BETWEEN AFFECT AND COGNITION: PROVING AT UNIVERSITY LEVEL

Fulvia Furinghetti* & Francesca Morselli **

*Dipartimento Matematica Università Genova **Dipartimento Matematica Università Torino. Italy

In this paper we report on a case study of a university student (third year of Mathematics course). She was engaged in proving a statement of elementary number theory. We asked her to write the thoughts that accompanied her solving process. She was collaborative and her protocol is suitable to study the interrelation between affect and cognition. We seize in her performance a poor control of the solving process intertwined with emotions and beliefs on the mathematical activity.

INTRODUCTION AND BACKGROUND

It is widely recognized that in mathematical activity "'purely cognitive' behavior is extremely rare" (Schoenfeld, 1983, p.330). Already in the first half of XX century mathematicians such as Henri Poincaré and Jacques Hadamard reflected on the nature of mathematical activity and singled out aspects that nowadays could be ascribed to the affective domain[1]. Nevertheless, traditionally research on students' performances has concentrated primarily on cognition, less on affect, and still less on interactions between them. This way of looking at students' behavior engaged in mathematical tasks has shown its limits. According to Goldin (2002, p.60) "When individuals are doing mathematics, the affective system is not merely auxiliary to cognition – it is central". Leron and Hazzan (1997, p.266) claim that:

> In many cases, cognitive factors may be just the tip of the iceberg. Rather, forces such as the need to make sense, trying to hang on to something familiar, the pressure to produce something (anything!) to fulfill the expectations of the instructors – in short, trying to cope – are dominant.

Recent research, see (Araújo et al., 2003), (Gómez-Chacón, 2000), (Schlöglmann, 2003), addresses to analyze the interrelation of affective and cognitive factors. Following this stream of research in this paper we report on a case that seems to us suitable to integrate affective and cognitive domains. It concerns a university Mathematics student engaged in the proof of the following statement (compare with Euclid, VII, 28):

> Prove that the sum of two relatively prime numbers is prime with each of the addends
> (Two natural numbers are relatively prime if their only common divisor is 1)[2]

Our analysis of the way she faces the proof is in line with the observation made by DeBellis and Goldin (1997, p.211), who think it is essential to focus on "subtle

[1] It is well known that there is a problem of terminology, see (Leder, Pehkonen & Törner, 2002). We adopt the view of McLeod (1992), who claims that "the affective domain refers to a wide range of beliefs, feelings, and moods that are generally regarded as going beyond the domain of cognition. H. A. Simon (1982), in discussing the terminology used to describe the affective domain, suggests that we use affect as a more general term; other terms (for example, beliefs, attitudes, and emotions) [are] used as more specific descriptors of subsets of the affective domain." (p. 576)

[2] The definition of relatively prime numbers was included in the statement to prevent difficulties linked to ignoring it.

emotions, such as puzzlement, curiosity, frustration, or confidence, inherent in solving mathematical problems." Through a meticulous scrutiny of the student's protocol we look for these "subtle emotions", which contribute to shape the student's process of proving.

Our working hypothesis is that in proving a statement a student follows two intersecting pathways: the cognitive, and the affective. The cognitive pathway encompasses steps such as reading the text of the statement, understanding it, designing the plan and developing it through different proving techniques. Our statement, which encompasses just arithmetic and algebraic concepts, could orient students to expect that the proof may be carried out by developing familiar routines of elementary arithmetic and algebra, that is to say through "sequential procedures in which each mathematical action cues the next", see (Barnard & Tall, 1997, p.42). But this 'automatic-like' style is not enough for proving our statement. The proving process is more complex and students have to control and to link together cognitive units, since "mathematical proof requires the synthesis of several cognitive links to derive a new *synthetic* connection" (*ibidem*, p. 42). It is also required to activate anticipatory thinking, see (Boero, 2001). The impossibility of proving only automatically makes essential to mobilize other resources useful to go on, among them to review the different methods of proof (by counterexamples, by contradiction, by induction…) and to choose the most suitable to the purpose. The statement proposed could have been proved rather easily by contradiction: we will see that one of the reasons of difficulty for our student resides in having started the proof without a preliminary reflection on methods of proving, which could have oriented her toward this method.

The cognitive pathway towards the final proof presents stops, dead ends, impasses, steps forwards. The causes of these diversions reside only partially in the domain of cognition; they are also in the domain of affects. Thus, beside the cognitive pathway, we have to consider the affective pathway, which is described by DeBellis and Goldin (1997, p.211) as "a sequence of (local) states of feelings, possibly quite complex, that interact with cognitive representational configurations". The focus of our work will be on the modalities of this interaction.

A main point of our analysis will be the role of beliefs. According to (Schoenfeld, 1983; 1987; 1992) beliefs appear to be an overriding factor in students' performances, more rooted than emotions and attitudes. Beliefs may concern the mathematical activity or the person. Beliefs about self are closely related to notions of metacognition, self-regulation and self-awareness. Among the beliefs about self McLeod (1992) mentions confidence in learning mathematics (e.g. a belief about one's competence in mathematics). This author also takes into account causal attributions (what subjects perceive as cause of their success and failure). They are categorized along three main dimensions: locus (internal vs external), stability (stable vs unstable), controllability (controllable vs uncontrollable) of the causal agent.

We also take into account the categorization of proof schemes given by Harel and Sowder (1998). These schemes are grouped in three main classes: external (ritual, authoritarian, symbolic), empirical (inductive, perceptual), analytical (transformational, axiomatic). In the following we will be concerned with ritual proof scheme (which manifests itself in the behavior of judging mathematical arguments only on the basis of their surface appearance) and symbolic scheme (proof is carried out using symbols without reference to their meaning).

METHODOLOGY

In our study we consider a student attending the final year of the university course in Mathematics. She attended all basic courses (algebra, geometry, analysis…) and some advanced courses in mathematics. Her curriculum encompasses one course of mathematics education, in which our experiment was carried out. In this course students are regularly engaged in activities of proving, developed as follows:

- a problem is given
- the students are aware that the problem is at their grasp
- the students are asked to write the solving process and contemporarily to record the thoughts that accompany their work
- the students work out the problems, solving them individually
- the individual protocols produced are analyzed by all students.

The goal of these activities is not the proof by itself, nor marks are given to the performances. Instead, the students are asked to focus on what they think and do when proving. They are allowed to use pseudonyms (usually they do it). Emotions and feelings are not explicitly mentioned as required information, nor they have been mentioned in the mathematics education course. In order to avoid the influence of time in performances, see (Walen & Williams, 2002), it is given to the students as much time as they need. The case study we refer to (student "Fiore") is set in this context: she worked with a will, was very collaborative, and provided us with rich information. The whole experiment is reported in (Morselli, 2002).

In our analysis Fiore's protocol is split into its component steps, signed by us with numbers (bold font). In this paper the sentences (translated by us) are typed (italic font) in a shape as similar as we can to that in her protocol (signs, symbols, layout are kept). In our comments we will attempt to understand the factors that shaped the student's performance rather than judging it from an expert's perspective.

FIORE'S PROTOCOL AND OUR ANALYSIS

1. *Help! I'm not familiar with prime numbers! ←(but it does not matter)*

Fiore's first reaction is emotional ("*Help!*"): she expresses panic. This panic may be seen as a consequence of her low self-confidence. In the very moment she approaches the statement of the problem she is already judging it out of her grasp. Since she takes for granted that she will have difficulties in solving the problem, she is looking for these difficulties rather than attempting to actually understand the problem. Her concern is to check whether she has the knowledge connected to each word, thus she isolates each word loosing the general meaning of the statement. The result is that her first reading is superficial and not goal oriented. Polya (1945) has pointed out the importance of an efficient reading of the text as the starting point in mathematical performances. The first passage in the protocol brings to the fore affective aspects as a cause of non efficient reading of the text.

Fiore's initial focus is on the word "prime", while the adverb "relatively" is neglected. Her reaction is negative ("*I'm not familiar with prime numbers!*"): she compares what she thinks is needed to solve the problem and what she feels to know: her conclusion is that she is not adequate to the task. We may say that the affective factor (low self-confidence) turns to be a cognitive factor as it influences the way she reads the text. On the other hand her superficial reading enhances her feeling of inadequacy to the task.

The student's behavior reminds us a widespread belief, see (Schoenfeld, 1992), that doing mathematics requires to memorize rather than to understand. Possibly Fiore is not aware of holding it, but we feel that this belief accompanied her along all the solving process, both in the choice of strategies and in the reactions to difficulties. After this first phase of panic she comes back to the text and reads it more carefully. The critic word "prime" is no more frightening, because she caught that the real clue is "relatively prime" and not "prime" alone. The new clue is more comfortable, since being relatively prime refers to a concept (divisibility) that has been treated from early school times and for which she met specific algorithms, while the concept of prime number keeps a degree of mystery and uncontrollability (no way to elementarily decide whether a number is prime or not, etc.). Fiore expresses relief ("*but it doesn't matter*") and writes:

2. *I do a few numerical checks:*
2 and 7 2+7=9. Are 9 and 2 relatively prime? Yes.
Are 9 and 7 relatively prime? Yes

Fiore uses this numerical example just to get in touch with the problem, but she does not see anything in it because she confines herself to one example (which, in addition, is trivial because is based on two prime numbers). Fiore is not really committed because she doesn't rely on exploration; she already has the idea of using algebra as a tool for working on the problem, as shown in the following step:

3. *I stop. It's fine. I'm already thinking of the way of representing two relatively prime numbers through the algebraic language. I have to think for a while.*

Again we find a behavior which brings to the fore Fiore's beliefs about mathematical activity. On one hand the explorative aspect (here numerical exploration) is completely neglected in her way of conceiving mathematical activity, on the other hand she trusts in algebra as a powerful tool to obtain proofs in an automatic-like way. The consequence is that she privileges syntactic aspects of reasoning (manipulation of symbols) rather than semantic ones and appears to be not concerned with meaning. Fiore offers us a telling example of behavior whose aspects refer both to ritual and symbolic proof schemes, see (Harel & Sowder, 1998).

Now the critical point is how to represent two relatively prime numbers, as evidenced by Fiore's sentence "*I have to think for a while*." The cognitive difficulty in finding a representation pushes Fiore to look back to her memories on the subject, but the problem is that she has not an efficient strategy; she goes on trying to recall facts and definitions at random.

The following steps will show an additional problem, i.e. Fiore's low mastering of algebra. She tries to recall the concept of factorization, but she can not master it in an operative way, thus she has to resort to numeric examples.

4. I *have n. m cannot be a multiple of n.* [She calls the two numbers m and n]
Do I have to use the factorization of a number...?
← *natural!*
But how?

2=2·1 and 7=2·3+1 (no, 7 is prime!)
3=3·1 and 8=2·2·2

At this point Fiore's goal has shifted from the main task to the task of recalling the factorization of numbers. She tries to apply the algorithm of factorization to 7 that is a prime number, and this shows that she doesn't associate meaning to what she is doing. We note again that prime numbers are a source of confusion.

5. *6 = 2·3 and it* [the second number m] *cannot be a multiple of 2, it cannot be a multiple of 3…. The first number fitting the requirement is 5, which is prime*

The conclusion of this part of Fiore's reasoning is:

6. *Help! I cannot do it, I still do not see anything. The deepest darkness*

We see here an example of

> the interplay between affective states and the heuristic or strategic decisions taken by students during problem solving [….]. For example, feelings of frustration while doing mathematics may encode (i.e., represent) the fact that a certain strategy has led down a succession of "blind alleys", and (ideally), these feelings may evoke a change of approach. (Goldin, 2002, p.61)

It is remarkable the use of the metaphor based on "*see*" and "*darkness*", which is very efficient for transmitting Fiore's panic when her expectation of immediate feedback and clues is not realized. Again it emerges her view of mathematical activity as an automatic activity, which does not require personal elaboration and creativity. She expects the solving process to be linear, say made up of a sequence of stages that follow each other in a linear and continuous way. When she chooses a path she expects to arrive at the end, that is to say she does not contemplate the possibility of dead ends and failures. When she meets a situation that requires a time of reflection she feels lost ("*The deepest darkness*"). Schoenfeld (1992) points out that one of the typical students' beliefs is that "Students who have understood the mathematics they have studied will be able to solve any assigned problem in five minutes or less" (p.359). Fiore shows the persistency of this belief even at university level: she is a mathematics student, thus, in principle, she is not hostile to this discipline, but this does not prevent her from keeping this misleading belief about mathematical activity. Her underestimation of the role of reflection in the solving process may come from her very poor experience in actively doing mathematics during her school and university career. She has always seen mathematical facts presented as finished products, where the complex process that has brought to the solution is hidden. This is the reason why she lives every step of her solving path that has not an immediate consequence (because it is wrong or because it needs some deepening) as a failure and not as a physiological component of the process.

After this moment of discouragement Fiore goes back to a numerical example:

> **7.** *3 and* $14=7\cdot 2$

Once again the example is trivial and she does not reflect on it (see the further step). This step makes even clearer her view of exploration through numbers: it is a kind of *rite*, which is applied without considering it an actually efficient strategy. The use of numerical examples is simply a way of restarting the process of solution. This restart constantly marks the moments when she feels the failure. After she writes:

> **8.** *n m such that*
> $n = p_1^{r1} p_2^{r2} p_3^{r3}$
>
> *(yes, in general a natural number is a product like this, it is a product of powers of prime numbers)*
>
> $m = p_4^{r4} p^{r5}$
> $n+m = p_1^{r1} p_2^{r2} p_3^{r3} + p_4^{r4} p_5^{r5}$

This representation could hint that if *n*+*m* is not prime with *n* there is a factor, say p_1^{r1}, which is common to *n*+*m* and *n*. It is easy to see that this factor is also a factor of *m*. This contradicts the hypothesis that *n* and *m* are relatively prime. But Fiore, faced with this representation, can not see this development. We hypothesize that her blindness is due to the fact that, after having reached her goal of factorizing, she expects to go on in an almost automatic way. We stress again that this expectation intertwines with the absence of a useful strategy (proof by contradiction) and of critic control of her proving process.

> **9.** *But in algebra* [she means university course of algebra]*, indeed, we have worked a lot on these things! May I have forgotten everything?*
> *[But does this mean I have not understood what I studied???*
> *What a troubling question! I hope it is matter of memory]*

Fiore ascribes the cause of her failure to internal and personal reasons, such as lack of remembering and understanding (internal causal attribution) and not to the lack of a strategy in proving. Fiore focuses on memory because she has the stable belief about mathematical activity as an activity relying on applying rules and algorithms. This, together with the belief about self, generates the internal causal attribution ("*May I* *have forgotten everything?*”). The attribution concerning remembering is rather stable: in the initial step **1** Fiore refers to familiarity that is linked to the past experience, in the last step **9** again she recalls forgetting/remembering. There is a further internal causal attribution linked to understanding (“*But does this mean I have not understood what I studied*???”). This latter causal attribution apparently is less stable or at least, Fiore, being aware of its gravity, attempts to reject it: as a matter of fact she ends by expressing the hope that only remembering is the problem (“*I hope it is matter of memory.*”) Fiore’s belief about remembering is not unusual among university students. Burton (1999, p.31) reports that “68% of the students in the ‘most-able’ class, that is presumably and possibly the future mathematicians, prioritized memory over thought.”

We could say that Fiore is always oriented to reproduce rather than to create, she lacks of mathematical creativity. Maslow (1962) considers self-actualizing creativity, which does not come from a particular talent (“genius”), but from personality. He claims that while ordinary peoples feel uncomfortable with unknown and uncertain situations, creative peoples live such situations as pleasant challenges. Fiore makes explicit her anguish about past (remembering, understanding) and feels unsafe and uncomfortable toward future (the beginning of each step.)

FINAL REMARKS

Our initial assumption on the intertwined nature of cognitive and affective factors has led us to scrutinize through two different lenses (cognitive and affective) Fiore’s protocol.

Through cognitive lens	**Through affective lens**
• poor exploration and production of examples	• low self confidence
• low mastering of algebra	• internal causal attribution
• scant anticipatory thinking	• view of mathematical activity
• no reflection on proving strategies	• low creativity

Only one type of lenses would have deprived us of important details and given an unfaithful picture of the situation, while the two different lenses allow to see causal links among affective and cognitive components of the student’s behavior. From the cognitive point of view it clearly emerges that the discriminating element in filling or not filling the task is the mastering of proving strategies. This finding was quite obvious and predictable, but becomes more interesting if we ask ourselves why Fiore, who, in theory should be ‘expert’, does not master proving strategies. To answer we resort to the domain of affect. In our analysis we have stressed the moments where the beliefs about mathematical activity may have influenced the outputs. As Schoenfeld (1987, p.34) claims, “Beliefs have to do with your mathematical *weltanshauung* or world view. The idea is that your sense of what mathematics is all about will determine how you approach mathematical problems”. In addition to this fact, Fiore shows a low capability to manage her personal emotional responses and their interactions with cognition. Emotions about emotional states and emotions about cognitive states, which are such an important component of her affective pathway, reveal themselves as a burden in her cognitive pathway.

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