ON STUDENTS’ CONCEPTIONS IN VECTOR SPACE THEORY

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The study we are going to present is part of a research project which aimed to identify and analyze undergraduate and graduate students’ difficulties and errors in solving Vector Space Theory problems. We report on some students’ errors related to very basic notions of that mathematical domain. More precisely we introduce the ck¢ model, and then we investigate, with the aid of such theoretical framework, the possibility to interpret some students’ emerged errors as instantiations of knowing.

INTRODUCTION

The importance of Linear Algebra in many fields of mathematics, science and engineering is widely acknowledged by both mathematicians and scientists, who consider Linear Algebra as an important mathematical prerequisite for undergraduate students in science and technology. Coherently Linear Algebra courses are basic for a wide variety of disciplines at the tertiary level such as mathematics, physics, computer science and engineering.

Nevertheless Linear Algebra education is a quite recent research field. In 2000 Dorier edited an extensive overview on the state of research in Linear Algebra education at tertiary level, later revised by Dorier himself and Sierpinska (2001) on the occasion of an ICMI Study on math education at tertiary level. From those surveys it emerges that researchers seemingly share the view that difficulties in Linear Algebra (no matter what concepts are involved) are due to general features of the field or to the axiomatic approach usually followed in teaching Linear Algebra. As a matter of fact, in their review Dorier and Sierpinska do not mention any studies focusing on specific concepts of Linear Algebra; and indeed, as far as we know, the only exception is constituted by Nardi’s study on students’ concept-images of span and spanning set (Nardi, 1997).

This apparent ‘characteristic’ of research in Linear Algebra education contrasts, for instance, with research in Calculus where many studies are devoted to the analysis of the cognitive difficulties related to specific concepts such as those of limit, continuity, function, derivative and so on.

OUR STUDY

In this paper we report on a part of our doctorate research project (Maracci, 2005). The general goal of that research was to identify undergraduate and postgraduate students’ errors and difficulties in solving Vector Space Theory (VST from now on) problems.
More in detail, our study focuses on students’ difficulties and errors related to basic notions of VST: linear combination, linear dependence/independence, spanning set, basis, and dimension.

**Methodology**

Our research is articulated in two different but interlaced phases: (a) the analysis of undergraduate textbooks, and (b) the observation and qualitative analysis of undergraduate and graduate students’ behaviours to solve VST problems. This report focuses only on the findings of this latter phase.

The study involved 15 students in Mathematics: 8 first year undergraduates, 4 last year undergraduates and 3 PhD students. The methodology of investigation was that of the clinical interview (Ginsburg, 1981; Swanson et al., 1981; Cohen & Manion, 1994); each student was presented with two or three different problems to be solved in individual sessions; no time constraints was imposed over the problem solving sessions, which were recorded.

The use of the clinical interview is motivated by its flexibility which makes this methodology highly suitable for uncovering phenomena, providing rich descriptions and generating hypotheses (Swanson et al., 1981).

The analysis of the transcripts of the interviews highlighted a number of students’ difficulties concerning basic notions of VST. Some preliminary results have been discussed in Maracci, 2003 and 2004.

Here we present some of the observed difficulties and errors of which we propose an analysis in terms of the ck¢ model (Balacheff, 1995 and 2000; Gaudin, 2002). We are going to introduce such model in the next section.

**THE CK¢ MODEL**

The ck¢ model is an attempt to model the subject’s knowing of mathematics within the theory of situations (Brousseau, 1997). This model explicitly takes in charge the assumption – widely accepted in the community of mathematics educators – that:

> ‘errors are not only the effect of ignorance, of uncertainty, of change, […] but the effect of a previous piece of knowledge which was interesting and successful, but which now is revealed false or simply unadapted.’ (Brousseau 1997, p. 82)

and it attempts to acknowledge both the possible lacking of global coherence and the local efficiency of the subject’s knowing.

The problem of elaborating such a model is faced by formally defining the notion of ‘conception’

1^VST is a subject matter of the first year undergraduate courses in the Mathematics Faculties of the Italian Universities.

2^ck¢ is the acronym for conception, knowing and concept.

3^Artigue (1991) remarks that the notion is widely used even if rarely defined in math education.
of a knowing which, as such, hasproved to be efficient with respect to a certain domain. Within the model, a conception is defined as a quadruplet constituted by:

- **a set P of problems**, on which the conception is efficient – also said the sphere of practice of the conception;
- **a set R of operators**, i.e. a set of both physical and mental actions which the individual can perform to solve a problem;
- **a system L of signifiers**, which allows to represent both problems and operators;
- **a control structure Σ**, which is usually implicit and allows to choose operators, decide their relevance, evaluate their efficiency and decide whether a problem is solved or not.

The first three components are those to which Vergnaud refers in order to define a concept (Vergnaud, 1991); to these components the control structure is added. Once conceptions are defined, one can also formally define knowings and concepts, anyway we won’t present here the complete modelization which can be found in Balacheff, 1995 and 2000.

Let us remark that a given problem may not belong to the sphere of practice of any conception. On the other hand, we can attest a conception because it emerges as a means to solve a problem:

‘c’est sa manifestation en tant que moyen de résolution dans le problème qui nous permet d’attester d’une conception’ (Gaudin 2002, p. 37).

The need emerges to precise the relationship among problems (and their solutions) and conceptions. According to the ck¢ model, the solution of a given problem is a sequence of operators of possible different conceptions which transform the problem itself into one belonging to the sphere of practice of a conception.

As consequences, the subject’s errors in solving problems might be interpreted in terms of conceptions, i.e. in terms of knowings efficient on certain sets of problems.

**OUR QUESTIONS**

We can now explicitly pose the questions we address in this report:

Is it possible to interpret subjects’ difficulties in terms of operators and corresponding controls?

More precisely, is it possible to recognize hypothetical operators and controls with a ck¢-conception like internal consistency?

Let us note that we are not facing the problem of fully characterizing the conceptions to which operators and controls could be referred. To what extent a conception can be characterized on the basis of the analysis of students’ behaviours is an interesting point which we can not address here. Up to now just a few studies have been carried on within the ck¢ model, as a consequence many relevant methodological questions need to be deepened.
THE PROBLEM

The problem we will refer to during our discussion is the following:

Problem. Let \( V \) be a \( \mathbb{R} \)-linear space and let \( u_1, u_2, u_3, u_4 \) and \( u_5 \) be 5 linearly independent vectors in \( V \). Consider the vector \( u = \sqrt{2}u_1-1/3u_2+u_3+3u_4 - \pi u_5 \).

- Do there exist two 3-dimensional subspaces of \( V \), \( W_1 \) and \( W_2 \), such that \( W_1 \cap W_2 = \text{Span} \{ u \} \)?
- Do there exist two 2-dimensional subspaces of \( V \), \( U_1 \) and \( U_2 \), which do not contain \( u \) and such that \( u \) belongs to \( U_1+U_2 \)?

The answer to both the questions is that such subspaces of \( V \) exist. In order to successfully approach the problem one might try to describe the conditions which the subspaces must fulfil in terms of their possible generators. For instance, \( \text{Span}\{u, u_1, u_2\} \) and \( \text{Span}\{u, u_3, u_4\} \) verify the conditions posed in the former question and \( \text{Span}\{u_1, u_2\} \) and \( \text{Span}\{u_3+3u_4, u_5\} \) verify the conditions posed in the latter one.

Although different other approaches to the problem are possible (and many other couples of subspaces could be found) all the interviewed students followed the one sketched above.

DATA ANALYSIS

In this section we will show and analyze few excerpts from the transcripts of the interviews. Before that, we are going to specify the methodology followed for the analysis\(^4\).

Methodology of analysis

We articulate our analysis in 3 steps:

1. coherently with the ckc definition of solution of a problem, we look for possible operators among what the subjects said and did to solve the given problem.
2. Then we take as possible controls those results (definitions and propositions) of VST which are ‘coherent’ with the highlighted operators.
3. We express operators and controls in the same semiotic system whenever it is possible and suitable.

The choice made explicit in step 2 is motivated by the hypotheses that:

1. one constructs operators and controls also attending lectures, studying textbooks, lecture notes, that is by studying the mathematical theory itself;
2. being knowing, operators and controls share potentialities coherent with the mathematical theory.

\(^4\)As previously remarked, also because of the low number of studies with the ckc model, we think that methodological aspects are particularly interesting and relevant.
Moreover as far as operators or controls are compatible with results of the mathematical theory, they also share at least part of the domain of validity of those results.

An example

In this paragraph we present and analyze a brief excerpt of Nic’s interview. Nic, a last year undergraduate student, has correctly answered the former question of the problem. When she approached the latter, she asserted since the very beginning that the answer is negative.

83. Nic: I think it is not possible... because... because in order to write \( u \) I need 5 linearly independent vectors, in order to write it as [element of the] sum of two 2-dimensional vector spaces I can at most use 4 vectors, because they are linearly independent...

Nic spent several minutes to investigate the second question of the problem, without questioning this assertion neither succeeding to elaborate more deeply on it. The argument exposed in this item represents the core of her solution to the problem.

In the quoted item we can recognize the mobilisation of at least two different operators:

- **r₁**: in order to write \( u \) I need 5 linearly independent vectors
- **r₂**: in order to write it as [element of the] sum of two 2-dimensional vector spaces I can at most use 4 vectors

As for \( r₂ \), it is perfectly coherent with many results (not stated by the subject) of VST, among the others let us quote:

- **s₂a**: The dimension of the sum of two 2-dimensional vector spaces is less then or equal to 4.
- **s₂b**: A subspace \( W \) of a given vector space \( V \) is itself a vector space.
- **s₂c**: The dimension of a vector space is the number of vectors of its bases.
- **s₂d**: Given a basis of a vector space, its vectors can be expressed as linear combination of the vectors of the basis.
- **s₂e**: The dimension of a vector space is the highest possible number of linearly independent vectors.

In fact consistently with the five above statements, one can conclude that the elements of a 4-dimensional vector space, such as \( U₁+U₂ \), can be expressed as linear combination of the elements of one of its bases, which contains 4 vectors. Moreover, a 4-dimensional vector space does not contain any linear combination of 5 linearly independent vectors because it does not contain systems of 5 linearly independent vectors at all.

Though \( r₂ \) is coherent with many results of VST, it does not appear adequate to solve the given problem. In our opinion such inadequacy may derive from the fact that \( s₂b \) induces to consider a subspace \( (U₁+U₂) \) as a vector space neglecting the peculiarities of being a subspace: that is the existence of an ‘over’ vector space where systems of
5 (or more) linearly independent vectors may exist as well as linear combination of more than 4 linearly independent vectors.

Even if $r_2$ is not adequate for the given problem, it is anyway adequate when referred to vector spaces. Finally, we want to stress the importance within VST of $s_2b$, which has played a central role in our analysis: indeed $s_2b$ allows to ‘transfer’ notions and properties (i.e. notions of dimension, basis, spanning set) from vector spaces to subspaces and it makes the notion of subspace itself meaningful and relevant.

Let us now discuss $r_1$.

$r_1$: in order to write $u$ I need 5 linearly independent vectors

Possible controls coherent with $r_1$ may be:

$s_1a$: If $v_1,\ldots, v_k$ are linearly independent vectors of a given vector space $V$ over a field $K$, and $a_1,\ldots,a_k, b_1,\ldots,b_k$ are scalars in $K$ such that $a_1v_1+\ldots+a_kv_k=b_1+\ldots+b_k$ then $a_j=b_j$ for each $j=1,\ldots,k$.

$s_1b$: Given a basis of a vector space, each vector of $V$ may be expressed in a unique way as a linear combination of the elements of that basis.

The two statements express similar results: the former one is more coherent to $r_1$ which does not explicitly mention bases, whereas the latter one shares with $r_1$ the same semiotic system of representation. According to $s_1a$ and $s_1b$ a linear combination $a_1v_1+\ldots+a_nv_n$ of $n$ linearly independent vectors $v_1,\ldots,v_n$ cannot be written as linear combination of a subset of those vectors themselves (but some of the scalars $a_1,\ldots,a_n$ are zero). That is, coherently with $s_1a$ and $s_1b$, the number of vectors in a linear combination can not be decreased.

Here again, $r_1$, $s_1a$ and $s_1b$ constitute a coherent system which is coherent with VST too, even if inadequate to solve the given problem.

Throughout this section we have spoken of adequacy or inadequacy of operators and controls to solve problems. When we say that operators and controls are adequate or inadequate to solve a problem, we express the point of view of a conception — the observer’s one – over another conception – the subject’s one: operators and controls mobilised by the subject cannot be inadequate from her own perspective.

More excerpts

For the sake of brevity we cannot discuss other examples so in detail, anyway we present some more excerpts from the collected data which, in spite of slight differences, reveal the mobilisation of operators and control consistent with the ones just discussed.

Let us quote the cases of Fra and Lau, respectively first year and last year undergraduate students: they both correctly answered to the former question of the
problem and then they both failed to solve the latter one. The following items contain the main arguments on which Fra’s and Lau’s respective solutions are based.

74 Fra: I think that it is not possible because \( u \) is linear combination of 5 linearly independent vectors… and if one can write it as… that is, it should be an element which can be written as the sum of an element of \( U_1 \) and of an element of \( U_2 \), and then it should be linear combination of at most 4 linearly independent vectors …

86 Fra: […] anyway \( u \) is written as linear combination of 5 linearly independent vectors… then I cannot write \( u \) with only 4 linearly independent vectors

286 Lau: \( u \) is linear combination of 5 linearly independent vectors, how can I find the fifth if I have at most 4 linearly independent vectors [in \( U_1+U_2 \)]?

295 Lau: \( u \) is linear combination of 5 linearly independent vectors… yes, no, well, 5 linearly independent vectors which I cannot find in the sum [of \( U_1 \) and \( U_2 \)], because the sum is made by 4, at most 4 linearly independent vectors.

The operators respectively mobilised by Fra and Lau evoke more possible controls in addition to the previously discussed ones; unfortunately we cannot analyze them in detail. Just to give some more hints of a possible further analysis, we specify an operator mobilised in the item 74 and we quote some of the corresponding controls.

\[ r_3: \] \( u \) can be written as the sum of an element of \( U_1 \) and of an element of \( U_2 \), and then it should be linear combination of at most 4 linearly independent vectors .

Many of the above discussed controls are more or less directly related to \( r_3 \), to them we can add at least the following:

\[ s_{3a}: \] The elements of \( U_1+U_2 \) can be written as the sum of an element of \( U_1 \) and of an element of \( U_2 \).

\[ s_{3b}: \] The sum of 2 linear combinations of 2 vectors each, is a linear combination of 4 vectors.

\[ s_{3c}: \] Four two by two linearly independent vectors may be not linearly independent.

**SUMMARY**

In the previous section we reported a few excerpts which reveal similar errors and difficulties of different students. The highlighted errors and difficulties concern very basic notions of VST: linear combination, linear independence, basis, spanning set.

As for the analysis of such errors and difficulties within the ckè model, we highlighted systems of operators and controls which present the internal consistency of a conception, in ckè terms, and we showed that some of the emerged difficulties may be interpreted in terms of such operators and controls.

The hypothesized systems of operators and controls are at some extent coherent with definitions and propositions of VST: in fact problems exist to which such operators and controls give solutions consistent with and acceptable within VST.

Therefore these systems of operators and controls express a knowing which shares potentialities consistent with the mathematical theory. Difficulties and errors are due
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to the inadequacy – from the point of view of VST – of this knowing to solve the
posed problem.

References


