# PROCESS-OBJECT DIFFICULTIES IN LINEAR ALGEBRA: EIGENVALUES AND EIGENVECTORS

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*Many beginning university students struggle with the new approach to mathematics that they find in linear algebra courses. These courses may focus on conceptual ideas more than procedural ones, with the ideas arriving one after another and building upon each other in a rapid fashion. This paper highlights the example of the conceptual processes and difficulties students find in learning about eigenvalues and eigenvectors, where a word definition may be immediately linked to a symbolic presentation,* $Ax = \lambda x$*, and its manipulation. The results describe the thinking about these concepts of a group of first year university students, and in particular the obstacles they faced, and the emerging links some were forming between the parts of their concept images forming from embodied, symbolic and formal worlds.*

## INTRODUCTION

Many university students are introduced to the formal presentation of mathematics through a first course in linear algebra. Unlike calculus, that often emphasises manipulation of symbols in order to solve problems, the focus in linear algebra is on the description of concepts, often through word definitions, and derivation of further concepts from these. Considering the problems of understanding more advanced mathematics Tall (1998) described an *enactive* approach to learning about differential equations (DE's) in which one builds an *embodied* notion of the solution to a DE in contrast with an algebraic introduction that stresses analytic procedures without first giving a feeling for a DE and its solutions. In recent papers Tall (Tall, 2004a, b) has developed these ideas into a theory of the cognitive development of mathematical concepts. He describes learning taking place in three worlds: the embodied; the symbolic; and the formal. The embodied is where we make use of physical attributes of concepts, combined with our sensual experiences to build mental conceptions. The symbolic world is where the symbolic representations of concepts are acted upon, or manipulated, where it is possible to "switch effortlessly from processes to *do* mathematics, to concepts to *think* about." (Tall, 2004a, p. 30). Movement from the embodied world to the symbolic changes the focus of learning from changes in physical meaning to the properties of the symbols and the relationships between them. The formal world is where properties of objects are formalized as axioms, and learning comprises the building and proving of theorems by logical deduction from the axioms.

The DE situation above is exactly analogous to what often happens when eigenvalues and eigenvectors are introduced to students. While the concept definition may be given in words the student is soon into manipulations of algebraic and matrix

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representations, e.g. transforming $Ax = \lambda x$ to $(A - \lambda I)x = 0$. In this way the strong visual, or embodied metaphorical, image of eigenvectors is obscured by the strength of this formal and symbolic thrust. However, an enactive, embodied approach would first give a feeling for what eigenvalues, and their associated eigenvectors are, and how they relate to the algebraic representation. Such linking of multiple representations of concepts is an important idea, and it has been suggested that 'a central goal' of mathematics education should be to increase the power of students' representations (Greer & Harel, 1998, p. 22). Developing the *representational versatility* (Thomas & Hong, 2001; Thomas, in press) to make the links between the concepts of scalar, vector, equation, eigenvalue, and eigenvector, and their algebraic, matrix and geometric representations is not automatic, and yet often students are asked to deal with this before they have been introduced to linear transformations, as is the case in computation-to-abstraction linear algebra courses (Klapsinou & Gray, 1999). When we begin to consider how each of these concepts arises then things get more complicated. Dubinsky and others (Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas, & Vidakovic, 1996; Dubinsky, & McDonald, 2001) have described how actions become interiorised as processes that in turn may be *encapsulated* as objects, forming part of a schema. One serious problem with $Ax = \lambda x$ for students is that the two sides of the equation are quite different processes, but they have to be encapsulated to give the same mathematical object. In the first case the left hand side is the process of multiplying (on the left) a vector by a matrix; the right hand side is the process of multiplying a vector by a scalar. Yet in each case the final object is a vector that has to be interpreted as the product of the eigenvalue and its eigenvector. A second difficulty may be that the symbolic manipulation process may be obscuring understanding of the concept of eigenvector. Explanations of what an eigenvector is start with it as an object and then explain the effect of performing actions upon it; applying a transformation to it and multiplying it by a scalar. However, to find the eigenvector one must first find its associated eigenvalue, holding in obeyance any action it will perform upon the eigenvector until it's found. In this paper we use Tall's three worlds to analyse the way that some students think about these concepts and how they cope with the cognitive obstacles described above.

## METHOD

This research study took place in early 2005 and comprised an initial case study of first year Maths 108 mathematics and science students from The University of Auckland. Maths 108 is a first year computation-to-abstraction course covering both calculus and elementary linear algebra (systems of linear equations, invertibility of matrices, determinants, eigenvectors, eigenvalues and diagonalisation). The first-named author was one of the lecturers on the course, and she tried to emphasize a geometric, embodied approach. During the linear algebra lectures she took the students for a tutorial in a computer laboratory on two occasions and showed them how to use Maple for linear algebra. After this the students were given a questionnaire that asked them about their attitude to linear algebra (and Maple). Finally, at the end of the linear algebra lectures a written test on eigenvalues and eigenvectors was given to a group of 10 students who volunteered to take part in the research. Of these students six (numbered 1-6 below) had attended the researcher's lectures, while the rest attended other streams. The test (see Figure 1 for some of the questions) was not designed to assess course progress but was primarily to examine the students' understanding of the concepts of eigenvectors and eigenvalue, and their ability to carry out the process of finding them for a given 2x2 matrix (see question 2).

**Maths 108 Questions**

1. Describe the definitions of eigenvalues and eigenvectors in your own words.

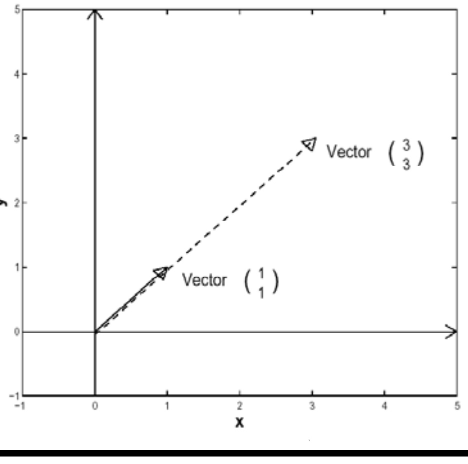
2. Let $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ be a 2 × 2 matrix. What are the eigenvalues and eigenvectors of matrix $A$?

3. Suppose $A$ is a matrix representing a transformation and: $A\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 3\begin{pmatrix} 2 \\ 3 \end{pmatrix}$. What does this tell us about the 3? Describe this geometrically.

4. How can we decide whether a given vector is an eigenvector of a matrix? Explain this in your own words.

8. How many different eigenvectors are there associated with a given eigenvalue?

11. Describe what the following diagram may represent as best as you can.



Note: Question 2 is a worked example in the course manual.

Figure 1. The questions from the test discussed in this paper.

## RESULTS

Eigenvectors and eigenvalues probably form the most difficult part of Maths 108 linear algebra course. Students may not agree, since they often focus on solution procedures rather than conceptual understanding. So in the questionnaire we asked them what they thought was the most difficult idea in the linear algebra course. While many students mentioned the somewhat involved procedures of row reduction and calculating inverses of matrices, etc., student 2, among others agreed with us.

> Student 2: Eigenvalues and eigenvectors, yes Maple help me understand how to get the eigenvectors from eigenvalues.
>
> Student 11: Inverse, eigenvalues, eigenvectors. Tutorial sessions helped. Continued computer sessions will help me more.

## Indications of an embodied perspective

There were some indications in the test responses of the students that they were using the embodied world to help build their thinking about eigenvalues and eigenvectors. For example question one asked them to describe the definition of the terms in their own words. Three of the students (1, 6, 8) mentioned the idea of the 'direction' of a vector. Although student 1 did not use it correctly, the other two had a clearer embodied aspect to their concept image of eigenvector.

> Student 6: After transformation the direction of eigenvectors will not change.
>
> Student 8: Eigenvector is a vector which does not change its direction when multiplied (or transformed) by a particular matrix. An eigenvector can change in length, but not in direction.

We see that student 8 has also added the embodied notion of change of length to her thinking. While the other students who answered the question referred to the procedural, symbolic manipulations in their answers, two of them had formed a mental model of the structure of this too (see Figure 2).

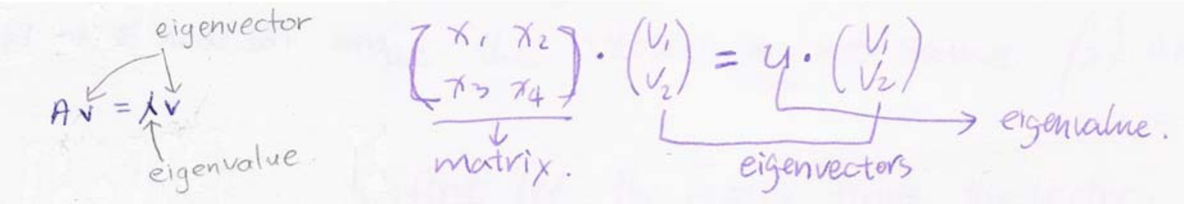


Figure 2. Students 5 and 9 use a structural model of the algebra.

In answering question 3 we also saw examples of the embodied nature of the students' thinking. Student 1 explains that "the eigenvalue changes the vector's direction. ie more steep.", using the embodied notions of 'change of direction' and 'steepness'.

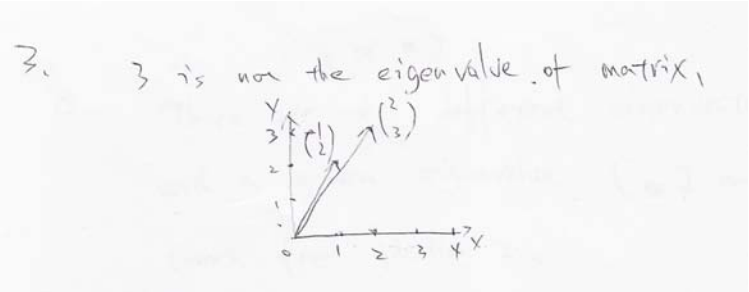


Figure 3. Student 6's embodied notions of 'change of direction' and 'steepness'.

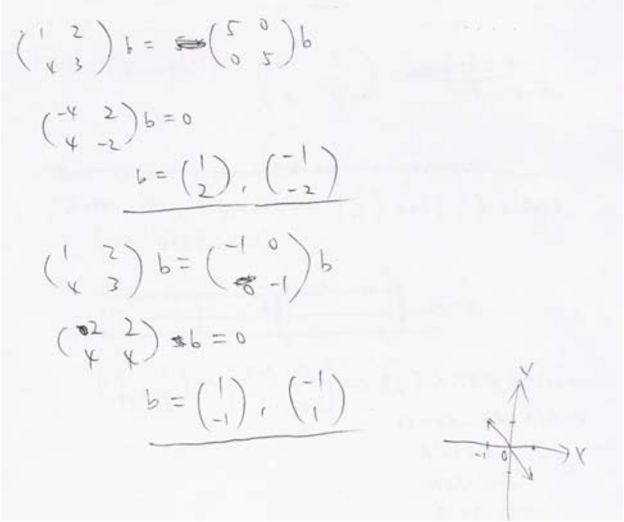
Student 4 also said that “3 is not an eigenvalue of the equation. Hence it changes the direction of the original vector.” Student 6 has a similar recourse to the embodied idea of change of direction of the vector, drawing the picture in Figure 3. In question 4, students 4 and 8 also referred to the idea of direction to decide on whether a vector is an eigenvector. In question 11, all the students except 7 and 10 linked the diagram to a vector (1, 1), an eigenvalue of 3 and a final vector (3,3). In doing so they again used embodied terms such as “being stretched” (1), “it makes eigenvector longer” (3), and “stretch the length of (1, 1)” (5). It seems that the researcher’s students did make more use of embodied ideas than the others.

## Conceptual process problems

As we have described above there is a possible tension in $Ax = \lambda x$ between the process of matrix multiplication on the left and the scalar multiplication on the right, both resulting in the same object of the transformed vector.

$$\begin{aligned} A\mathbf{v} &= \lambda\mathbf{v} \\ A\mathbf{v} &= \lambda\mathbf{v} \\ A\mathbf{v} - \lambda\mathbf{v} &= \mathbf{0} \\ (A - \lambda I)\mathbf{v} &= \mathbf{0} \qquad \text{[note the use of I here]} \\ B\mathbf{v} &= \mathbf{0} \qquad \text{[where } B = A - \lambda I\text{].} \end{aligned}$$

4a.



$\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} b = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} b$

$\begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix} b = 0$

$b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} b = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} b$

$\begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix} b = 0$

$b = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

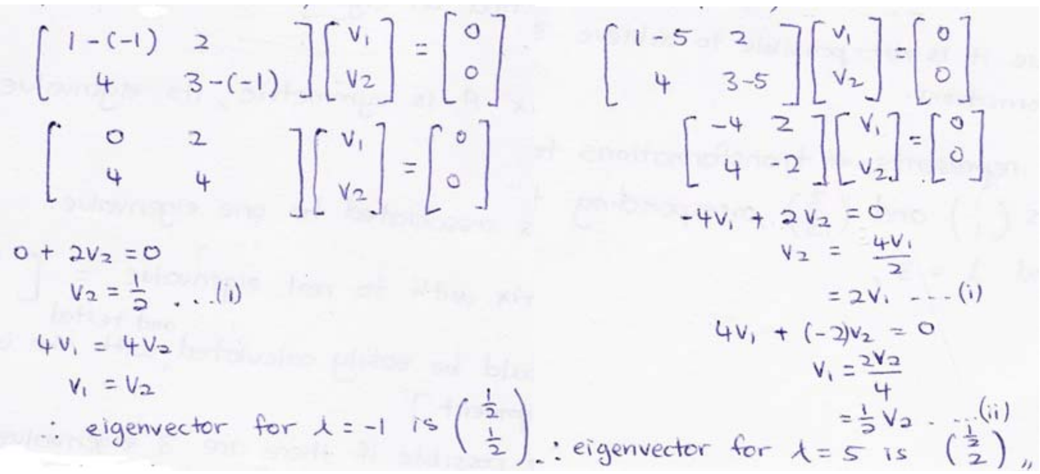
4b.

Figure 4. The course manual dealing with the two processes and Student 6’s solution to the problem.

The transformation of this equation to the form $(A - \lambda I)x = 0$ in order to carry out the process to find the eigenvalue $\lambda$ tends not to make explicit the change from $\lambda$ to $\lambda I$, from a scalar to a matrix. The section of the course manual where this is done is shown in Figure 4a. We see that the problem is skated over and the comment is simply made “note the use of *I* here.” In Figure 4b we see the example of student 6 who explicitly replaces the 5 and –1 in 5***b*** and –1***b*** on the right of the equation with the matrices 5*I* and –*I*. This no doubt helped him with equating the objects of the processes, but thinking of $Ax = \lambda x$ as $Ax = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} x$ may be an obstacle to understanding how the definition of an eigenvector relates to the algebraic representation.

### Symbolic manipulation action-process problems

Of the ten students we considered in detail, 5 were able to find correctly both the eigenvalues and eigenvectors for the matrix $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ in question 2. Of the others, two (students 2 and 9) found the eigenvalues but were unable to find the corresponding eigenvectors, and three (students 1, 3 and 10) were unable to make any progress. Student 3 wrote "I just forgot conseption [sic] of it" and student 10 "I would like to do this with the help of Maple."



$$\begin{bmatrix} 1-(-1) & 2 \\ 4 & 3-(-1) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$0 + 2v_2 = 0$

$v_2 = \frac{1}{2} \quad \ldots(i)$

$4v_1 = 4v_2$

$v_1 = v_2$

$\therefore$ eigenvector for $\lambda = -1$ is $\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$

$$\begin{bmatrix} 1-5 & 2 \\ 4 & 3-5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-4v_1 + 2v_2 = 0$

$v_2 = \frac{4v_1}{2}$

$= 2v_1 \quad \ldots(i)$

$4v_1 + (-2)v_2 = 0$

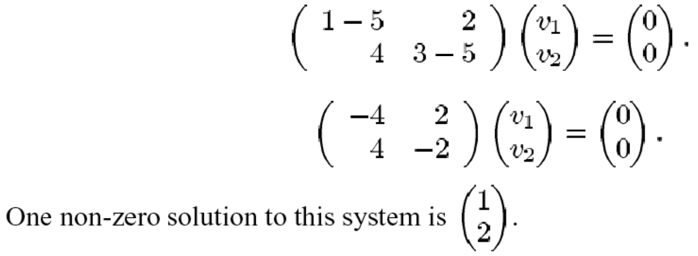
$v_1 = \frac{2v_2}{4}$

$= \frac{1}{2}v_2 \quad \ldots(ii)$

$\therefore$ eigenvector for $\lambda = 5$ is $\begin{pmatrix} \frac{1}{2} \\ 2 \end{pmatrix}$ ,,

Figure 6. Student 9's working to find the eigenvectors.

Student 9's working as she tries to find the eigenvectors is shown in Figure 6. The arithmetic and symbolic manipulation here contains a number of errors ($1-(-1)=0$; $0+2v_2=0 \Rightarrow v_2=1/2$; $4v_1+4v_2=0 \Rightarrow 4v_1=4v_2$; and $v_2=2v_1 \Rightarrow$eigenvector is (0.5, 2)), showing a weakness in such manipulation, rather than in the understanding of the conceptual process. Student 2 made a similar manipulation error, moving from writing the matrix $\begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix}$, without a vector, say, $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, or an '=0', to an incorrect vector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.



$$\begin{pmatrix} 1-5 & 2 \\ 4 & 3-5 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

One non-zero solution to this system is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Figure 7. Part of the course manual's method for finding an eigenvector.

The fact that the working in the final stages of the process of finding the eigenvector caused some problems is not surprising when we look at the course manual. We can see from Figure 7 that the final steps in the method are not delineated, but are presumably left for the student to complete. As we can see, they sometimes find this a problem. This omission proves, we think, to be even more costly in terms of conceptual understanding, as we explain below.

## Conceptual object problems

Questions 4 and 8 in the test addressed the conceptual nature of the eigenvector by considering two of its properties. A student with an object perspective of eigenvector might be expected to describe whether a vector is an eigenvector or not, without resorting to a procedural calculation (Q4), and to say that any scalar multiple of the eigenvector will also be an eigenvector (Q8). Students 2, 4, 5, 6, 7, and 8 correctly found the eigenvectors from the procedure. Of these three gave a procedural response to question 4, referring to key aspects in the symbolic world, rather than giving an object-oriented answer.

Student 5: First let the matrix times the vector…If the answer equal to $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ or n times $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ then $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is an eigen vector of the matrix.

Student 6: Using the equation Ab=λb to confirm the relation.

Student 7: Use the formular [sic] Av=λv.

In contrast, the others gave replies based on the word definition, showing some move away from the need to employ the symbolic world towards properties of a formal object:

> Student 4: When a given vector multiply with a matrix, if the direction of the vector doesn't change, only expanded or shrinked we can say the given vector is the eigenvector.
>
> Student 8: When its direction isn't changed when it's multiplied by the matrix.

In their responses to question 8, students 1, 2, 4, 5, and 9 stated that there is only one eigenvector associated with each eigenvalue. Stating, for example, "Each instance of an eigenvalue has one and only one eigenvector associated with it." (student 2) and "One eigenvector is associated to one eigenvalue." (student 9). However, students 3 and 7 said that there were an infinite number, writing "I think that for any eigenvalue can be infinitely [sic] number of eigenvector because [blank]." (4) and "infinity" (7).

As mentioned above (see Figure 7) the course manual did not put in all the details at the end of the method to find the eigenvector. As we see in Figure 8, 2 students (2 and 6,) followed this pattern and tried to write down the vectors from the matrix form of $(A-\lambda I)x=0$, one succeeding (6) and the other not (2). Others (students 7, 8), went further, and were sometimes unsuccessful due to manipulation errors (7, 9—see Figure 6), or managed it correctly (5, 8). In the case of student 5 this was accomplished using $v_1$ and $v_2$, but giving them the values 1 and 2 at a crucial point. However, only one student (4) managed to write the eigenvectors in the form $\begin{pmatrix} v_1 \\ 2v_1 \end{pmatrix} = v_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} v_1 \\ -v_1 \end{pmatrix} = v_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ before getting the eigenvectors correct. Unfortunately, this student was not among only two who were able to say that there are an infinite number of possible eigenvectors. This step of seeing that any scalar multiple $v_1$ of the vector satisfies the equation may be a direct consequence of understanding this last step in the symbolic world manipulation, missing from the manual.

## CONCLUSION

This study suggests that the students who received encouragement to think in an embodied way about eigenvectors found it a useful adjunct to the procedural calculations they carried out in the symbolic world. It seems that these manipulations in the matrix and algebraic domains caused some conflict with understanding the natural language definition of eigenvalues and eigenvectors, and that an embodied approach may mediate initial understanding. We have seen the importance of presenting complete procedures for finding eigenvectors, and of linking these to conceptual ideas such as the number of possible eigenvectors. The two different processes in $Ax = \lambda x$ may be preventing understanding of key ideas and the role of this obstacle requires further investigation. When asked whether they thought computers should be used in linear algebra lectures the majority of students agreed that such work was beneficial, and this may provide a way forward.

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