THEORETICAL BACKGROUND

Mathematics teachers and researchers agree that teachers learn through their teaching experiences (e.g., Cobb and McClain, 2001; Kennedy, 2002; Lampert & Ball, 1999; Lesh & Kelly, 1994; Mason, 1998; Ma, 1999; Shulman, 1986; Wilson, Shulman, & Richert, 1987). Teachers’ expertise is usually considered a function of their experience (e.g., Wilson, Shulman, and Richert, 1987; Berliner, 1987; Leinhardt, 1993).

The main source of teachers’ learning through teaching (LTT) is their interactions with students and learning materials (Leikin 2005, 2006). This clearly follows from cyclic models of teaching (e.g., Artzt & Armour-Thomas, 2002; Steinbring, 1998; Simon, 1997) which include expectation of development in teacher knowledge from this interactive process (see Figure 1).

Figure 1: Cyclic models of teaching (from Leikin, 2005a)

Epistemological analysis of teachers’ knowledge reveals significant complexities in its structure (e.g., Scheffler, 1965; Shulman, 1986; Wilson, Shulman, & Richert, 1987). Addressing these complexities and combining different approaches to the
classification of knowledge, Leikin (2006) identified three dimensions of teachers’ knowledge as follows:

Dimension 1 – KINDS OF TEACHERS’ KNOWLEDGE – is based on Shulman’s (1986) classification: Teachers’ subject-matter knowledge comprising their own knowledge of mathematics; Teachers’ pedagogical content knowledge including knowledge of how students approach mathematical tasks, as well as knowledge of learning setting; Teachers’ curricular content knowledge including knowledge of different types of curricula and understanding different approaches to teaching mathematics.

Dimension 2 – SOURCES OF TEACHERS’ KNOWLEDGE – is based on Kennedy’s (2002) classification. Teachers’ systematic knowledge is acquired mainly through studies of mathematics and pedagogy in colleges and universities, craft knowledge is largely developed through classroom experiences, whereas teachers’ prescriptive knowledge is acquired through institutional policies. In the discussion of teachers’ learning through teaching craft knowledge is of the main interest.

Dimension 3 – FORMS OF KNOWLEDGE – refers to differentiation between teachers’ intuitive knowledge as determining teachers’ actions that cannot be premeditated and their formal knowledge, which is mostly connected to teachers’ planned actions (Atkinson & Claxton, 2000, Fischbein, 1984). Additionally this includes distinction between knowing and believing (Scheffler, 1965). Knowing has “propositional and procedural nature” whereas believing is “construable as solely propositional” (p. 15, ibid.). In the framework of learning through teaching transformations of intuitive knowledge into formal knowledge is a foci study point.

![Diagram](image.png)

Figure 2: Three dimensions of teachers’ knowledge (from Leikin, 2006)

While it is evident that people learn from their practice in general and, in particular, teachers learn from their teaching, what exactly is being learned is often not evident. Leikin (2006, 2005a, 2005b) explored what changes in teachers’ knowledge occur
through teaching, how development of teachers’ knowledge of mathematics and that of their knowledge of pedagogy in the field of mathematics relate to each other, and what mechanisms support those changes.

MECHANISMS OF LTT

Qualities of instructional interactions determine potential of the lesson for students’ and teachers’ learning (Leikin, 2005). In this context *initiation* of interaction by the teacher or by the students, as well as *motives* for interacting, determine learning processes in the classroom. The motives may be external if they are prescribed by the given educational system, or internal, being mostly psychological, including cognitive conflict, uncertainty, disagreement, or curiosity. Piagetian disequilibration, is the main driving force in intellectual growth or learning. For teachers, unexpected, unforeseen or unplanned situations are the cause of disequilibration and the sources for learning. These sources surface via interaction with students and via reflection on this interaction.

Development of teachers’ mathematical knowledge depends on their flexibility (Leikin & Dinur, 2003). By opening opportunities for students to initiate interactions and by managing a lesson according to students’ ideas teachers open opportunities for their own learning.

Teachers’ noticing and attention (Mason, 1998, 2001) are also of great importance. Of particular interest here is attending to students’ responses, both correct and incorrect. When observing students’ mistakes during interactions, the teachers search for new explanations or clarifications in order to correct student’ understanding, so in the course of the lesson they may construct new mathematical connections (example 1 below). The other interesting source for teacher’s LTT is learning from students’ unexpected correct ideas (example 2) or from students’ surprising questions (example 3) (for elaboration of this phenomenon see Leikin & Dinur, 2003; Leikin & Levav-Waynberg, in press, Leikin, 2005b).

WHAT CHANGES IN TEACHERS’ KNOWLEDGE OCCURS THROUGH TEACHING?

Within the space constrains of this paper we mainly focus on teachers learning of mathematics.

**From intuitions to formal knowledge and beliefs:** Teachers learn mainly in unpredicted (surprising) situations. As Atkinson and Claxton (2000) show in “intuitive practitioner”, many of the teachers’ actions when teaching are intuitive and unplanned. Teachers’ craft knowledge develops as the transformation of their intuitive reactions into formal knowledge or into beliefs. In terms of the relation between knowledge, intuitions, and beliefs suggested in the 3D model of teachers’ knowledge (figure 2), the research mainly outlined the transformation of mathematical intuitions into formal mathematical knowledge whereas pedagogical intuitions were transformed into beliefs (Leikin, 2006).
Development of new mathematical knowledge takes place at all the stages of teachers’ work: planning, performing and analyzing a lesson. When planning the lesson teachers clearly express their “need to know the material well enough” and their “need to predict students’ possible difficulties, answers, and questions”. At the planning stage the teachers are involved in designing activities that allow them to reach new insights. Hence new pieces of information are sometimes collected and some familiar ideas are refined (Leikin, 2006, Leikin, 2005a). The need to “know better than the students” stimulate teachers’ thinking about possible students’ difficulties. When predicting them teachers reflect on their own uncertainties, thus solve their own questions when planning the lesson. Through interaction with students teachers become aware of new (for them) solutions to known problems, new properties (theorems) of the mathematical objects, new questions that may be asked about mathematical objects and in this way they develop new mathematical connections. In what follows we exemplify this newly acquired awareness.

Example 1: Learning from a student’s mistake

Lora, an instructor in a course for pre-service elementary school teachers, taught a lesson on number theory. The following interaction occurred:

Teacher: Is number 7 a divisor of K, where $K = 3^4 \times 5^6$?
Student: It will be, once you divide by it
Teacher: What do you mean, once you divide? Do you have to divide?
Student: When you go this [points to K] divided by 7 you have 7 as a divisor, this one the dividend, and what you get also has a name, like a product but not a product…

Lora’s intention in choosing this example was to alert students to the unique factorisation of a composite number to its prime factors, as promised by the Fundamental Theorem of Arithmetic, and the resulting fact, that no calculation is needed to determine the answer to her question. This later intention is evident in her probing question.

What Lora learned from the above interaction?

She learned that the term “divisor” is ambiguous and a distinction is essential between divisor of a number, as a relationship in a number-theoretic sense and divisor in a number sentence, as a role played in a division situation. She learned that the student assigned the meaning based on his prior schooling and not on his recent classroom experience in which the definition for a divisor was given and usage illustrated. Before this teaching incident Lora used the term properly in either case, but was not alert to a possible misinterpretation by learners. The student’s confusion helped her make the distinction, increased her awareness of the polysemy (i.e. different but related meanings) of the term divisor and definitions that can be conflicting. This resulted in developing a set of instructional activities in which the terminology is practiced (Zazkis, 1998).
Example 2: Learning from a student’s solution

Shelly, a teacher with 20 years of experience in secondary school, solved with her Grade 12 students the following problem:

\[
\text{Prove that: } 1 + 2t + 3t^2 + 4t^3 + \ldots + nt^{n-1} = \frac{1-t^n}{(1-t)^2} - \frac{nt^n}{1-t}
\]

She expected her students to prove this equality using mathematical induction but unexpectedly one of the students (Tom) suggested the following solution:

\[
S(t) = 1 + 2t + 3t^2 + 4t^3 + \ldots + nt^{n-1} = F'(t), \quad \text{when}
\]

\[
F(t) = t + t^2 + t^3 + \ldots + t^{n-1} + t^n = \frac{t^{n+1}-t}{t-1}, \quad \text{thus}
\]

\[
S(t) = F'(t) = \ldots = \frac{1-t^n}{(1-t)^2} - \frac{nt^n}{1-t}
\]

Shelly’s reflective reaction was: “How could I miss this? Oh well, the problem is from the mathematical induction topic and I did not think about derivative at all. The solution is clear, but I did not think about it”.

What Shelly learned in this episode?

A connection between the fields of induction and calculus was new for her. She knew about use of mathematical induction in geometry, for example, in proving a theorem about the sum of interior angles in a polygon. Mathematical induction, for her, was also was connected to divisibility principles, since many divisibility rules may be proved using induction. As such, it was naturally connected to the topic of sequences and series, because of the multiple proofs using induction in these topics. She was also aware that many problems in mathematical induction could be solved using different methods.

Shelly: Even in the matriculation exams they say ‘prove using induction or in a different way’ like 3 is a divisor of \(n^3-n\) because \(n^3-n=(n-1)n(n+1)\)

However, when preparing this lesson Shelly did not think about this solution. Moreover during more than 29-years of experience she never connected induction with calculus.

Tom’s solution added a new mathematical connection to her subject matter knowledge and this, in turn, became part of her repertoire of problems with multiple solutions drawn from different areas of mathematics.

Example 3: Learning from a student’s question

During a geometry lesson Eva, a teacher with 15 years of experience in secondary school, proved with her Grade 10 students the following theorem:
If AD is a hypotenuse of an external angle CAF in a triangle ABC then \[\frac{AB}{AC} = \frac{BD}{CD}\] (Figure 3a).

After the theorem was proved one of the students asked: “What happens if AD is parallel to BC (Figure 3b)?” This question led to the classroom discussion in which students drew a conclusion that the theorem was correct for non-isosceles triangle.

\[BD = BC + CD\]

Figure 3

Eva, in her reflective analysis of the situation, reported that she “never had thought about correctness of the theorem for isosceles triangle”. Furthermore, when analyzing this situation with the researcher she unexpectedly connected this geometry topic with the topic of limits:

“When AD is parallel to BC \[\lim_{AC\to AD} \frac{BD}{CD} = 1\].

Since BD=BC+CD, this situation can demonstrate the rule: \[\lim_{x\to\infty} \frac{x+c}{x} = 1\].”

What Eva learned through this lesson?

The connection appeared to be surprising both to Eva and our research team. First of all this lesson led her to develop a “novel formulation of a theorem.” Eva commented that “this theorem was never mentioned in any familiar textbook or mathematics course”. She noted that next time, if students will not consider an isosceles triangle when proving the theorem, she will lead student towards consideration of this special case.

As mentioned earlier, a student’s question served as a trigger, but it is the teacher’s curiosity and deep mathematical knowledge that led her to develop new connections.

IS THIS KNOWLEDGE NEW? IS THIS MATHEMATICS OR PEDAGOGY?

Teachers learn both mathematics and pedagogy when teaching. In many situations teachers’ pedagogical content knowledge developed when they become aware of students’ unpredicted difficulties. Further, through analysing sources of the difficulties and misconceptions teachers appreciate better the structure of mathematical thought. Example 1 is a case of developing such awareness: in order to help students adopt the meaning of the term implied in a given situation the teacher had to first clarify the disparity in different uses of ‘divisor’ for herself.

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