A TEACHER'S TREATMENT OF EXAMPLES AS REFLECTION OF HER KNOWLEDGE-BASE

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In this paper we present an approach to examining teachers’ practice by focusing on one teacher’s choice and use of examples in her lesson. By analysing and characterizing the teacher’s treatment of examples, we are able to get a handle on significant aspects of her mathematical and content pedagogical knowledge-base that may support or limit students’ learning.

EXAMPLES IN MATHEMATICS LEARNING AND TEACHING

Examples are an integral part of mathematics and an important component of expert knowledge (Michener, 1978). Examples play a critical role in learning, and in particular form the basis for generalization, abstraction and analogical reasoning. Studies on how people learn from worked out examples suggest that effective instruction should include multiple examples, with varying formats, that support the appreciation of deep structures rather than excessive attention to surface features (Atkinson et al, 2000). Studies related to concept learning suggest that examples and non-examples be introduced in a carefully thought way, to support the distinction between critical and non-critical features and the construction of a rich and appropriate concept image and example spaces (e.g., Vinner, 1983; Watson & Mason, in press; Zaslavsky & Peled, 1996). A number of studies deal with the contribution of carefully sequenced sets of examples on learning (e.g., Petty & Jansson, 1987). In spite of the critical roles examples play in learning and teaching mathematics, studies focusing on teachers’ choice and treatment of examples are scarce. Rowlan et al (2003) identify three types of elementary novice teachers’ poor choices of examples, which concur with the concerns raised by Ball et al (2005) regarding the knowledge base teachers need in order to carefully select appropriate examples that are “useful for highlighting salient mathematical issues” (ibid). Clearly, the choice of examples in secondary mathematics is far more complex. Zaslavsky & Lavie (submitted) point to the possible complex web of considerations underlying teachers’ choice of examples. In our study we illustrate the complexity of treatment of examples in an 8th grade pre-algebra course.

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TEACHERS’ KNOWLEDGE BASE

In our study we focus mainly on teachers’ mathematical content knowledge and on their pedagogical content knowledge (Shulman, 1987). By pedagogical content knowledge we refer mainly to the teacher’s knowledge of how to transform mathematics into forms that are “pedagogically powerful and yet adaptive to the variations in ability and background presented by the students” (ibid, p. 15). In particular, we regard examples as possible elements of pedagogically powerful tools for students to develop their ways of understanding and ways of thinking (Harel, in press). There is an interplay between the knowledge base that a teacher needs in order to construct powerful instructional examples (e.g., Ball et al, 2005) and the knowledge that is reflected through the use of examples. Our study is an attempt to deal with this interplay.

THE INTERPLAY BETWEEN TEACHERS’ TREATMENT OF EXAMPLES AND THEIR KNOWLEDGE BASE

A teacher’s treatment of examples in the classroom can reveal a critical aspect of the proof schemes they possess as well as the proof schemes they target for their students. Specifically, a teacher may use a sequence of examples to help students reveal an underlying pattern of a mathematical phenomenon. On the other hand, a teacher may use such a sequence as evidence for the truth of the phenomenon. These two teaching practices reflect fundamentally different ways of thinking: while the former is conducive to the deductive proof scheme, the latter to the empirical proof scheme (Harel, 2001). Later, in this paper, we demonstrate an application of the former practice by the teacher who participated in this study.

Framework and goals

This paper is part of a series of reports, in progress, on the results of an NSF-funded project whose aim is to investigate the development of teachers’ knowledge base under a particular instructional intervention, called DNR-based instruction (Harel, in press). The data reported in this paper is drawn from classroom observations of one of the ten teachers from a Southwestern U.S. urban area, who participated in this project. One segment of this research project focuses on identifying the teaching practices of the teachers who participated in the project and the alignment of these teaching practices with DNR. The study reported in this paper focuses on the teaching practices associated with the use of examples by teachers as a pedagogical tool.

Procedure

Three lessons of one of the teachers, Marjorie, who participated in the large study, were chosen. The first two we term “Lesson 1”, since they were taught the same day.

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2 This part of the study was inspired by and based on the conceptual framework of a research project funded by the Israel Science Foundation (grant 834/04, O. Zaslavsky PI) that investigates teachers’ use of examples in mathematics.
separated only by lunch and addressed the same topic taught in Marjorie’s first year in the project. The third lesson, which we term “Lesson 2”, was taught a year later, on the same topic, to different students. These lessons were part of a pre-algebra course for 8th grade students in a low performing socioeconomically disadvantaged school. This was Marjorie’s first year teaching mathematics.

As mentioned above, our approach for this portion of the study was to analyse the lessons from the point of view of the teacher’s treatment of examples, in an attempt to find links to her knowledge-base. Thus, the lessons were divided into episodes according to their goal and main idea. Within each episode, the examples used were identified and characterized according to several constructs (e.g., the specific choice of examples; the sequence, range and variation among the specific cases; the type of examples used; the purpose the example served; the extent to which the example or set of examples may support the development of the mathematical idea). Some comparisons were drawn between the teacher’s treatment of examples in the two lessons. Based on the previous stages, we identified elements of the teacher’s knowledge-base that are reflected through her choice and treatment of examples. Finally, a stimulated recall interview was conducted with the teacher in order to validate or refute some conjectures that were raised regarding her considerations underlying her treatment of examples.

We turn to a description of Marjorie’s treatment of examples in Lessons 1 & 2. The description is not complete or extendedly detailed, yet it depicts the main moves of Marjorie related to her choice and use of examples. This description conveys what Thompson (2005) terms the ‘lesson logic’.

**Marjorie’s treatment of examples in Lessons 1 & 2**

**Lesson 1:**

Marjorie began the lesson by putting up front on the board three examples (Figure 1) leading to the formulae for the areas of a rectangle and a triangle. She moved from a rectangle and its area calculation – already familiar to her students, to a right triangle that is clearly half of the rectangle, to a more general triangle (not clear how generic it is). She kept the given measurements constant. This allows a better focus on the varying elements, e.g., the type of figure, the connection between a side and its corresponding height. Then, with the ‘help’ of some students that she invited to the board, she moved from one case to the next, building on the previous ones.

![Figure 1: Marjorie’s initial set of examples and calculations associated with each case](image-url)
Figure 1 displays a fair account of what was written on the board at a certain point, after going through all three cases.

Note that in the third case – the more general triangle – the class discussed how they might ‘split’ the side of measurement 6. Following the majority of the suggestions, Marjorie split it into 2 and 4, and built on the previous case where there was a right triangle.

After establishing the way to calculate the area of a triangle, Marjorie presented the class with the topic of the lesson – the Pythagorean Theorem – and the general goal of the lesson, which was to find the relationship between the legs and the hypotenuse of a right triangle. The goal formulation created a need to introduce the notions of a right triangle and its legs and hypotenuse, which Marjorie did through some examples. Then she formulated a more specific question, related to the main goal of the lesson:

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\text{Consider a right triangle with legs of length of 1. Suppose you draw squares on the hypotenuse and legs of the triangle. How are the areas of these three squares related?}
\]

Figure 2 illustrates the teacher’s moves in constructing the squares in a way that the relationship between their areas would be obvious. She moved from constructing (on a grid transparency) a right triangle with the given legs, to constructing the squares on each leg, pointing to their area measurements; then constructed the square on the hypotenuse, using the grid to construct right triangles congruent to the given one. The final stage was to divide the square on the hypotenuse into parts, the area of which could be calculated. In this case it was done by sketching the two diagonals, and pointing to the congruence between the given triangle and the 4 ‘inner triangles’.

Then the set of examples that Marjorie chose for illustrating of the relationship under investigation was presented in a form of a table (Table 1).

Having selected the specific cases for investigation, Marjorie used each one to illustrate not just the end result but the entire process leading to it, as illustrated in Figure 3 (for the first 4 examples). She repeated the reasoning for each step of the construction and accompanying calculations for all cases in Table 1, and filled the table with the data, pointing to the pattern that emerged from the process.
<table>
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<th>Area of Square on Leg1</th>
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Table 1: Specific cases for explorations through area considerations

Figure 3: Repeated illustration of the process of obtaining the relationship between the areas of the squares on the sides of a right triangle, for the first 4 cases in Table 1 (A denotes Area)
Lesson 2:
This lesson began with an exploration task that stated the topic of the lesson, that is, the Pythagorean Theorem as follows (Figure 4):

![Figure 4: The Exploration Task assigned to the students](image_url)

Similarly to Lesson 1, the task created a need to introduce the notions of a right triangle and its legs and hypotenuse, which Marjorie also addressed in this lesson through some examples, although different than those in Lesson 1.

The lesson progressed as the students sketched the three right triangles on grid paper and measured their hypotenuses. Interestingly, they all came up with the ‘correct’ measure. Marjorie filled the two empty columns in the table on the board based on the data the students reported.

THE TEACHER’S KNOWLEDGE-BASE AS RELECTED BY HER TREATMENT OF EXAMPLES

There are several elements of Marjorie’s knowledge-base that can be induced by examining her choice and use of examples in the two Lessons that were analysed. We allude to several of them and will elaborate on them further in our presentation. In our presentation we will also suggest links between Marjorie’s knowledge-base and the professional development framework in which she took part.

Looking at Lessons 1 & 2, in terms of the examples that Marjorie introduced, it is clear that she draws on her broad example-space and that she is flexible in her choices. Moreover, she is able to vary the examples she uses according to the lesson setting and approach she adapts. Thus, in Lesson 1, she applied an approach to the Pythagorean Theorem that is based on a process that can be justified and repeated for any right triangle. More specifically, this approach would work for any randomly chosen integer measures for the triangle legs (she needs the integers in order to be able to illustrate the construction on the graph paper). This is reflected in the large
number of cases they examine (7), and the systematic choice of these cases (1,1; 1,2; 2,2; 1,3; 2,3; 3,3; 3,4). By this she conveys the generality of their conclusions. On the other hand, when she decides to use another approach as in Lesson 2 (for reasons that we later learned have to do with external constraints), she realizes that she can no longer select the examples freely. In order to make sure that the students can reasonably measure the hypotenuse with grid paper, she needs to make sure all sides are integers (while in Lesson 1 the hypotenuse could be a non-integer). Thus, she must rely on her knowledge of Pythagorean triplets. The constraints of the paper, allow her to make only a very limited number of examples (3), all very special cases. This approach does not convey a process and does not support the generalization. In short, the craft of carefully selecting examples seems to be part of Marjorie’s knowledge-base, opposed to what Rowland et al (2003) report regarding unfortunate randomly chosen examples by novice teachers.

Through Marjorie’s treatment of examples, a lot of her mathematical knowledge comes to play. In Lesson 1 she is very fluent in the way she demonstrates the construction that actually constitutes one of the visual proofs of the Pythagorean Theorem, based on the following equality: $a^2 + b^2 = 4 \cdot \frac{1}{2}(a \cdot b) + (b-a)^2$. It appears that even if Marjorie does not think of what she demonstrates in this symbolic form, she does see the general geometric pattern of the visual proof. On top of this sophisticated mathematical knowledge, Marjorie is familiar with some Pythagorean triplets and may even know how to generate a new triplet based on a known one, e.g., $(3, 4, 5) \Rightarrow (3 \cdot 3, 3 \cdot 4, 3 \cdot 5)$.

From the analysis of Marjorie’s treatment of the initial set of examples (Figure 1), it appears that she understands to a certain extent the invariance of area of triangles with a given side and the corresponding height. Thus, when splitting the side in the 3rd example, she used the wording “call this 2 and call this 4”. So to her it is just a placeholder and could work for any other way of splitting it (we validated this interpretation in the stimulated recall interview).

In spite of her rather deep mathematical knowledge related to the topic she taught and of many manifestations of attentiveness to students and sensitivity to their ways of learning, Marjorie also manifested some missed or even mis-leading opportunities. For example, in the case of the invariance of triangle area – although she was careful in her wording – this most probably did not come across to the students. Trying out other possible splits and realizing that the area remains the same could have been more supportive.

To conclude, characterising Marjorie’s treatment of examples and using it as a means to learn about her knowledge-base appears to be both feasible and worthwhile in terms of the complexity and richness it conveys. Through her treatment of examples we were able to get a handle on significant aspects of her mathematical and content pedagogical knowledge, along the lines of Ball et al (2005). The approach suggested in this paper concurs with Hiebert et al’s ideas of learning from practitioners (2002).
Zaslavsky, Harel & Manaster

References


Zaslavsky, O. & Lavie, O. (submitted). *What is entailed in choosing an instructional example?*