In this paper, we investigated the extent of knowledge in mathematics and pedagogy that prospective middle school teachers have learned and what else they may need to know for developing effective classroom instruction. We focused on both prospective teachers’ (PT) own perceptions about their knowledge in mathematics and pedagogy and the extent of their knowledge on the topic of fraction division. The results reveal a wide gap between these PT’s general perceptions/confidence and their limited knowledge in mathematics and pedagogy for teaching, as an example, fraction division. The results also suggest that PTs need to master specific knowledge in mathematics and pedagogy for teaching in order to build their confidence for classroom instruction.

Accumulated research findings in past decades have led to the understanding that teachers’ knowing mathematics for teaching is essential to effective classroom instruction (e.g., Ma, 1999; RAND Mathematics Study Panel, 2003). Corresponding efforts have also been reflected in teacher preparation programs that call for more emphasis on prospective teachers’ learning of mathematics for teaching (CBMS, 2001; NCTM, 2000). Such efforts can presumably increase the quality of pre-service teacher preparation and prospective teachers’ confidence and ultimate success in future teaching careers. Yet, much remains to be learned about the extent of knowledge in mathematics and pedagogy that prospective teachers acquire and what else they may need to know for developing effective classroom instruction. As a part of a large research study of prospective middle school teachers’ knowledge development in mathematics and pedagogy, this paper focuses on a group of prospective middle school teachers’ knowledge of mathematics and pedagogy for teaching in general and on the topic of fraction division, in particular.

The topic of fraction division is difficult in school mathematics not only for school students (Carpenter et al., 1988), but also for prospective teachers (Ball, 1990; Simon, 1993). Mathematically, fraction division can be presented as an algorithmic procedure that can be easily taught and learned as “invert and multiply.” However, the topic is conceptually rich and difficult, as its meaning requires explanation through connections with other mathematical knowledge, various representations, and/or real world contexts (Greer, 1992; Ma, 1999). The selection of the topic of fraction division, as a special case, can present a rich context for exploring possible depth and limitations in prospective teachers’ knowledge in mathematics and pedagogy. Specifically, this study focuses on the following two research questions:
(1) What are the perceptions of prospective middle school teachers regarding their knowledge in mathematics and pedagogy for teaching?
(2) What is the extent of prospective middle school teachers’ knowledge in mathematics and pedagogy for teaching fraction division?

CONCEPTUAL FRAMEWORK

The conceptual complexity of the topic of fraction division is evidenced in a number of studies that documented relevant difficulties pre-service and in-service teachers have experienced (e.g., Ball, 1990; Borko et al., 1992; Contreras, 1997; Simon, 1993; Tirosh, 2000; Tzur & Timmerman, 1997). Although both pre-service and in-service teachers can perform the computation for fraction division, it is difficult for teachers, at least in the United States, to explain the computation for fraction division conceptually with appropriate representations or connections with other mathematical knowledge (Ma, 1999; Simon, 1993). Teachers’ knowledge of fraction division is often limited to the invert-and-multiply procedure (e.g., Ball, 1990), which restricts teachers’ ability to provide a conceptual explanation of the procedure in classrooms (e.g., Borko et al., 1992; Contreras, 1997). Because the meaning of division alone is not easy for pre-service teachers (e.g., Ball, 1990; Simon, 1993), fraction division is even more difficult (Ma, 1999). Based on the findings from studies on teachers’ knowledge and difficulties with division and fraction division, it can be summarized that teachers often have the following five types of difficulties:

(a) How to explain the computational procedure for “division of fraction” with different representations (e.g., Contreras, 1997; Ma, 1999)
(b) How to explain why “invert and multiply” (e.g., Borko et al., 1992; Tzur & Timmerman, 1997)
(c) Mathematical relationships between fraction division and other mathematical knowledge (e.g., fraction concept; addition, subtraction, and multiplication of fractions) (e.g., Ma, 1999; Tirosh, 2000)
(d) Related misconceptions (e.g., can not divide a small number by a big number, division always makes a number smaller) (e.g., Greer, 1992)
(e) Solving problems involving fraction division (e.g., Greer, 1992)

The identification of these five types of difficulties provided a general framework for the current study and served as a guideline for examining the nature of prospective middle school teachers’ possible difficulties with fraction division.

METHODOLOGY

Subjects

The participants were prospective middle school teachers enrolled in a mathematics and science interdisciplinary teacher education program at a southwestern U.S. university. These prospective teachers were in their last stage of study in the program. They had already taken all of the required mathematics courses and were completing mathematics methods course at the time of their participation in this study. A
majority of the participants were seniors with only a few juniors. A total of 46 prospective teachers participated in the study for data collection.

**Instruments and data collection**

Two instruments were developed for this study. The first instrument was a survey of prospective teachers’ general knowledge in mathematics and pedagogy. Many items were adapted from TIMSS 2003 background questionnaires (TIMSS 2003).

The second instrument was a math test that focused on prospective teachers’ content knowledge and pedagogical content knowledge of fraction division. It contained items targeted to prospective teachers’ possible difficulties as specified in the conceptual framework. While some items were adapted from school mathematics textbooks and previous studies (e.g., Hill, Schilling, & Ball, 2004; Tirosh, 2000), others were developed by the researchers of the current study.

All 46 prospective teachers enrolled in the mathematics methods course were invited to participate in this study. The participants were notified that both the survey and the test were for research purposes only and should be completed anonymously. The survey and the tests were administrated at the last class of the senior methods course. Participants were requested to complete the survey first, then the mathematics test.

**Data analysis**

Both quantitative and qualitative methods were used in the analysis of the participants’ responses. Specifically, responses to the survey questions were directly recorded and summarized to calculate the frequencies and percentages of participants’ choices for each category. To analyze participants’ solutions to the problems in the mathematics test, specific rubrics were first developed for coding each item, and subsequently, the participants’ responses were coded and analyzed to examine their use of specific concepts and/or procedures.

**RESULTS AND DISCUSSION**

In general, the results presented a two-sided picture that illustrated the importance of examining and understanding prospective teachers’ knowledge in mathematics and pedagogy for teaching.

On one side, the results from the survey indicated that (1) participating prospective middle school teachers in this sample knew about their state curriculum framework in general; (2) they were confident in the preparation they received in mathematics and pedagogy for future teaching careers; and (3) they had developed general pedagogical understanding for mathematics classroom instruction.

On the other side, however, this group’s performance on the mathematics test revealed that their knowledge in mathematics and pedagogy for teaching fraction division was procedurally sound but conceptually weak. The apparent inconsistent patterns in their responses suggested that these prospective teachers did not know what they would be expected in order to develop effective teaching. Their confidence
was built upon their limited knowledge in mathematics and pedagogy. The following sections are organized to present more detailed findings corresponding to the two research questions.

**Prospective teachers’ perception of their knowledge preparation in mathematics and pedagogy for teaching**

Prospective teachers’ responses to the survey were quite positive. The following items were selected from the survey to illustrate prospective teachers’ perception of their knowledge preparation needed for teaching, as related to fraction division.

For item 1: How would you rate yourself in terms of the degree of your understanding of the Mathematics Curriculum Framework in your state? On a scale of four choices (High, Proficient, Limited, Low), 9% and 91% of the participants chose “High” and “Proficient”, respectively. None of the prospective teachers perceived themselves to have limited or low understanding of their state mathematics curriculum framework.

For item 4-(6): Choose the response that best describes whether students (grades 5-8) in your state have been taught the topic - *Computations with fractions*. In a scale of five choices (Mostly taught before grade 5, Mostly taught during grades 5-8, Not yet taught or just introduced during grades 5-8, Not included in the state math framework, Not sure), 96% participants indicated that the topic of fraction division is “mostly taught during grades 5-8”, while the remaining 4% chose the first response (“Mostly taught before grade 5”). This result, in conjunction with the participants’ response to question 1, suggested that this group of prospective teachers had general knowledge about their state’s mathematics curriculum.

For item 5-(2): Considering your training and experience in both mathematics and instruction, how ready do you feel you are to teach the topic of “Number – Representing and explaining computations with fractions using words, numbers, or models?” On a scale of three (Very ready; Ready; Not ready), 60% of the participants thought they were “ready”, while 38% chose “very ready” and 2% “not ready.” The results indicated that this group of prospective teachers was confident in their preparation for teaching fraction computations, including fraction division. This result was further supported by their general pedagogical understanding for teaching mathematics.

In particular, item 6 on the survey contained several sub-items that examined participants’ attitudes toward mathematics teaching. For example:

- To what extent do you agree or disagree with the following statement?
  - (1) More than one representation (picture, concrete material, symbols, etc.) should be used in teaching a mathematics topic.
  - (3) A teacher needs to know students’ common misconception/difficulty when teaching a mathematics topic.
  - (9) Modeling real-world problems is essential to teaching mathematics.
With a scale of four choices (Agree a lot; Agree; Disagree; Disagree a lot), the following table summarizes the responses.

<table>
<thead>
<tr>
<th>Item</th>
<th>Agree a lot</th>
<th>Agree</th>
<th>Disagree</th>
<th>Disagree a lot</th>
</tr>
</thead>
<tbody>
<tr>
<td>6-(1)</td>
<td>89%</td>
<td>11%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>6-(3)</td>
<td>69%</td>
<td>27%</td>
<td>4%</td>
<td>0%</td>
</tr>
<tr>
<td>6-(9)</td>
<td>78%</td>
<td>22%</td>
<td>0%</td>
<td>0%</td>
</tr>
</tbody>
</table>

Table 1: Percentages of participating prospective teachers’ responses

The results showed that this group of participating prospective teachers had developed a positive attitude toward mathematics teaching.

In general, prospective middle school teachers’ responses to the survey suggested that they were confident in their preparation and they were ready to teach. In fact, these results are similar to what has been found from the US eighth-grade mathematics teachers in the TIMSS 2003 study (Mullis, Martin, Gonzalez, & Chrostowski, 2004). The consistency in responses between prospective teachers in the current study and US eighth-grade mathematics teachers in the TIMSS 2003 study suggests that US teachers develop their confidence quite early and hold their confidence for what they can do in teaching mathematics.

The nature of prospective middle school teachers’ knowledge in mathematics and pedagogy for teaching fraction division

The prospective teachers’ responses to the mathematics test allowed a closer look at the participants’ knowledge in mathematics and pedagogy for teaching, especially on the topic of fraction division. Results indicated these prospective teachers did very well in computing fraction division. For example, for the problem “find the value of $\frac{7}{9} + \frac{2}{3}$”, 93% of the participants solved the problem correctly. However, when the problem was changed slightly with a conceptual requirement, their performance decreased. As an example, to find “How many $\frac{1}{2}$’s are in $\frac{1}{3}$?”, only 52% answered correctly. Many gave an answer as either “none” or “0”. In fact, this problem is a typical problem in mathematics textbooks for middle school students. These prospective teachers’ responses revealed their possible misconception related to division and their weakness in understanding fraction division in the verbal format.

Moreover, these prospective teachers experienced difficulty in solving problems that involved fraction division, especially for some of the multi-step problems. For example, only 39% participants solved the following problem correctly.

Johnny’s Pizza Express sells several different flavors of pizza. One day, it sold 24 large-size pepperoni pizzas. The number of large-size plain cheese pizzas sold on that day was $\frac{3}{4}$ of the number of large-size pepperoni pizzas sold, and was $\frac{2}{3}$ of the number of large-
size deluxe pizzas sold. How many large-size deluxe pizzas did the pizza express sell on that day?

The prospective teachers were also asked to explain given computations of fraction division. In particular, the problem of “How would you explain to your students why $\frac{2}{3} \div 2 = \frac{1}{3}$? Why $\frac{2}{3} \div \frac{1}{6} = 4$?” (adapted from Tirosh, 2000) was included in the test. It was found that about 26% participants drew and used pictorial representations (e.g., fraction bar, pie chart) to explain the division procedure (e.g., how to divide 2/3 by 2 to get the answer 1/3), and 22% explained with “flip and multiply.” Most (46%) other participants failed to provide a complete explanation to both computations. Surprisingly, none of these prospective teachers tried to explain the computations as why you can flip and multiply (e.g., why you can transform “divide 2/3 by 2” to “multiply 2/3 by 1/2”).

These participating prospective teachers seemed to have even more difficulty when the computation procedure for fraction division was presented in a different way. In solving the following problem (adopted from Tirosh, 2000):

You are discussing operations with fractions in your class. During this discussion, John says

It is easy to multiply fractions; you just multiply the numerators and the denominators. I think that we should define the other operations on fractions in a similar way:

Addition $\frac{a}{b} + \frac{c}{d} = \frac{(a + c)}{(b + d)}$

Subtraction $\frac{a}{b} - \frac{c}{d} = \frac{(a - c)}{(b - d)}$

Division $\frac{a}{b} \div \frac{c}{d} = \frac{(a + c)}{(b + d)}$

How will you respond to John's suggestions? (Deal with each separately.)

About 90% of the participants indicated that the given computations for fraction addition and subtraction were not correct, and only 2 out of the 46 prospective teachers stated that the given computation for fraction division was correct. The majority of others stated that the fraction division should be “flip and multiply” or “KFC” (i.e., keep the first, flip the second, and change the sign). The results suggested that these prospective teachers actually had very limited procedural understanding of fraction division, especially when related to other mathematical knowledge.

The results from these prospective teachers’ responses on the mathematics test revealed their difficulties in all five types, as specified in the framework. However, prospective teachers’ difficulties across these five types varied to a certain degree. It appeared that these prospective teachers can do a relatively better job when their thinking and explanation are aided by drawing pictorial representation, a result that is
consistent with existing findings about US students’ preference in using visual representation (e.g., Cai, 1995). However, performance became much less satisfactory when multiple mathematical relationships or mathematical ideas in an abstract format were presented.

**CONCLUSION**

The findings from this study show two different and seemingly contradictory sides of prospective teachers’ knowledge in mathematics and pedagogy for teaching. The more positive perspective is revealed by the prospective teachers’ responses to the selected survey questions. Certainly, these positive perceptions and attitudes can possibly help drive prospective teachers in their future efforts in developing effective classroom instruction. At the same time, however, their positive perceptions and attitudes are likely built upon insufficient (or limited) mathematical knowledge and pedagogical knowledge in mathematics. As revealed by their performance on the mathematics test, these prospective teachers had many difficulties with fraction division that they may not realize or recognize as deficiencies in their knowledge base. It is not realistic to expect prospective teachers to determine by themselves what they need to learn for future teaching career. Instead, as teaching mathematics requires a special set of skills (Viadero, 2004), it becomes necessary and important for teacher educators to identify what knowledge in mathematics and pedagogy prospective teachers need to learn through their program study. Ideally, prospective teachers would build their positive perceptions and attitudes upon their solid understanding of specific knowledge in mathematics and pedagogy for teaching.

**References**


Li & Smith


