“MAKING PROOF CENTRAL TO PRE-HIGH SCHOOL MATHEMATICS IS AN APPROPRIATE INSTRUCTIONAL GOAL”: PROVABLE, REFUTABLE, OR UNDECIDABLE PROPOSITION?1

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Many researchers and curriculum frameworks, especially in the United States, recommend that proof become an integral part of all students’ mathematical experiences as early as the elementary grades. However, the development of proof in school mathematics has been uneven. Proof has historically been associated only with tenth-grade courses on Euclidean geometry. This historical tradition, coupled with the difficulties that even advanced students face with proof, seem to challenge the appropriateness of the goal to make proof central to pre-high school mathematics. In this paper, we use mathematics education and psychological research to examine the appropriateness of this goal. In a PME conference, this paper can initiate fruitful discussions among researchers from different countries and with diverse perspectives on the issue of incorporating proof into pre-high school mathematics.

INTRODUCTION

Many researchers and curriculum frameworks, especially in the United States (U.S.), recommend that proof become an integral part of all students’ mathematical experiences and across all grades (e.g., Ball & Bass, 2003; NCTM, 2000; Schoenfeld, 1994; Yackel & Hanna, 2003). In the U.S., indicative of the current focus on making proof central to school mathematics is the following excerpt from the Principles and Standards for School Mathematics, an influential curriculum framework recently released by the National Council of Teachers of Mathematics (NCTM, 2000):

Instructional programs from prekindergarten through grade 12 should enable all students to recognize reasoning and proof as fundamental aspects of mathematics, make and investigate mathematical conjectures, develop and evaluate mathematical arguments and proofs, select and use various types of reasoning and methods of proof…. Reasoning and proof cannot simply be taught … by “doing proofs” in geometry…. (p. 56)

Although the current policy and research discourse (at least in the U.S.) is tilted in favor of a widespread presence of proof in the school curriculum, historically proof has been associated only with tenth-grade courses on Euclidean geometry. This historical tradition, coupled with the well-documented difficulties that even high school and university students face with proof (e.g., Healy & Hoyles, 2000; Moore, 1994; Reiss et al., 2002), seem to challenge the appropriateness of the goal to make proof central to pre-high school mathematics. Therefore, the examination of the following question becomes critical: Is pre-high school students’ learning of proof

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Is It Appropriate for School Mathematics Instruction to Promote Proof in the Pre-High School Grades?

Our thesis is that it is appropriate for instruction to promote proof in the pre-high school grades. This position derives from five arguments, which we discuss separately. The first three arguments relate more to whether promoting proof in pre-high school is a worthy educational objective with high priority. The last two arguments relate more to whether this objective is feasible for the given age group.

An argument based on the purported relationship between the structure of the discipline of mathematics and the school mathematics curriculum

The first argument goes as follows: Proof deserves a central place in the K-12 mathematics curriculum and, thus, in students’ mathematical education from the start of their schooling, because: (1) proof holds a central place in the discipline of mathematics, and (2) the school mathematics curriculum should represent the structure of the discipline of mathematics in an undistorted way from the very
beginning. Below we elaborate on the two foundations of the argument, but before we do so we note that the second foundation of the argument is essentially a value statement and, as such, cannot be corroborated or refuted by research.

The first foundation of the argument is based on the fact that proof serves two central functions in the discipline: (a) it is the principal means by which mathematicians establish mathematical truth and derive new knowledge from old, and (b) it promotes mathematical understanding and fosters connections among mathematical ideas.

The second foundation of the argument is based on ideas set forth by general educational scholars such as Bruner and Schwab. Bruner (1960) asserts that there should be “a continuity between what a scholar does on the forefront of his discipline and what a child does in approaching it for the first time” (pp. 27-28). Likewise, Schwab (1978) argues for a school curriculum “in which there is, from the start, a representation of the discipline” (p. 269), and in which students have progressively more intensive encounters with the inquiry and ideas of the discipline.

An argument regarding the role of proof in students’ learning of mathematics

The second argument focuses on the role that proof can play in students’ mathematical education – not as an esoteric or advanced skill, but as a central component of students’ learning of mathematics. Specifically, proof can promote in even young students’ mathematical activity functions similar to those it promotes in the discipline of mathematics: it can serve as a mode of establishing what is acceptable mathematically in a classroom community, as a means of supporting students’ process of mathematical discovery, and as a tool for promoting sense making (see, e.g., Ball & Bass, 2003; Yackel & Hanna, 2003).

An argument regarding the difficulties that even advanced students of mathematics face with proof

The third argument turns the problem of even advanced students’ insufficient performance with proof on its head: high school and university students have difficulties with proof not because they are incapable of engaging successfully with it, but because of their abrupt and unsupported introduction to proof at high school (e.g., Moore, 1994; Usiskin, 1987). For example, Usiskin (1987) notes:

[O]f all mathematical areas, justifying, discussing logic and deduction, and writing proofs are major goals only in geometry. Ideas of logic and deduction need not wait until secondary school …. [I]t is still important to give children experience in drawing inferences. Part of the difficulty in dealing with a systematic approach to geometry in secondary school is surely due to the ignoring of any sort of system earlier. (pp. 27-28).

The argument described above suggests that the pre-high school grades can play an important role in the efforts to promote students’ competency in proof. However, a possible criticism of this suggestion is that, even if proof were central to pre-high school students’ mathematical experiences, these students could still not become sufficiently competent in proof when they enter high school. This criticism invites examination of the following question against the existing body of research: Can pre-
high school students reason deductively and engage successfully in proof? The following two arguments focus on the issue of feasibility raised by this question.

**An argument based on mathematics education research regarding the feasibility of the goal to promote proof in the pre-high school grades**

This argument is based on existing research evidence that, as early as the elementary grades and in supportive classroom environments, pre-high school students can engage successfully in deductive reasoning and proof (e.g., Ball & Bass, 2003; Stylianides, in press; Zack, 1997). Important to note, however, is that the teaching practices documented in these studies were not typical. Almost all of them were the practices of teacher-researchers. Thus, a major issue here is how deductive reasoning and proof can become central to pre-high school students’ learning of mathematics on a large scale. This issue, however, goes beyond the scope of our paper.

**An argument based on psychological research regarding the development of students’ proof competence and reasoning skills essential for proof competence**

The last argument we present in this paper is based on psychological research that has focused on the development of students’ proof competence and reasoning skills (notably deductive reasoning) that are essential for proof competence. Although there is no consensus among researchers on the ages at which students master different reasoning skills, several studies suggest that it is possible to expect from pre-high school students to engage successfully in tasks involving deductive reasoning and proof. Also, a major idea supported by the psychological research in this domain is that the development of students’ reasoning skills follows a trajectory that begins from the early elementary grades and continues until the end of the high school grades. Thus, it is imperative that instruction nurtures the development of students’ reasoning skills from the start of their schooling, thus helping students to increasingly master these skills. By operating on the erroneous assumption that students are incapable of this kind of reasoning in the pre-high school grades, instruction does not help students develop naturally their emerging reasoning capabilities.

Below we summarize findings of some major psychological studies in this domain. We begin with studies related to the development of proof competence. Next, we present studies that investigate the development of children and adolescents’ competence in deductive reasoning and other related reasoning skills (e.g., correct application of rules of inference) that are important for proof competence.

**Development of proof competence**

Piaget (1987) suggested that the development of students’ ability for proof construction follows a path toward logical necessity. In his studies, he found that elementary school students generated logical proofs. However, children of this age did not recognize the sufficient character of their proofs. It is only in adolescence that a deductive system is developed and students recognize proof as being sufficient.

Foltz et al. (1995) in a study with fifth- and eighth-graders examined inductive and deductive approaches to the construction of proofs. The lack of significant differences in deductive reasoning ability for the two age groups led the researchers
to cluster together the subjects across grades into categories of formal (deductive), transitional, and nonformal reasoning. The results from this study demonstrate that deductive reasoning competence is associated with a deductive approach to proof construction. Thus, Foltz et al.’s findings support Piaget’s earlier findings about an intimate relation in the development of the ability for deductive reasoning and proof construction. However, the findings of the two studies do not seem to agree on the stages of this developmental progression.

Lester (1975) investigated the development of students’ ability to write valid proofs by examining developmental aspects of problem-solving abilities (number of tasks solved, number of correct applications of rules of inference, difficulty per task attempted, and total time per task attempted) in an experimental mathematical system. Lester’s study included 80 students in the following grade groupings: 1-3, 4-6, 7-9, and 10-12. Lester’s findings showed that middle school students (grades 7-9) were as capable of solving problems in the experimental mathematical system as high school students (grades 10-12). Also, when students in the upper elementary grades (grades 4-6) were given some extra time, they were able to solve problems just as successfully as the secondary school students. Students in the lower elementary grades appeared to be less successful. These results suggest that, “certain aspects of mathematical proof can be understood by children nine years old or younger.” (p. 23)

**Development of reasoning skills that are important for proof competence**

A significant body of research suggests that students’ reasoning skills develop with age. Specifically, students’ ability for deductive reasoning passes through a developmental progression that extends over the whole range of school years (e.g., Ward & Overton, 1990). However, there is no agreement on when different levels of mastery of deductive reasoning are achieved. Some studies suggest that sophisticated mastery of deductive reasoning and explicit understanding of logical necessity do not emerge before early adolescence (e.g., Markovits et al., 1989; Overton et al., 1987). Some other studies suggest that even preschoolers are capable of drawing deductively valid conclusions (e.g., Hawkins et al., 1984; Richards & Sanderson, 1999).

In order to address the gaps in the existing research, Galotti et al. (1997) investigated students’ reasoning skills in kindergarten and grades 2, 4, 6. They concluded:

[Y]oung children can do more than draw deductive and inductive inferences. Even by second grade, they show the beginnings of implicit recognition that these two types of inferences are different and, as is appropriate, show more consistent answering, and higher confidence, in deductive inferences than in inductive inferences (although confidence in deductive answers is not as high as it should be). (p. 77)

Next, we turn to findings of psychological studies on the development of students’ ability for other related reasoning skills that are important for proof competence. Klaczynski and Narasimham (1998) in a study with preadolescents, middle adolescents, and late adolescents (mean ages 10, 14, and 17, respectively) report the following about the development of students’ ability to draw rules of inference and conditional rules:
The rules for drawing MP [modus ponens] and MT [modus tollens] inferences appear well-developed by early adolescence. Similarly, the rules for generating indeterminant conclusions seem well-defined by early adolescence. This is not, however, to say that conditional reasoning abilities are fully developed before the onset of adolescence. (p. 878)

Braine and Rumain’s (1983) review of developmental psychology research on logical reasoning shows that the rule for drawing MP inference can be available to 6-year-olds, which is earlier than what is suggested by Klaczynski and Narasimham (1998). Although performance on MT is usually poorer than that on MP, MT can be acquired by second graders. Falmagne (1980) reports further that training can help improve children’s ability for MT and to make this ability applicable across different contexts.

With regard to students’ ability for direct reasoning, Osherson’s (1976) investigations suggest that the ability to chain inferences together in short steps of direct reasoning is available to 10-year-olds. Some strategies for indirect reasoning – reasoning that departs outside the given starting information – such as setting up all alternatives a priori and working out their consequences to see if they all lead to the same conclusion, seems to become available to children at around the age of 10 (Braine & Rumain, 1983). However, some other strategies for indirect reasoning, such as the reductio ad absurdum, seem to cause more difficulties. In particular, Braine and Rumain (1983) note that, “in maximally propitious circumstances, a half to two thirds of adults and a minority of 10- and 11-year-olds find the reductio ad absurdum solution to Modus Tollens” (p. 290). This result indicates that even more sophisticated forms of indirect reasoning strategies begin to emerge in late childhood.

When the previous statement is taken together with Falmagne’s (1980) finding that training can improve children’s responses on MT and increase their generalization across different contexts, we can infer that supportive classroom environments can facilitate the development of students’ reasoning strategies.

CONCLUSION

This paper has focused on whether making proof central to pre-high school mathematics is an appropriate instructional goal. Our analysis, which utilized research from education and psychology, offers support for the appropriateness of this goal. Nevertheless, translating theoretical ideas and even empirical findings from different studies into instructional practices that ordinary teachers can use to cultivate proof in the classroom requires considerable work. For example, it requires identification of teaching practices that can support effective instruction of proof in the early grades as well as enactment of professional development programs that will help teachers master these practices. A major challenge, but also a primary urgency, for this work is to find fruitful ways to coordinate relevant research in mathematics education and psychology and also to integrate findings from these two fields. As we have shown in this paper, there is a rich body of psychological research that provides a broad portrait of students’ emerging reasoning skills that can be used to inform our

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2 Modus ponens states that if the antecedent of a conditional is true, then the consequent is also true; modus tollens states that if the consequent of a conditional is false, then the antecedent is also false.
understanding of when students are expected to overcome innate constraints related to their ability for deductive reasoning and proof. An interdisciplinary and collaborative approach to the problem of promoting proof in even young students’ learning of mathematics promises major advancements.

References


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