

DIDACTIC MODEL – BRIDGING A CONCEPT WITH PHENOMENA

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The article focuses on a specific method of constructing the concept of function. The core of this method is a didactic model that plays two roles together– on the one hand a role of a model of the concept of function and on the other hand a role of a model of physical phenomena that functions can represent. This synergy of modeling situations and constructing mathematical language to describe them is an initial issue in a Learning System.

DIDACTIC MODEL AS A COMPONENT IN A LEARNING SYSTEM

A didactic model is a means for learning a new mathematical concept. It consists of objects familiar to the learner with well-defined operations on them. The operations performed on the objects are rigorously defined and are fully and uniquely mapped onto the formal mathematical operations and syntax of this yet unknown mathematical concept. These operations support the construction of an understanding of more formal reasoning about mathematical concepts and about their essential properties. An example of such model is the Cuisenaire Rods for supporting the foundation for learning the concept of number (described by Nesher 1989).

An attempt to use the didactic models of mathematical concepts -- to address the complexity of establishing a meaningful formal language is described by Nesher (ibid.). At the heart of Nesher's Learning System mathematical models are a major means toward understanding the properties of formal mathematical language. As in traditional approaches, the physical world enters the sequence only after the formal language has been meaningfully understood. Thus, while the didactic modeling offers a significant ways to grasp the properties of the concept of number (e.g., comparison of quantities as objects, group of binary operation) and connect them to formal language, the link between the physical field and the mathematical concept is not firmly established.

The Realistic Mathematics Education (RME) approach suggests that students investigate real contexts on their own (Gravemeijer & Doorman 1999). Within their perspective, the model and the situation modeled co-evolve and are constituted mutually within the course of the modeling activity.

Our approach to function based algebra learning by “guided inquiry”, combines elements of the “guided reinvention” environment (RME approach) and of the Learning System approach. Like the RME developers, we studied the students' ability to re-invent and construct mathematical notations and mathematical concepts (Yerushalmy, 1997). Like the Learning System developers, we take responsibility for

the design of mathematical didactic models that are the core of the conceptual knowledge to be acquired.

Elsewhere we describe in details the rationale for a function approach to algebra that starts with qualitative modeling (Yerushalmy & Shternberg 2001). Here we will describe the design of the didactic models –DM - that promote and enable learning in this approach. About each component of the DM we will make clear to which mathematical object and action it corresponds.

The offered DM has two parts according to two directions for analyzing functions and their changes: from a function to its change and from a change to appropriated functions.

The first part - from a function to its change - consists of a set of seven graphical icons; each icon is a line with a different behavior of change. The other part of DM allows defining function by using another set of icons, which determines discrete “changes” and presents the function as accumulation of these changes. The DM is in fact a piece of software, and as such it is naturally dynamic. For example, each line can be stretched as far as it preserves the behavior of the change (for instance, it remains ascending with a descending change), or the parameters of each change can be altered manually and the function is adjusted accordingly.

Figure 1 presents briefly the main components of the first part of the DM that is relevant to the study concerning in this paper.

Learning with the DM the students have a chance to begin constructing basic mathematical concepts of calculus at their first steps in high school algebra. The meaning of function grows in parallel with the meaning of phenomena this function could model. Far away from being familiarly with symbolical presentation of function, students get mathematical tools for inquiring meaningful mathematics and phenomena.

The different components: icons, verbs, and stairs are eventually adopted as manipulable objects that support students in constructing complex mathematical models. The learning system is designed to map the terrain of the concept of function in a simplified and abstract way. It provides mathematical terminology and an overview of the possible behavior of processes. The icons, that are models of functions and are used for reasoning about functions, are also mathematical models of situations, and together with the stairs model are used for reasoning about phenomena. Thus, the same model serves as the DM of the concept of function and as the mathematical model of a physical situation. While mathematical modeling cannot be fully accomplished by this qualitative sign system, the issue for this study is to identify essential contributions of modeling using didactic models.

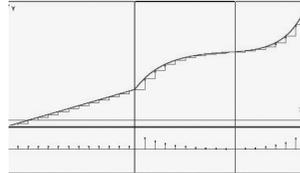
A Context

During the summer cows produce small quantities of milk. As winter nears, **the quantity of milk increases** to the point where the cows reach peak production. In midwinter there is a decrease in the quantity of milk. **The pace of the decline speeds up until the following summer.** Describe how the pace of milk production depends on the time that elapsed from one summer to the next.

The DM:

A model for the concept of function and functional change

The objects of the DM:
seven graphic icons;
each icon is a line representing different behaviors of change
Action on the objects (lines):
constructing different graphs from the lines and
constructing stairs to create a model of a concept of change



and

A model of acting within the contexts that a function could model

Above the lines are the properties of functions in terms of the situation (e.g.: the pace of milk production is increasing). Placing points on the properties (or the domains), one gets the appropriate segment in the context.

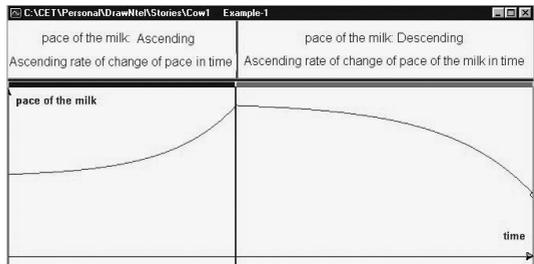


Figure 1: The main components of the DM

Applying the didactic model enables one to analyze phenomena, to describe them mathematically and to communicate about the phenomena and about the functions that represent them. As an illustration, we will introduce the case of pre-algebra students who, before studying symbolic expressions, apply the concept model of the concept of the function to describe a phenomenon.

We will argue that a Learning System that combines the concept model of the concept of function with the mathematical model of the physical situation can help students develop a profound understanding of the mathematics of functions and of its role in describing physical situations.

In the following section we will make some distinctions for describing pre-algebra students' modeling attempts.

REASONING WITH DIDACTIC MODELS: THE CASE OF INSTANTANEOUS SPEED

A group of 34 pre-algebra 7th grade students was interviewed as they were solving seven tasks. The tasks required constructing mathematical models of phenomena characterized by their function or by their changes and distinguished one from the other by the way of presenting the phenomenon and by the aspect of the rate of change that was outlined in the task.

“Motorcycle” was one of the suggested tasks. It presented a continuous kinetic phenomenon by a graph that describes the length of the route of the motorcycle in time (figure 2).

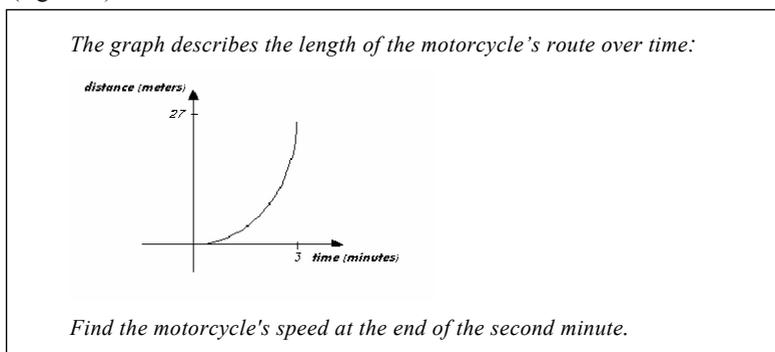


Figure 2: The “Motorcycle” task

It was a new type of problem for the students, because it asked about change at a point (unlike the DM that deals with change over an interval), and because it required a numerical solution (unlike the modeling tasks that ask for graphical and verbal descriptions with which the students were already familiar from their previous class work).

In class, students learned to use the DM to analyze and compare situations, usually in their graphic or verbal forms and with no concern as to their numeric values. “Motorcycle” challenged the students to use the DM in a way that differed from that for which it was designed. In their regular math course, these students had not yet coped with functions in any presentation except the graphic one. A common strategy used by the class, which was also applied to the “Motorcycle” task, was to analyze the behavior of a graph by visually comparing Δf (heights) or $\Delta f/\Delta x$ (slopes). Although this strategy was inadequate for finding the speed at $t=2$, it was obviously very applicative at different steps of the solution process. It was interesting, therefore, to observe whether the didactic model would play any role in the definition of the new concept, its construction, and the process of seeking the solution for the task, i.e. measuring the speed at the second minute.

The examples below demonstrate the use of the DM in defining a new concept about which the students had only some previous intuitive insight.

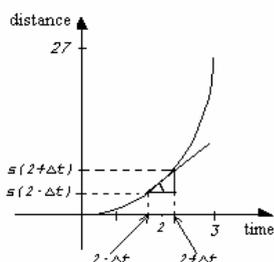
- Defining speed as the distance within the second minute

Tzor: From it (points in the stairs) one can infer the speed. Suddenly I *see* the speed in the graph. The stairs do not continue equally, they are increasing. You can see that at this time (points to a width of a stair) you move this distance (points on the height of the same stair) and from this you derive the speed.

Tzor “sees” speed just as the distance within the whole second minute, as if the speed would be constant during this period.

- Defining speed as the distance within a small unit near $t=2$

Almost all the students defined speed as the distance within a unit of time. Moreover, most of them improved their definition by dividing the 3 minutes into more than 3 units referring to a small unit near $t=2$.

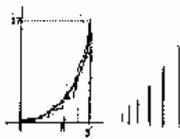


Marina and Rina were first to initiate an explicit discussion about the definition of speed.

Marina: What is speed?

Rina: Speed is the distance traveled in an allocated period of time. In the second minute it means something like this (draws a stair)... In the last minute he traveled almost twice the distance covered in the second minute. Try to make the stairs less wide so that the slope is not too drastic.

Marina: You don't have anything exact between one stair and another. I think that the difference will increase all the time (draws the differences). It comes out that in the second minute he passes something like 5 meters.

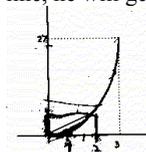


From Marina's drawing we can learn that she divided the 3 minute period into more than 3 stairs and referred to the one that was close to the point of $t=2$.

✦ Referring to the average speed

With regard to average speed, none of the 68 students mentioned the misleading possibility of finding the speed at $t=2$ as the average of the speed during the 3 minutes. All the students indicated that the actual speed, unlike average speed, was not constant.

David: When you ask about the speed in the second minute, do you mean the average in two minutes? But the speed is different each second! If one draws here a straight line, he will get the average speed.



Some of the students went further and tried to define a specific rule for the change of speed, keeping the average as a reference.

Nir: The average for the three minutes is nine.

Shani: But at the beginning he moves a bit slow and then a bit faster. We should therefore add to nine here, and subtract from nine here. You should add and subtract equally.

Nir: No, here the acceleration is fast and you cannot do it equally.

Shani: We check the distance he traveled in the first minute. On average he traveled 9 km in a minute. But the first incline should be a bit less than the average, and then a bit more than average.

Nir: The rise between one and the other cannot be constant. Shani conjectured and I refuted it. He said we should add and subtract equally. It would be 6, 9, 12. It would fit if... it would increase at a constant rate, but the distance does not increase at a constant rate. Actually it cannot be right. If it were a straight line, then the speed would be constant. And here the speed begins slowly and slowly increases.

DISCUSSION

The stairs component of the DM offers presentations of the average rate of change in an interval, and does not offer any presentation of instantaneous speed, such as a tangent. Although most of the students knew intuitively that in order to determine the speed they needed to divide the distance by time, they realized that this would not work when the speed was not constant.

Unable to discover another way to find the speed, they tried using very small intervals of time, and even then they understood that the division would give them the average speed, not the actual one. Applying the DM of stairs, the students created a mathematical model that uses the concept suitable to the processes in a situation, “saw” the speed, and came up with an operative definition for it. The DM linked the speed to the accumulated distance and created a visual presentation of the speed as the change of the function. Some of the students even tried to define a recursive rule

for the change of the function, attempting to present each height of a stair as a function of the height of the previous stair. A striking impact, which was probably a result of using this strategy, was the absence of 'average' considerations that typically appear when considering linearity. Not one of the 68 students used the linearity argument.

The use of the stairs to find the number value of the speed might result in grasping speed as a constant at each unit of time. This did not happen because the students were used to analyzing the change of function as a continuous smooth magnitude.

In the "Motorcycle" task, the students spontaneously gave the DM new roles: as a model of speed and instantaneous speed, as a tool for definition of new concepts and as a presentation of average speed. Thus, the students used the DM not only to analyze the processes, as they were accustomed, but also to define concepts and to do computations. They really connected between the different presentations of speed and took a step forward from viewing speed as a process to defining it as an object.

Carlson et al (2002) reported on an investigation of high-performing 2nd-semester calculus students, where students appeared to have difficulties in forming images of a continuously changing rate and could not interpret accurately the increasing and decreasing rate of dynamic function situations. The 7th grade students in this study, in contrast to the above mentioned calculus students, demonstrated an ability to figure out co-varying quantities, to represent constant and non-constant changes and to make the link between the graph of accumulated quantity to the graph of change. Turner et al (2000) believe that these abilities play an essential role in understanding the concept of change.

The DM helped the students to develop an understanding of the main properties of the concept of instantaneous speed and to build together the concept of the mathematical signifier – a small stair near the point -- and the signified object in the phenomenon –instantaneous speed. It seems that they took a meaningful step towards understanding some important subjects in calculus, such as the average change of function and instantaneous change and the linkage between them.

The students demonstrated an understanding of the relationship between the graphical presentation of speed and its numerical characteristics and communicated about each of the presentations in terms of the modeled situations. They related to the speed as to a process of accumulating distance in a unit of time and also as an object (defining its properties such as its ability to increase). All this allows us to claim that these students took a step forward towards the profound understanding of important concepts in calculus, such as the average change of function, instantaneous change, and the linkage between them.

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