantee that students could benefit from geometry or number approach because three main obstacles and two students' individual reasons existed. Students with the main obstacles only did a pattern generalization, not a sequence generalization. (b) Students will have potential for developing generalization, if they apply the geometry approach.

INTRODUCTION

Mathematics can be thought of as a search for patterns and relationships (Biggs & Shaw, 1985); Mathematics is described as "a science of pattern and order (Van De Walle, 1998). Thus, a more apt definition of mathematics becomes fully apparent; that is, mathematics is the science of patterns. The mathematicians seek patterns in number, in space, in computers, in science, and even in imagination.

Patterns involve a progression from step to step, and in technical term these are called "sequence". Patterning activities develop directly a sense of pattern and regularity, and practice the skills of searching for pattern, extending patterns, and making pattern generalization. These processes will involve in variable and the concept of function. As a result, Usiskin (1995) states the view that algebra is the language through which we describe patterns. Patterning activities play a significant role for primary graders to establish the algebra foundation (Herbert & Brown, 1997;, 2002).

Algebra is a source of considerable confusion and negative attitudes among students (Cockcroft, 1982). Some students experience difficulty making the transition from arithmetic-based programs to the ideas of algebra (Greenes & Findell, 1999). "Expressing generality" is described as one of four different roots of algebra (Mason, Graham, Pimm, & Gowar, 1995). Thus, the use of patterns, leading to an improvement in this unhappy situation, has become an ordinary route into expressing generality within school mathematics curricula (A. Orten & J. Orten, 1999).

Hargreaves, et al. (1999) denotes that the need to generalize about the given terms might have two meanings. One is "a pattern generalization," it is to see more in the set of numbers than is given. The other is "a sequence generalization," it is to go beyond the set of numbers. Ma and Wu (2006) show that in the process of expressing generality, most fifth and sixth grades were unaware of linking patterns to algebraic

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concepts; what they did only was "a pattern generalization". Few could obtain the concept of a function that describes the relationship between any object and its position in a sequence; what they did was "a sequence generalization".

The school practice involving generalization in algebra often starts from figures or numeric sequences. Children need to realize that there are two representations of the same situation, and need to enable to switch from one to the other (Ursini, 1991). J. Orten, A. Orten, and Roper (1999) suggest that there are three purposes of setting patter tasks within pictorial context. One is for those who could possibly find support or their thinking from a more geometrical approach. Thus, it might be assumed that pictorial context adds meaning to the task. Second is pictorial context might be more elementary than purely symbolic context. Third is just to vary the format to create more of a problem to be solved. The students might use different method to convert pictorials to numbers sequence. Based on Orten et al. (1999), there are three methods of translating pictorial to number. One method is to count the dots for each shape presented in the task, then immediately converting the shapes into a number sequence. A second method is to look at how many more dots each new shape requires. A third method is based on seeing the shapes.

Quadratic sequences are those where the difference of the differences (i.e., the second difference) is constant, nevertheless, the majority of the sequences used in the textbooks are linear, where the difference between successive terms is constant. For a deeper appreciation and knowledge of children approaches to sequences they less met, this study was conducted focusing on pupils' generalization about and process approach to quadratic sequences. Especially the patterns in the sequence were set with pictorial and numerical contexts, which were two representations of the same situation. What processes would be involved when students work with quadratic sequences? What would be students' obstacles along the road to successful generalization while they observe and summarize patterns? Which method would they adopt while they convert pictures to numbers sequences? Which approach (geometry or number) could students benefit from? However, pupils were not expected to produce a formula for the general, or *n*th, term, while they did not yet receive formal algebra curricula. The purposes were as follows:

(1) To analyse the processes and how these might relate to generalizations

(2) To investigate the obstacles along the road to successful generalization;

(3) To explore the approaches students could benefit from while they work at patterning activities.

METHOD

An internet discussion board would be given an educational meaning while shifting the mathematical activities to it (Ma, 2001, 2004, 2005). The participants in this study operated on the pattern activities via an internet discussion board. Except for the traditional functions of word typing and recording, it also includes the functions of picture and chart pasting. Figure 1, for example, shows a screenshot on the internet discussion board. Each of eight problems, pattern question, was posed on the internet every two to three weeks. Among them problem 1, 3, 5, and 7 were with pictorial contexts, while problem 2, 4, 6, and 8 were with numerical contexts. Problem 1 and 6 are two representations of the same situation. They are quadratic sequences.

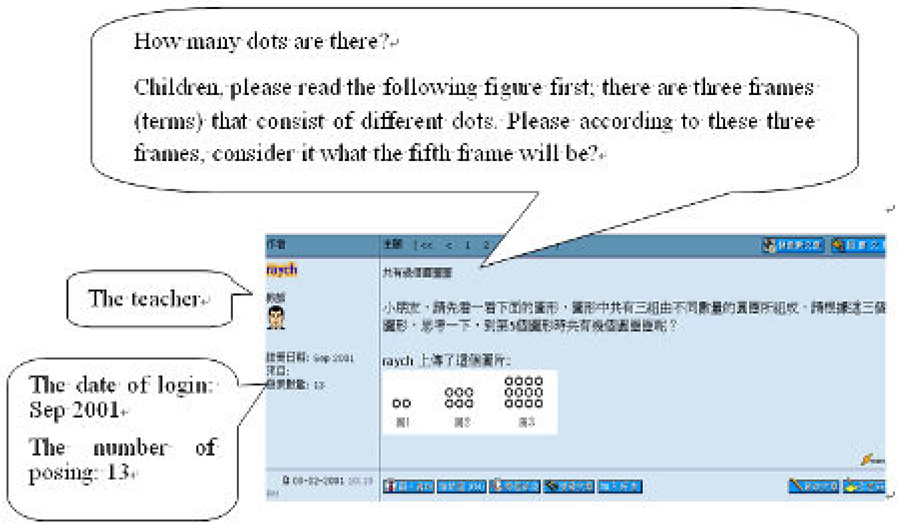


Figure 1: A screenshot on the internet discussion board

The participants in this study were 28 sixth graders and 12 fifth graders from Taiwan. They had basic computer skills and used the internet regularly. Each participant was anonymous but had fixed code. Three codes or four codes (adding "m" in front of three codes) represented the sixth and fifth graders respectively. Among three codes, the first symbolized the sex (b: boy, g: girl), the second symbolized the mathematics achievement (h: high, m: middle, l: low), and the last was only serial number. For example, bh1 represented the first (1) high-achieved (h) boy (b) from grade six, mgl2 represented the second (2) low-achieved (l) girl (g) from grade five (m).

For solving the problems the students were asked to search for pattern, extend patterns, and make pattern generalization. They were allowed to work on the problems at anytime and from anyplace. After having completed the activity on the internet, some students were interviewed to understand their thinking by teachers. Data relating to students' understanding of pattern in the sequence were collected in both forms. One is written form with students' responses to describing the rule for the patter on an internet discussion board, and another is oral form by interviewing students on a one-to-one basis. This activity lasted from September, 2001 to March, 2002.

DISCUSSION

The written responses of problem 1 and 6, two representations of the same situation, were examined to gain insights into the kinds of approach used, the processes applied and how these might relate to generalizations. Problem 1 with pictorial contexts was in figure 1 above, and problem 6 with numerical format was as 2, 6, 12, and 20. The pupils were asked to predict the number for the next, fifth, tenth, and hundredth in the sequence. Five students were chosen as examples in this study. For convenience, the researcher would use "P" and "N" to represent the pictorial and numerical context respectively, and use number (1, 2, and 3,) to arrange in students' responses order. For example, "P-1 bl1" would represent the first response of bl1 to pictorial context.

Protocol 1

P-1 bl1: I add 2 on 4. 6 times 5 are 30. There will be 30 dots in the fifth.

N-1 bl1: 1×2=2, 2×3=6, 3×4=12, 4×5=20, 5×6=30,…, 10×11=110.

Student bl1 preferred to geometry approach, where he showed the analytic thinking. For example, he viewed "12"as "3×4" (N-1). Thus, he could see that both sequences were equivalent. By interviewing, bl1 expressed that the row and the column respectively required more one dot from shape 1 to shape 2, and it required more two dots from shape 3 to shape 5. Thus, His method was based on seeing the shapes before converting pictorial to number (P-1). However, he only could find more terms in a sequence such as the fifth and the tenth. He did not try to do far generalizing tasks (e.g., the hundredth), Stacey (1989) describes. The individual interviews conformed that his arithmetical incompetence could be his obstacles.

Protocol 2

P-1 gm4: 2, 6, 12. 3 multiplied by 2 are 6. 6 multiplied by 2 are 12. They go up in 3, 6, 9, and 12. 2×9=18. 2×12=24. There will be 24 dots in the fifth.

P-2 gm4: There will be 480 dots in the hundredth. There are 24 dots in the fifth, 100 divided by 5 equals 20, and then 24 times 20 equal 480.

N-1 gm4: Number 2 is 2 plus 4, number3 is 6 plus 6, number 4 is 12 plus 8,…, and number 10 is 90 plus20. It will be 110 in the tenth.

N-2 gm4: Number 10 is 110, and 100/10=10. It will be 1100 (110×10) in the hundredth..

The student gm4 preferred to numerical approach, where she immediately translated the shapes into a number sequence by counting the dots (P-1). She adopted idiosyncratic methods unpredictably to extent the pattern, such as 2×9=18, 2×12=24 (P-1). Unfortunately, the method was not useful to make generalization. She fixedly swapped to a short-cut method for far generalization after finding more terms in a sequence. For example, A100= A5×20= 24×20=480 (P-2) and A100=A10×10= 110×10=1100 (N-2). This suggested that gm4 did not perceive that both sequences were two representations of the same situation. Thus gm4 had obstacles along the road to successful far generalization.

Protocol 3

P-1 gh1: 2 dots plus 4 dots equals to 6 dots, and 6 dots plus 6 dots equals to 12 dots. They go up in 4, 6, 8, and 10. 12 dots plus 8 dots equals to 20 dots. 20 dots plus 10 dots equals to 30 dots. There will be 30 dots in the fifth.

P-2 gh1: There are 900 dots in the *99*th. I add 200 dots to the *99*th shape. There will be 1100 dots in the *100*th shape.

N-1 gh1: …..They go up in 2s, i.e., 4, 6, 8,..,18, 20. 30+12=42, 42+14=56,…., 90+20=110. Number 10 is 110.

N-2 gh1: 2, 6, 12, 20, 30, 6/2=3, 12/2=6, 20/2=10, 30/2=15. (3, 6, 10, 15) I worked it out like I counted in 3 to 6 is 3, 6 to 10 is 4, 10 to 15 is 5. (3, 4, 5) 3 to 4 is 1, and 4 to 5 is also 1.

Student gh1 was affected by different context. Her method of converting pictorial to number was to look at how many more dots each new shape requires (P-1). She could see how the addend was related to the position of the term, for example, adding 200 dots to the 99th shape (P-2). At last, she could not get the number of the 99th shape (900 dots was assumed) because she depended on a recursive approach. In addition, she described quadratic sequence by different way, that is, the constant difference (i.e., 1) of differences (i.e., 3, 4, 5) was between first quotients (i.e., 3, 6, 10, 15) (N-2). Unfortunately the quotients came from unpredictably personal methods of gh1, and it was not useful to make generalization. She had the fixation with a recursive approach in extending a pictorial or number sequence (P-1, N-1). She was used to looking for a local rule rather than a global rule. Thus gh1 had obstacles along the road to successful far generalization.

Protocol 4

P-1 mgl2: 2, 3+3, 4+4+4, 5+5+5+5, 6+6+6+6+6. Thus, there will be 30 dots in shape 5.

P-2 mgl2: It would be 101 dots. It requires more one dot from shape 99 to shape 100.

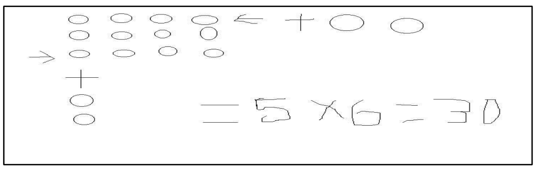
N-1 mgl2: Number 1 is 2, number 2 is 6, number 3 is 12, number 4 is 20, and number 5 is 30. 2+4+6+8+10+12+…+30+…. Keep adding them up and you will get the hundredth.

Student mgl2 preferred to geometry approach, where she showed the analytic thinking. For example, she viewed "12"as "4+4+4" (P-1) or "2+4+6" (N-1). Her method was based on seeing the shapes before converting pictorial to number. She focused on component parts of shapes. For example, the third contains three rows, and each row covers four dots, thus there is "4+4+4"(P-1). She made a correct verbal statement (101 dots), although she did not continue to work out the number of the hundredth shape (P-2). The individual interviews conformed that she could not handle the big number (i.e., 101+101+…+101). In addition, she made a creditable attempt at the approach to algebra, where she expressed the number in the form of "2+4+6++…+30+…"(N-1). In both contexts mgl2 noticed the method, not the answer. She had potential for developing far generalization although she only did near generalizing tasks.

Protocol 5

P-1 bh2: 3 dots in the left would be width and 4 dots in the bottom would be length, if we viewed shape 3 as the rectangular arrays of dots. 3 plus 2 equals 5, 4 plus 2 equals 6, and 5 times 6 equals 30.

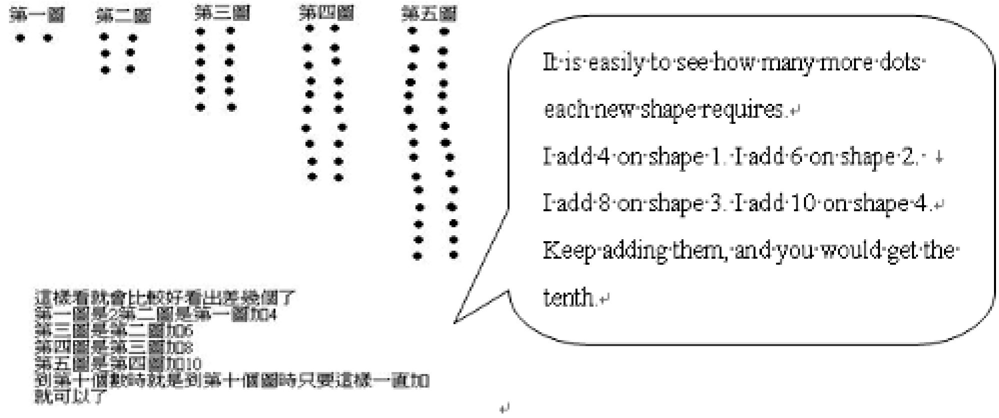
P-2 bh2:



P-3 bh2: Frame 1: 1×(1+1). Frame 2: 2 × (2+1). Frame 3: 3 × (3+1). Frame 4: 4 × (4+1). Frame 5: 5 × (5+1), 5×6=30 (dots), Frame 100: 100 × (100+1)= 10100 (dots) Frame n: n×(n+1).

N-1 bh2: Number 1 is 2. I add 4 on number1. 2+4=6. I add 6 on number2. 6+6=12. Keep adding even number and you will get the hundredth.

N-2 bh2: It will be easily to count, if we convert number to picture. The way is:



Student bh2 was affected by different context, but he could only benefit from geometry approach. His method was based on seeing the shapes before converting pictorial to number (P-1, P-2). He made far generalization because he could see how the numbers of width and length of rectangular were related to the position of the term in the sequence (P-3). It might prove the assumption of pictorial context adding meaning to the task, possibly enlivens it or simplifies it. In fact, Ma (2002) indicated that there were 84.4% students prefer to pattern with pictorial contexts, while there were only 15.6% like numerical contexts. The reasons of the former were easier, more creative and interesting, or giving extra hints.

Student bh2 used two methods to extend the number sequence. One is numerical way (N-1); another is to switch to figure (N-2). He illustrated the notion of equivalence with visual materials, in which each shape fell into two lines and new shapes required more 2×2, 2×3, 2×4, and 2×5 dots. He made a creditable attempt at the approach to figure, but what he did was just like the numerical way. He only recognized the recursive nature of the pattern, and thus only produced a local rule (N-2).

CONCLUSION

The results from this research were the following.

1. Few students perceived that both sequences are two representations of the same situation. Some students' preference was for geometry approaches (e.g., bl1, mgl2), and yet others students' preference was for number approaches (e.g., gm4).

It was not to guarantee from which approach students could benefit. There were three main obstacles along the road to successful "far generalization".1. Students merely produced a local rule (i.e., a recursive formula), not a global rule (e.g., bh2). 2. Students always thought about the numerical answer, not the method itself. The answer-driven approach leads student away from thinking about the methods of arithmetic and what they might mean. 3. Students applied a short-cut method, an inappropriate but simple method (e.g., gm4). As a result, these students with obstacles above could not complete far generalizing tasks. That is, they only did a pattern generalization, not a sequence generalization.

In addition, students' individual reasons, such as arithmetical incompetence (e.g., bl1, mgl2) and unpredictably idiosyncratic method (e.g., gm4, gh1), could influence the development for near or far generalization. The student gm4 and gh1 described the sequences by different way, that is, they divided each term by "the first term" and got the quotients (e.g., 6/2=3, 12/2=6).

2. Students will have potential for developing generalization, if they apply the approach based on the pictures, not on the equivalent number sequence. For example, bh2 could do far generalization (i.e., Frame 100: 100× (100+1) = 10100) with sequence presented in picture format. Even bl1 and mgl2, low achievers, might have opportunities to do far generalizing tasks. Their thinking was related to "analytic thinking". For example, "12" could be represented as "3×4" by bl1 and as "4+4+4"or "2+4+6" by mgl2. These methods with geometry approaches students applied were based on seeing the shapes of sequence. Bednarz, Kieran and Lee (1996) denote that certain geometry approaches appear to be a possible precursor to the emergence of analytic thinking in learning of algebra.

3. The students who had geometry approaches but were not yet aware of seeing relationship of patterns might easily progress from a recursive approach to an explicit approach after suggesting. For example, mgl2 might easily progress from seeing patterns as 2, 3+3, 4+4+4, 5+5+5+5,…or 2, 2+4, 2+4+6, 2+4+6+8,… to viewing them as 2×1, 3×2, 4×3, 5×4, or 2×1, 2+2×2, 2+2×2+2×3, 2+2×2+2×3+2×4,…. Therefore, mgl2 will be able to find an algebraically useful pattern such as frame 10: 11×10, or 2+2×2+2×3+…+2×10. Student bh2 switch number to figure sequence, and could see patterns within patterns such as 2×1, 2×3, 2×6, 2×10, 2×15 (N-2 bh2). He might easily progress from seeing patterns as 2, (shape 1) + 4, (shape 2) + 6, (shape 3) + 6,…to viewing it as 2, 2+2×2, (2+2×2)+2×3, (2+2×2+2×3)+2×4,…. Thus, bh2 will enable to obtain an algebraically useful pattern such as frame 10: (2+2×2+2×3+…+2×9) +2×10.

Thus, students need progress from a recursive approach to an explicit approach. In the process, students will focus attention on the method, not the answer. This suggests that there is potential development for "far generalization". Thus, in patterning activities teachers should encourage students to work at expressing their own generalization through geometry approaches, and then their algebraic thinking will take place.

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