TEACHERS’ CONCEPTIONS OF MATHEMATICAL
CHALLENGE IN SCHOOL MATHEMATICS

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This study arose from our belief that mathematics should be challenging in any
mathematics classroom. We analyse conceptions of mathematical challenge of two
groups of experienced mathematics teachers. The first group was asked to define the
notion of mathematical challenge and give examples of challenging mathematical
tasks ($N_1 = 9$). The second group of teachers was presented with a questionnaire
based on the replies of the teachers from the first group ($N_2 = 41$). We found that the
teachers have a broad conception of mathematical challenge, appreciate relativity of
mathematical challenge, but are not always convinced of the possibility of
incorporating challenging mathematics in everyday classroom.

BACKGROUND

The role of challenge in mathematics education

This study is inspired by our participation in ICMI-16 study group "Challenging
mathematics in and beyond the classroom". Peter Taylor (2006) in his presentation
of the agenda of the conference wrote: "Challenge is not only an important component
of the learning process but also a vital skill for life. People are confronted with
challenging situations each day and need to deal with them. Fortunately the processes
in solving mathematics challenges (abstract or otherwise) involve certain types of
reasoning which generalise to solving challenges encountered in every day life”.

In Cambridge Advanced Learner's Dictionary 'challenge' is defined as 'difficult job',
something needing great mental (or physical) effort in order to be done successfully.
Incorporation of a mathematical challenge in learning/teaching process involves both
psychological and didactical considerations.

Principles of 'developing education’ (Davydov, 1996), which integrate Vygotsky’s
(1978) notion of ZPD (Zone of Proximal Development) claim that to develop
students mathematical reasoning the tasks should not be too easy or too difficult and
the learners have to approach any task through meaningful activity. Another
perspective on mathematical challenge may be seen in differentiation between
‘exercises’ and ‘problems’. According to Polya (1973), Schoenfeld (1985), and
Charles & Lester (1982) mathematical task is a problem when it incorporates a
challenge for learners. It should (a) be motivating; (b) not include readily available
procedures; (c) require an attempt; and (d) have several approaches to solution.
“Obviously, these criteria are relative and subjective with respect to a person’s
problem-solving expertise in a particular field, i.e. a task that is cognitively

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demanding for one person may be trivial (or vice versa) for another” (Leikin, 2004, p. 209).

Teachers' role in devolution of challenging mathematical tasks

Jaworski (1994) claimed that mathematical challenge together with sensitivity to students, and management of learning are core elements of teaching. In order to develop pupil's mathematical understanding a teacher must create situations that demand from the students' mental effort. Teachers' choices of mathematical tasks for their classes and the ways in which these tasks are introduced to students determine the quality of mathematics in the classroom (e.g., Simon, 1997; Steinbring, 1998). However, many teachers choose conventional tasks for their lessons and guide students towards 'standard' solutions (Leikin & Levav-Waynberg, accepted). Simply providing teachers with ready-to-use challenging mathematical activities is not sufficient for their implementation. The teachers should be aware and convinced of the importance of mathematical challenge in teaching and learning mathematics, they should ‘feel safe’ when dealing with such kind of mathematics (mathematically and pedagogically) and have autonomy (Krainer, 2001) in employing this kind of mathematics in their classes.

Teachers' knowledge and beliefs

Teachers' knowledge and beliefs are interrelated and both have very complex structure that determines teachers' decision making when planning, performing, and reflecting. In this study we consider mathematical challenges as an integral part of teachers' content knowledge (Shulman, 1986). From this point of view teachers’ subject-matter knowledge comprises their own understanding of the essence of mathematical challenge, their knowledge of challenging mathematics, their ability to approach challenging tasks. In this study teachers' conceptions of mathematical challenge are an integral part of teacher's subject-matter knowledge attributed to their meta-analysis of mathematical content to be taught. Teachers’ pedagogical content knowledge includes knowledge of how students cope with challenging mathematics, as well as knowledge of appropriate learning setting. Teachers’ curricular content knowledge includes knowledge of different types of curricula and understanding different approaches to teaching challenging mathematics.

THE STUDY

The purpose and the questions

This is the first stage of an ongoing study that analyses development of teachers’ conceptions of mathematical challenge. At this stage we explored teachers’ views of mathematical challenges through the content analysis of their definitions and examples. The main research question here is: What kinds of mathematical challenge teachers mention in their definitions? What types of mathematical problems teacher provide as examples of challenging tasks? How teachers rank different criteria of
mathematical challenge? How teachers rank mathematical tasks with respect to the challenge they embrace?

**Procedure and tools**

Two groups of in-service secondary school mathematics teachers with at least 10 years of experience took part in our study. Group A included 9 teachers, Group B included 41 teachers. There were two stages in this research:

**Stage 1:** The teachers from Group A were asked to complete an open Questionnaire-1 that included 2 assignments: (1) to define mathematical challenge; (2) to give two examples of mathematically challenging tasks and explain how they illustrate mathematical challenge (for the effectiveness of examples as a research tool see Zazkis & Leikin, accepted). After completing Questionnaire-1 the teachers discussed their definitions and examples. They were asked to solve the tasks given as examples by other teachers and evaluate challenge incorporated in the tasks. Lastly the teachers were asked to choose (at home) 3 types of mathematical challenge and present an example of each one. All the teachers' work was collected and the discussion was recorded in hand-written protocol. Teachers' written responses and their discourse during the discussion were analysed by two researchers independently. Based on this analysis we identified 12 categories of challenging mathematical tasks as viewed by teachers (see Table 1). Based on this categorization Questionnaire-2 was constructed. We also included in Questionnaire-2 three problems that were suggested by the teachers from Group A as “challenging at different levels” and “corresponding to different characteristics of mathematical challenge”.

**Stage 2:** Forty-one teachers from Group B were presented with Questionnaire-2 and asked (1) to rank types of the tasks with respect to the level of their challenge from 1 (the most challenging) to 12 (the least challenging); (2) to solve and rank the three chosen tasks from 1 (the most challenging) to 3 (the least challenging) with respect to teaching 9th grade students.

**RESULTS**

**Characteristics of challenging mathematical tasks**

Teachers’ definitions included various criteria for mathematical challenge. For example, Roni and Sohila addressed solving problems in different ways, Soha considered non-conventional problems, Tami clearly indicated that using combination of different topics in one solution is challenging. Sohila also addressed finding mistakes in solutions of the problems as a mathematical challenge.

- **Roni:** Mathematical challenge for me is a problem that has several stages in solution or problem that has different solutions. The solution is not obvious.

- **Soha:** Mathematical challenge requires thinking at the level higher than regular. A challenging problem is a problem whose solution or topic is non-standard. I think that mathematical challenge is relative. It is difficult to characterize a problem as challenging or not challenging without
considering population. The problem can be challenging for the 1st grade students and not challenging for the 3rd graders. The problem may be challenging for 'regular' people but not challenging for mathematicians.

Tami: Mathematical challenge is a combination of a number of different methods and topics together. It means that solving challenging problem I use different mathematical principles and topics (e.g. algebra and geometry or analytical geometry)

Sohila: For me mathematical challenge is (a) looking for different solutions of a problem or (b) looking for the mistakes in solutions.

As mentioned earlier we identified 12 categories based on the analysis of teachers’ definitions of challenging mathematical tasks (see Table 1).

Table 1: Frequency of different characteristics in Group A and ranking the characteristics by Group B.

<table>
<thead>
<tr>
<th>No of the criteria in quest-2</th>
<th>Characteristics of mathematical challenge</th>
<th>Group A</th>
<th>Group B</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Criteria mentioned in the definition Lesson -1</td>
<td>Problems of a particular type Lesson -1</td>
<td>Problems of a particular type HW-1</td>
</tr>
<tr>
<td>6.</td>
<td>A problem that requires combination different mathematical topics</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>2.</td>
<td>A problem that requires logical reasoning</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1.</td>
<td>A problem that has to be solved in different ways</td>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td>12.</td>
<td>An inquiry based problem</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>7.</td>
<td>A non-conventional problem</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>10.</td>
<td>A problem that requires generalization of problem results</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3.</td>
<td>Proving a new mathematics statement</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>9.</td>
<td>A problem that requires auxiliary constructions</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>11.</td>
<td>Finding mistakes in solutions</td>
<td>1</td>
<td>10.15</td>
</tr>
<tr>
<td>8.</td>
<td>A paradox</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>5.</td>
<td>A problem that requires knowledge of extra-curricula topics</td>
<td>1</td>
<td>10.56</td>
</tr>
<tr>
<td>4.</td>
<td>A problem with parameters</td>
<td>2</td>
<td>10.78</td>
</tr>
</tbody>
</table>

Table 1 presents frequency of the appearance of different characteristics in the definitions and problems given by teachers in group A and ranking the criteria by Group B. Table 1 also presents distribution of the examples of challenging problems given by teachers in group A with respect to different characteristics. This characterization of the problems was performed by the teachers themselves and by the two researches independently and then was discussed and refined in group
discussion. Within the space limit of this paper we cannot consider the differences between the kinds of problems the teachers presented on the spot and at home. Note that whereas only one teacher mentioned inquiry as a characteristic of challenging problem, ten of forty-five problems suggested by the teachers were open inquiry problems (e.g., Task 3).

**Ranking the characteristics**

The teachers in group B were asked to score the criteria that arose from the analysis of the responses of teachers from group A. We found that teachers' views on mathematical challenge as followed from the teachers ranking were consistent with the views of teachers from group A. The minimal average rank was given to combination of different topics and tools in one particular solution. Logical reasoning was ranked as most suitable criteria for mathematical challenge by 11 of 41 teachers. The three other criteria that were highly ranked by teachers from group B were problem solving in different ways, mathematical inquiry and non-conventional tasks. We were surprised by the fact that non-conventional task were scored lower than other abovementioned criteria. However, from teachers' discussion it became clear that non-conventional tasks were less popular since teachers often did not feel 'safe' enough to use this type of tasks in their classes.

**Examples of challenging problems**

<table>
<thead>
<tr>
<th>Task 1:</th>
<th><img src="image" alt="Figure 1" /></th>
</tr>
</thead>
<tbody>
<tr>
<td>In a regular octagon all the diagonals from vertex A are constructed. By this construction six disjoint angles are obtained near the vertex A. What can you say about these angles: ( \angle A_1, \angle A_2, \angle A_3, \angle A_4, \angle A_5, \angle A_6 )</td>
<td></td>
</tr>
</tbody>
</table>

**Task 2:**

Find the product of the two numbers: \( 407 \times 393 \)

**Task 3:**

The distance between school and Tom’s home is 9 km and the distance between the school and Jerry's home is 7 km. What is the distance between Tom’s and Jerry's homes?

Figure 1

All the problems given by the teachers in group A (including Tasks 1, 2 and 3) were discussed by them in the whole group discussion. Before the discussion they were asked to solve the problems given by other teachers and evaluate the level of their challenge. Often, when solving, the teachers did not find problems of other teachers challenging at all. They claimed the problems were too easy. However, after the 'authors' of the problems explained why they considered the problems challenging the teachers accepted all of them as examples of challenging tasks.
As a result of the whole group discussion (in group A) the teachers developed their comprehension of the relativity of mathematical challenge. Similarly to considerations presented by Soha in her definition of mathematical challenge, teachers agreed that in order to evaluate mathematical challenge of a problem it should be considered with respect to students' age, knowledge, ability, expertise and creativity.

Tami: You cannot say whether the task is challenging if you do not know who the students are, what they know, where they lean and moreover who is the teacher that teaches them. For some students this will be a challenge, for other students not at all.

Figure 1 presents three problems that were provided by the teachers in group A and chosen for Questionnaire-2. The reasons for this choice were the following: The three tasks represented different types of challenges as shown above, they belong to different fields of school mathematical curriculum, and were found challenging for the same grade level: 9th grade.

Task 1 has different solutions, requires use of different topics in one solution (i.e. circle and inscribed regular polygons or calculation of angles and equilateral triangles and trapezoid, properties of an angles bisector). This problem also requires auxiliary constructions.

Task 2 is not standard task since its 'elegant' solution is based on use of reduced multiplication formulas.

Task 3 is an example of an open problem that has infinite number of solutions that should be presented in the form of inequation ("something that you do not see in the textbook"). This is an inquiry problem whose formal solution is not trivial even for the teachers (see Figure 2). Teachers agreed that this problem required logical consideration in the process of solution.

The teachers in group B ranked the challenge of the tasks (from 1 – the most challenging to 3 – the least challenging). Table 2 shows teachers’ performance in the three tasks and their ranking. According to the teachers’ ranking Task 3 appeared to be the most challenging one (1.65). The least challenging was Task 2 (2.19).

We found clear relationship between the difficulty of the tasks for the teachers and their perception of the level of their challenge. Only 3 of 41 teachers gave completely correct answer to Task 3 whereas 38 of 41 teachers solved Task 2 correctly including 34 teachers who used the 'elegant' solution.

Evaluation of the challenge of the tasks by teachers in group A correlated with teachers' views in group B: the more challenging characteristics a task comprised in the eyes of the teachers’ from group A the more challenging it was ranked by the teachers from group B. Additionally, the most challenging task (task 3) fitted five
most 'popular' characteristics of challenge as ranked by teachers from group B whereas Task 2 fitted only two of these characteristics.

Table 2: Teacher's solutions and challenging rank of 3 tasks (Group B)

<table>
<thead>
<tr>
<th>Task 1</th>
<th>Task 2</th>
<th>Task 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Answer</td>
<td>No of teachers</td>
</tr>
<tr>
<td>Correct solution</td>
<td>22.5°</td>
<td>20</td>
</tr>
<tr>
<td>Partly correct solution</td>
<td>$22.5°$, no justification</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>$\angle 1 = \angle 6$, $\angle 2 = \angle 5$, $\angle 3 = \angle 4$</td>
<td>8</td>
</tr>
<tr>
<td>Incorrect solution</td>
<td>No answer</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
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<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Challenge rank (from 1 the highest to 3 the lowest):

- Task 1: 2.06
- Task 2: 2.19
- Task 3: 1.65

Concluding remarks

The experiment presented in this paper is only the first stage in the study on teachers’ conceptions of challenging mathematics. Our study demonstrates that teachers (as a group) held a broad conception of mathematical challenge. During the group discussion they refine and verify their conceptions and attain a shared meaning. We were happy to see that many of the criteria teachers suggested for mathematical challenge had been used by Krutetskii in his study on students' mathematical abilities (Krutetskii, 1976).

We found that teachers connect pedagogy and mathematics (Jaworski, 1992; Shulman, 1986) incorporated in the mathematical tasks: when reasoning about mathematical challenge the teachers were sensitive to individual differences among their students. This connection was also evident in their reasoning about the relativity of the challenge. During the discussions the teachers clearly stated that they value mathematical challenge and agreed that challenging mathematics is important for ‘deepening and broadening of students’ mathematical thinking’.

At the time of this study the teachers from Group A participated in MA program for mathematics teachers. We were glad to see that they used the content of the courses in their reasoning about challenges, since most of the courses in the program were challenging to them. For example, the frequency of the requirement for different solutions (mentioned by 7 of 9 teachers) may be explained by the fact that teachers in
group A were aware of the special interest of the researchers in different ways of problem solving (e.g., Leikin & Levav-Wayneberg, accepted). Additionally, one of the tasks presented by the teachers as a challenging task was similar to the one used in the studies by Verschaffel & De Corte (e.g., Verschaffel & De Corte, 1997). This implementation of knowledge obtained in a different context indicates for us the importance of the systematic teachers education.

**Bibliography**


