

## PROSPECTIVE TEACHERS' IMAGES AND DEFINITIONS: THE CASE OF INFLECTION POINTS

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*This paper describes prospective secondary school mathematics teachers' images and definitions of inflection points. Our data indicate that prospective teachers tended to regard  $f'(x)=0$  and/or the location "where the graph bends" as necessary/sufficient conditions for inflection points. The solutions were based on previous investigations of functions and on daily images associated with driving a car, like "where one has to turn the wheel when driving on a curved track... while still going in the same... general direction". These types of solutions were given even by participants who correctly defined the notion.*

There is wide agreement that teachers should encourage students to present their solutions, raise assumptions, and evaluate each others' suggestions (e.g., Cooney, & Wiegel, 2003; NCTM, 2000). Clearly, teachers conducting such lessons should be able to determine for themselves the validity of students' suggested ideas. Research findings, however, indicate that teachers' and prospective teachers' mathematical knowledge of the topics they teach is not always satisfactory (e.g., Cooney, 1994). There is a call to promote this knowledge (e.g., NCTM, 2000). Any attempt to take this recommendation from theory to practice points to certain prerequisites, such as, familiarity with teachers' common, correct and incorrect solutions, and with possible reasons for their errors as well as with different approaches to promote this knowledge. Here we focus on the first of these prerequisites.

The present study was part of a wider project, designed (a) to extend the existing body of knowledge regarding prospective secondary school mathematics teachers' conceptions of functions, and (b) to examine ways for raising their awareness of their own correct and incorrect ideas. The concept of function was chosen due to its importance in many branches of mathematics and to its central role in the secondary mathematics curriculum in Israel.

We used Tall and Vinnars' (1981) terminology of *concept image* and *concept definition* as a theoretical framework to analyze prospective teachers' conceptions. The term *concept image* is used "to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes", while *concept definition* is "a form of words used to specify the concept" (pp. 152). The authors further emphasize that "a *personal* concept definition can differ from a *formal* concept definition, the latter being a concept definition which is accepted by the mathematical community" (ibid., pp. 152). The terms *concept image* and

*concept definition* proved to be useful in analyzing students' conceptions of various mathematical notions and specifically those related to functions (e.g., limits and continuity: Tall & Vinner, 1981; tangent: Vinner, 1982; Tall, 1986).

While much is known about learners' difficulties when dealing with various notions related to functions, we looked at prospective teachers' grasp of less investigated notions such as inflection points and asymptotes. Here we limit ourselves to prospective teachers' *concept images* and *concept definitions* of inflection points.

Publications about students' conceptions of inflection points usually address issues concerning the related tangent, indicating, for example, that students commonly encounter difficulties in determining from a graph whether a given line was a tangent to a given function at an inflection point (e.g., Artigue, 1992).

In this paper we focus on the question: What *concept images* and what *concept definitions* of inflection points can be identified in prospective secondary school mathematics teachers' solutions to verbally and graphically presented tasks? What are possible reasons for these images and definitions?

## METHODOLOGY

### Participants

We investigated 56 prospective mathematics teachers, who participated in the course "Didactical issues of high school mathematics" (DIM), as part of their studies in a teacher education program for secondary school, at Tel Aviv University (one class of 22 and another of 34 prospective teachers). All but two participants had first degree in mathematics, mathematics education or computer science and a number of them were enrolled in M.A./Ph.D. programs. All in all, their mathematical background was solid, as was their motivation.

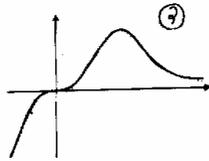
In Israel, functions usually receive considerable attention, and in the upper grades of high school they are commonly treated in calculus lessons in an algorithmic way. As inflection point is one of the notions addressed in high school, the participants in this study had met this notion in their high school studies and probably also in their more advanced studies at the university.

### Tools and Procedure

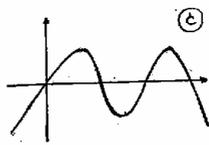
During the DIM course, the first 15-20 minutes of each 90-minute lesson were dedicated to the prospective teachers' individual work on worksheets which included mathematical tasks and occasionally also didactic dilemmas. When the prospective teachers submitted their completed worksheets, they usually continued working on certain tasks in small groups, and then all were engaged in a concluding whole-class discussion. The worksheets distributed in the first and the second lesson included the following three tasks:

**Task 1: The statement:**  $f: R \rightarrow R$  is a continuous, differentiable function. If  $A(x_0, f(x_0))$  is an inflection point, then  $f'(x_0) = 0$  is **correct** / **incorrect** (circle, explain).

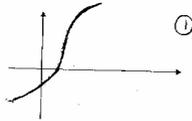
**Task 2:** Given are sketches of graphs. **Mark** on each graph all (possible) inflection points.



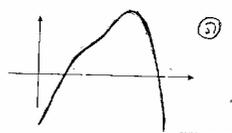
Graph B



Graph A



Graph D



Graph C

**Task 3:** What is an inflection point?

Based on the analysis of their solutions, participants were occasionally invited to individual, follow-up interviews, where they were usually asked by the researchers to elaborate on their written solutions. The interviews took 30-45 minutes, all were audiotaped and transcribed.

## RESULTS

### Prospective Teachers' Reactions to the Statement $f'(x_0) = 0$ (Task 1)

Table 1 indicates that about 40% of the participants correctly answered that the statement is not valid. Most of them incorrectly explained that “the condition should be  $f''(x)=0$ , and not  $f'(x)$ ”. Others, who answered correctly, usually accompanied this judgment with a valid counter-example. One prospective teacher gave  $y=x^3+3$  as an (improper) counter-example.

**Table 1: Prospective teachers' reactions to the statement ( $f'(x)=0$ )**

JUDGEMENT			
	JUSTIFICATION	N=53	%
<b>FALSE*</b>	The condition is... (e.g., $f''=0$ )	20	37.7
	Counter-example	16	30.2
<b>TRUE</b>		4	7.5
		<b>31</b>	<b>58.5</b>
	The definition is... (e.g., $f'(x)=0$ and $f''(x)=0$ )	9	17.0
	That's the definition...	15	28.3
	Algorithmic considerations...	2	3.8
	Irrelevant / No justification	5	9.4
<b>NO ANSWER</b>		2	3.8

\* correct judgment

It seemed that this participant knew that  $f'(x)=0$  is not a necessary condition for an inflection point and that a counter example is needed in order to refute the statement. Still, she was not careful in examining the example she chose.

About 60% of the participants incorrectly claimed that the statement was valid and provided four types of justifications to this judgment: (a) “The definition is...” – providing incorrect definitions, writing, for instance, “ $f'(x)=0$  and  $f''(x)=0$  are the conditions for an inflection point”; or “at an inflection point of  $f$ ,  $f'(x)=0$  and the function either keeps increasing or keeps decreasing”; or “an inflection point is a point where  $f'(x)=0$  and the graph bends”; (b) “That’s the definition” – declaring without actually defining; (c) Algorithmic considerations – addressing the methods in which they used to solve “investigate the function” tasks, where “[we] always found inflection points when looking for extreme points, thus starting this search with  $f'(x)=0$ ”; and (d) Irrelevant or no justifications.

### **Prospective Teachers’ Reactions to the Graphic Representation (Task 2)**

Table 2 indicates that only few participants provided a complete identification of the inflection points in graphs A ( $P_1$  and  $P_2$ ), C ( $P_6$  and  $P_7$ ), and D ( $P_8$ ), and no such solution was given to Graph B. In reaction to Graph A, most prospective teachers identified neither  $P_1$  nor  $P_2$ . That is, most participants identified no inflection points on Graph A. In reaction to Graph B, almost all the prospective teachers identified  $P_3$ , but only few identified  $P_4$  and none identified  $P_5$ . The reactions to Graphs C and D included three types of solutions: (a) correctly identifying all or one of the points; (b) identifying no inflection points, or (c) incorrectly marking the points “where the graph bends” ( $T_6$ ,  $T_7$ ,  $T_8$ ).

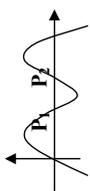
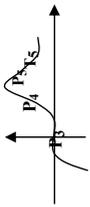
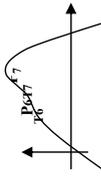
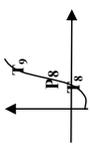
In their interviews, several of the latter participants, who incorrectly marked the points  $T_6$ ,  $T_7$ , or  $T_8$ , explained their solution in terms of “driving on curved tracks”. For example, “I imagine myself driving north, for instance, on a curved road... where I have to turn the wheel... but still go in the same direction... like... keep north... not turn back to the south... the point where I turn the wheel is an inflection point... in the graphical sense...”

A closer look at the data, as summarized in Table 3, shows that  $P_3$ , the inflection point where  $f'(x)=0$ , was identified by most prospective teachers as such,  $P_5$  the inflection point between a Maximum point and a horizontal asymptote was identified by no one (it was either overlooked or misplaced), and almost all prospective teachers misplaced the “vertical inflection point”  $P_8$ .

The points  $P_1$ ,  $P_2$ , (between two extremes) and  $P_4$ , (between an inflection point and a Max point) were correctly identified by a small number of prospective teachers and ignored by others. The points  $P_6$  and  $P_7$  gained a mixture of reactions: correct identifications, incorrect ones and no identification at all.

All in all, when examining the different types of solutions given by the prospective teachers to all the four tasks, two phenomena are evident: (1) the “horizontal inflection point” where  $y'=0$  was easily identified as such, and (2)

Table 2: Prospective teachers' identifications of inflection points (frequencies, %)

												
The Points	N	%	The Points	N	%	The Points	N	%				
<b>Complete solution</b>	$P_1 + P_2$	9	17.3	$P_3 + P_4 + P_5$	-	$P_6 + P_7$	4	7.7	$P_8$	9	17.3	
<b>Partial solution</b>												
	$P_3 + P_4$	3	5.8									
	$P_3$	43	82.7			$P_6$	5	9.6				
<b>Correct &amp; Incorrect Points</b>												
	$P_3 + P_4 + T_5$	2	3.8	$P_7 + T_6$	2	3.8						
	$P_3 + T_5$	1	2.0									
	$P_4 + T_5$	1	2.0									
<b>Incorrect points</b>												
				$T_6 + T_7$	7	13.5	$T_8 + T_9$	5	9.6			
				$T_6$	2	3.8	$T_8$	16	30.8			
				$T_7$	10	19.3	$T_9$	4	7.7			
<b>No Points</b>												
		43	82.7		2	3.8		22	42.3		18	34.6

$P_i$  - correct identification of a specific point       $T_i$  - incorrect identification of a point      N=52

most prospective teachers, at least once, erroneously regarded the “point where the curve bends” to be an inflection point (see + for  $T_1$ , under the borderline, Table 3).

**Table 3: Summary of Identification of Inflection Points (%)**

THE POINTS:								FREQUENCIES			
P <sub>1</sub>	P <sub>2</sub>	P <sub>3</sub>	P <sub>4</sub>	P <sub>5</sub>	P <sub>6</sub>	P <sub>7</sub>	P <sub>8</sub>	T <sub>5</sub>	T <sub>6</sub> /T <sub>7</sub>	T <sub>8</sub> /T <sub>9</sub>	N=52
-	-	+	-	-	-	-	-	-	-	-	21.0
-	-	+	-	-	+	-	+	-	-	-	5.8
-	-	+	-	-	-	-	+	-	-	-	2.0
+	+	+	-	-	+	+	+	-	-	-	2.0
-	-	+	-	-	-	-	-	-	+	+	19.3
-	-	+	-	-	-	-	-	-	-	+	9.6
-	-	+	-	-	-	-	-	-	+	-	7.8
-	-	+	-	-	-	-	+	-	+	-	5.8
-	-	+	-	-	+	-	-	-	-	+	3.8
+	+	+	+	-	+	+	-	-	-	+	2.0
+	+	+	+	-	+	+	-	+	-	+	2.0
+	+	+	-	-	+	+	-	-	-	+	2.0
+	+	+	+	-	-	+	-	+	+	+	2.0
+	+	+	+	-	-	-	-	-	+	+	2.0
+	+	+	-	-	-	-	-	-	-	-	2.0
+	+	+	-	-	-	-	-	-	-	+	2.0
-	-	+	+	-	-	+	+	-	+	-	2.0
-	-	-	+	-	-	-	-	+	-	-	2.0

+ correct identification      - no identification      Ti erroneous identification

**Prospective Teachers’ Reactions to “What is an inflection point?” (Task 3)**

The explanations of most prospective teachers included necessary and / or sufficient conditions for inflection points (Table 4). About 60% used notions of convex-concave: “a point where the function turns from concave to convex or vice versa”: about 10% of the participants presented tangent / slope ideas: “the point where the slope stops increasing and starts decreasing or the other way around”; and another 8% used  $f'(x)=0$  considerations, which are necessary but insufficient. However, about 15% of the participants expressed the erroneous view that  $f(x)=0$  or slope=0 are necessary conditions for inflection points.

**DISCUSSION**

The discussion addresses the two questions posed in the introduction: What *concept images* and what *concept definitions* can be identified in prospective secondary school mathematics teachers’ solutions to verbally and graphically presented tasks? What are possible reasons for these images and definitions?

***Concept images and concept definitions of inflection points***

Our data indicate that commonly the concept image of inflection points included

two types of points: those that fulfill the requisite of  $f''(x)=0$ , and those that are (mis)placed in the spot where the curve bends. Other points, like  $P_1$  and  $P_2$ ,

**Table 4: Prospective teachers' definitions for inflection points**

DEFINITION	FREQUENCIES	
	N = 52	%
Convex-Concave*	26	50.0
Convex-Concave* & $f'(x)=0^*$ / Increase-Increase	4	7.7
Convex-Concave* & Slope Zero	1	2.0
Tangent* / Slope*	5	9.8
$f''(x)=0^*$ & $f'(x)=0$ & Increase-Increase	1	2.0
$f''(x)=0^*$ & $f'(x)=0$	1	2.0
$f''(x)=0^*$	2	3.8
$f'(x)=0$ / Slope Zero & Increase-Increase	5	9.6
Increase-Increase	3	5.8
No answer	4	7.7

which fulfill neither  $f''(x)=0$ , nor are located in the spot where the curve bends, were frequently overlooked. These conclusions evolved from the prospective teachers' solutions to the graphical tasks and were supported by (a) their tendency to accept the statement: " $f''(x)=0$  in inflection points", and (b) their explanations during the individual interviews. These explanations gave us a clue regarding reasons that might underlie the prospective teachers' concept image of inflection points, so we shall address them in the following section.

In their responses to the question "What is an inflection point?" the participants gave their *personal concept definitions* to specify the concept. We did not explicitly ask for a mathematical definition. Since a description is meant to point to the notion under consideration, enabling to distinguish it from other notions, but not necessarily in the most economic manner. So we expected necessary, sufficient, and possibly some additional, yet relevant conditions. While most participants met this expectation, providing concave-convex plus other considerations, several participants suggested only an insufficient condition ( $f''(x)=0$ ), while others added the redundant condition of  $f'(x)=0$ . None of the participants referred to continuous and differentiable functions.

These data provided us with additional evidence about prospective teachers' tendency to regard  $f''(x)=0$  as a necessary condition for inflection points. It also showed that, like in other topics, the correct ideas presented in the definition (e.g., concave-convex) are not necessarily implemented when solving problems.

### Possible reasons for these images and definitions

We recognized two main sources for the prospective teachers' images of inflection points: one rooted in their previous mathematical studies, and the other in daily life. The unnecessary condition  $f''(x)=0$  may result from extensive algorithmic experience with investigations of functions, where the only

inflection points explicitly addressed were those that were found when " $f'(x)=0$ , but there is no extreme point". Although high school students also encounter inflection points of  $y=\sin(x)$  or  $y=\tan(x)$ , they rarely if ever address these inflection points in class. In this spirit, some prospective teachers explained "sure...  $y'=0$ ... this is how you start looking for inflection points... when looking for max... or min... you know... in the investigation of functions..."

On the other hand, the participants' daily experience with driving on curved roads seemed to underlie the erroneous location of inflection points in the "peak of a curve that keeps its up-up or down-down direction". These participants presented "driving and turning the wheel, while keeping the north", "imagine a turning point of a river", or "it's like the point where a plane changes the slope of its flight during its takeoff" considerations. These participants occasionally added some gestures with their hands to make this explanation more vivid.

Clearly, further research about prospective teachers' teachers' and students' concept images and their concept definitions of inflection points, is needed.

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