

# LEGITIMIZATION OF THE GRAPHIC REGISTER IN PROBLEM SOLVING AT THE UNDERGRADUATE LEVEL. THE CASE OF THE IMPROPER INTEGRAL

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*In this work we show some activities designed with a First Year group of the Mathematics Degree to give the graphic register back its mathematical status and promote its use on the part of the students. In particular, we have chosen the topic related to improper integration to reinforce the use of this register by the students.*

## INTRODUCTION AND BACKGROUND

Some research has stated the reticence Mathematics students show to use the graphic register when they have to solve problems or explain what they do. In particular, this reticence appears to be bigger at University level. On the one hand, the lack of practice in lower levels make it difficult for them to use this register in a natural way; on the other hand, in Higher Teaching this register is usually accused of being “not very mathematical”. However, its use may help to avoid numerous calculi or even may be used as a “control” and “prediction” register for purely algebraic work. Eisenberg & Dreyfus (1991) enumerate three reasons why visual aspects are rejected:

- Cognitive: visual thinking requires higher cognitive demands than the ones needed to think algorithmically.
- Sociological: visual aspects are harder to teach.
- Beliefs about the nature of Mathematics: visual aspects are not mathematical.

Mundy (1987) points out that students usually only have a mechanic comprehension of basic concepts of Calculus because they have not reached a visual comprehension of the underlying basic notions; in particular, he states that students do not have a visual comprehension of the integrals of positive functions being thought in terms of areas under a curve (which confirms Orton’s (1983) and Hitt’s (2003) outcomes on the dominance of a merely algebraic thought in students, even in teachers, when solving questions related to integration).

Other authors’ works (Swan, 1988; Vinner, 1989) reinforce the hypothesis that students have a strong tendency to think algebraically more than visually, even when pushed to a visual thought. These authors consider that many of the difficulties in Calculus may be avoided if students were taught to interiorise the visual connotations of the concepts of Calculus.

Among our results (González-Martín & Camacho, 2003), in accordance with the previous ones, we observe that, in general, students prefer algorithmic-type headings with clear instructions of what is demanded. Moreover, when the non-algorithmic questions use the graphic register, their resolution produces big difficulties in students (who do not use it regularly) or a high rate of no answers. Many students not even recognise the graphic register as a register for mathematical work. On the other hand, lack of coordination between registers produces difficulties to students and some paradoxes make them hesitate. This lack of coordination, or lack of an adequate use of one of the registers, takes them away both from anticipation tools of results to be obtained and control tools of obtained results. We also detect some difficulties and obstacles (González-Martín, 2002), some of them specific to the improper integral concept, as the *bond to compacity* (inability to conceive a volume, or an area, as finite unless the figure is closed and bounded) and the *homogenisation of dimensions* (a volume is attributed with the properties of the area that generates it by revolution, so it is thought that an infinite area will originate an infinite volume). They both may be aggravated by a lack of coordination between registers.

## THEORETICAL FRAMEWORK

Aiming to design our teaching sequence we carried out a cognitive analysis of the concepts at issue, trying to detect some difficulties, obstacles and errors that a traditional teaching generates in the students and a competence model was designed in order to assess several students' comprehension (Camacho & González-Martín, 2002). For this analysis we took into account, essentially, Duval's (1993) theory of semiotic representation systems, but we also considered other author's contributions on the role of errors and problem solving in the theory of representation systems (Hitt, 2000).

When it came to design our activities, we gave great importance to the variations of the typical didactic contract and to the construction of an adequate *environment*<sup>1</sup> for each situation (Brousseau, 1988), so that it produces contradictions, difficulties or imbalances. This initial condition of "no control" should produce an adaptation by the students to try to solve the problematic situation given. To promote this interaction, the environment has been designed in such a way that the student can use the knowledge he has to try to control it.

On the other hand, it has also been designed in such a way that allows the student's work to be as autonomous as possible and his acceptance of the given responsibility. This didactic contract is completely new for our students, so we begin with situations close to them to provoke a gradual acceptance of this new contract. The

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<sup>1</sup> We have chosen the term *environment* to translate the French *milieu*.

environments designed allow not only the building of knowledge, but also of mathematical knowledge capable of institutionalisation.

## METHODOLOGY

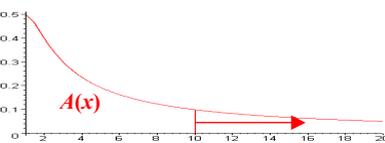
The second stage of our research consists on the design of a teaching sequence that helps to mitigate the numerous lacks observed in our students. However, this sequence also carries out a function of research methodology, which lets us observe and analyse the learning achieved by the students and assess it in an objective way.

The sequence was developed with First Year students of the Mathematics degree and about 25 students took part regularly. Some of the characteristics of our teaching sequence are the articulation of the graphic register with the algebraic one, the reconstruction of knowledge from previously studied concepts (series and definite integrals), the student is given a bigger responsibility in his learning process, the use of non-routine problems (Monaghan *et al*, 1999) and the systematic construction of examples and counter-examples in the two registers.

The graphic register is first presented to interpret some results and later to predict and apply some divergence criteria. On the other hand, we show the students some constraints of this register, which will make necessary the use of the algebraic register. This way, the use of the graphic register, with its potentialities and feeblednesses, together with the use of the algebraic register will facilitate the coordination between both registers.

The limitation to the study of positive functions, in a first moment, and the graphic interpretation of the calculus of areas may justify the definition by means of limits of the improper integral with unbounded integration

$$\text{interval: } \int_a^\infty f(x).dx = \lim_{b \rightarrow \infty} \int_a^b f(x).dx .$$



The study of the behaviour of these two integrals:

$$a) \int_0^\infty e^{-x} dx = 1 \quad b) \int_1^\infty x^{-1/3} dx = \infty$$

gives cause for observing that two functions with a very similar graph (in particular when handmade) may enclose quite different areas. The students may think over the possibility to predict *when the integral will diverge*. Is in this situation that the graphic register, if  $f(x)$  is positive, allows us to assure that if from a given value on  $f(x) \geq k > 0$ , *the integral will then be divergent*. This conclusion, together with the two already calculated examples, lets the students see the potentialities of the graphic register to

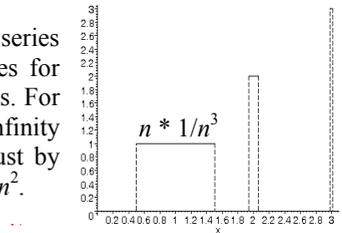
conclude divergence of a given integral and its feebleness to predict convergence, what justifies the development of more formal tools.

The construction of a table with the convergence of the integral of the most usual functions, as the one showed near, lets operationalise both the new definition for the improper integral and the divergence criterion obtained, reinforcing the mathematical status of the graphic register, since the divergence of the integrals indicated with a (\*) can be assured using the graphic register.

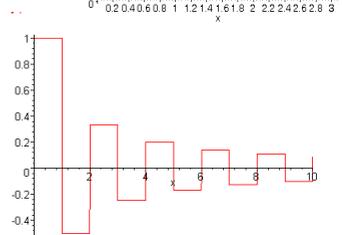
Funtion $f(x)$	Value of the integral from $\alpha > 0$ to infinity $\int_{\alpha}^{+\infty} f(x)dx$	
0	Convergent (= 0) *	
$\frac{1}{x}$	Divergent *	
$x^k, k > 0$	Divergent *	
$\frac{1}{x^k}, k > 0$	$k \leq 1$	Divergent
	$k > 1$	Convergent
$a^{kx}, a > 1$	$k \geq 0$	Divergent *
	$k < 0$	Convergent
$\ln kx, k > 0$	Divergent *	
$\sin kx$	Divergent	
$\cos kx$	Divergent	

Moreover, the fact that this table is constructed between all the students favours their implication in the construction of knowledge and the devolution of the given task, following Brousseau's (1988) ideas. Finally, this table will be used later, when the comparison criteria are studied, so the students will feel participants in the theoretical development of the concepts.

The graphic register and the use of the theory of series also allows the construction of useful counter-examples for questions that usually produce difficulties to the students. For instance, a non-negative function with no limit at infinity whose improper integral is convergent may be built just by constructing over each integer  $n$  a rectangle with area  $1/n^2$ .



Another quite useful counter-example is provided by the construction of a function whose integral converges, but not absolutely. The classic counter-example is the function  $f(x) = \frac{\sin x}{x}$ , out of the students' intuitive reach. Using the theory of series and the graphic register, it is much easier to build counter-examples of functions that converge conditionally. In particular, we show a piecewise



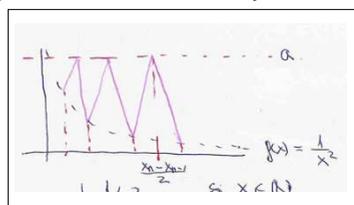
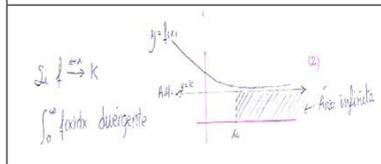
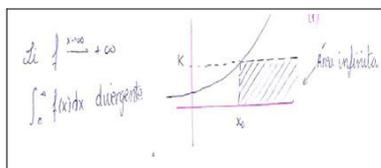
continuous function which over each integer interval  $[n, n + 1)$  equals  $(-1)^n \cdot \frac{1}{n+1}$ . Its integral converge, but the integral of the absolute value coincides with the harmonic series, divergent<sup>2</sup>.

## DATA COLLECTION, ANALYSIS AND DISCUSSION

Our sequence is assessed in several ways. During its implementation some working sheets are given to the students to be worked out in small groups, answering to new questions using the elements recently introduced; they are also asked to give the teacher a table of convergence of the integrals of the usual functions and the resolution of some problems. The sequence, globally, is evaluated by means of a contents test (with some questions previously used in our preliminary study: González-Martín, 2002; González-Martín & Camacho, 2003). Finally, the students are also given an opinion test about the most relevant aspects of our design.

In our classroom observations we can clearly tell the students' gradual acceptance of the graphic register in order to formulate some conjectures from the moment the *divergence criterion* is illustrated. At the moment of constructing the table of convergences, the students use graphic reasoning to conclude the divergence of the corresponding integrals and state it helps to avoid long calculi. Later, the work carried out in small groups is shared and the teacher gives his approval, what helps to institutionalise this register as a mathematical register. Afterwards, in the sheets given to the students we can see how they use much graphic reasoning. For instance, to analyse the different behaviours of a positive function in a neighbourhood of the infinity in order to conclude the convergence or divergence of its integral; also, to prove the falsehood of the following

statement: " $\sum_{n=1}^{\infty} f(n) < \infty \Rightarrow \int_1^{\infty} f(x)dx < \infty$ " a group of three students constructs the counter-example shown next<sup>3</sup>.



<sup>2</sup> In these activities, we emphasize the construction of the functions graphically and not the obtaining of their formulae.

<sup>3</sup> They create "triangles" joining the points  $(n, 1/n^2)$ ,  $(n + 1/2, a)$  and  $(n + 1, 1/(n + 1)^2)$ .

Furthermore, the students show their satisfaction with the use of the graphic register in their answers to the opinion test (completed by 24 of the students who took part in our sequence) to question 17: “ I think the use of graphics as a part of the mathematical work is”:

a)	0- Uninteresting. It is not formal	1- Not very interesting	2- Interesting	3- Very interesting and useful
b)	0- It has confused me	1- Indifferent	2- Helps a bit to understand things	3- Helps a lot to understand things

	PREG17A	PREG17B
N of cases	24	22
Minimum	2.000	2.000
Maximum	3.000	3.000
Median	2.000	3.000
Mean	2.458	2.545
Standard Dev	0.509	0.510

In their answers we can observe that two students did not answer the second part of this question. What is quite positive is that the minimums have been, in both parts, 2. Besides, the median in the second part is 3, so more than a half of the students think that its use “helps a lot to understand things”.

The exact distribution of the answers is the one shown next.

		17-a	
		2	3
17-b	No ans.	<b>1</b>	<b>1</b>
	2	<b>7</b>	<b>2</b>
	3	<b>5</b>	<b>8</b>

On the other hand, in the contents test, done by 26 students, the questions that needed the graphic register have been answered by a higher percentage than in a group that followed a traditional instruction.

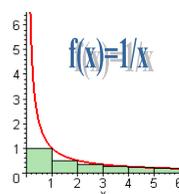
For instance, in the second question, which only had *one* correct answer between the 31 participants of the group with a traditional teaching (who interpreted the graph given and sketched a similar one to explain the behaviour of the second integral) we got the following answers:

**Question 2:**

We know that  $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$  and  $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ .

In view of these results, what can you say about the value of  $\int_1^{\infty} \frac{1}{x} dx$  y  $\int_1^{\infty} \frac{1}{x^2} dx$ ?

Use the graph given.



- Answers correctly to the first question using clearly the graph: 13
- Sketches a similar graph for the second integral: 6
- Answers correctly to the second question using graphic reasoning: 8

In the third question we got the following answers:

- Answers correctly with graphic reasoning (asymptote): 14
- Answers correctly with algebraic reasoning (criteria): 3
- Infers that the integral of a positive function cannot be negative (area): 2
- Sketches a graph of the function: 3

**Question 3:**

Can you find any mistake in the resolution of the following integral?

$$\int_2^5 \frac{dx}{(x-4)^2} = \frac{-1}{x-4} \Big|_2^5 = \frac{-1}{5-4} - \left( \frac{-1}{2-4} \right) =$$
$$= -1 + \frac{1}{-2} = -1 - \frac{1}{2} = -\frac{3}{2}.$$

## CONCLUSIONS

In this work we have shown some activities, related to the topic of improper integration, that try to reinforce the mathematical status of the graphic register in university students. In particular, we believe that the changes in the usual didactic contract (so that the students themselves can see the utilities and limitations of the graphic register) and the work constructing examples and counter-examples, together with the graphic interpretation of results, allow to recognise this register and to accept it. On the other hand, the approval of the teacher reinforces its mathematical status, allowing later institutionalisation.

As said before (Eisenberg & Dreyfus, 1991), unwillingness to the use of this register is quite strong, the cognitive demands it requires are higher. As a consequence, we feel that its use should not be done in an isolated way, but as an habitual part of the instruction, in such a way that the student accepts it and has the “approval” of the teacher. For this reason, its use as a part of an experience (as in our case) is positive, but may just become anecdotic if, once accepted, its use is not reinforced later.

Therefore, some of the open questions that remain are its use regularly during a whole semester, the analysis of the change of the students’ attitude towards it and whether they would use it in non-routine questions. Our results, although local, support the hypothesis that undergraduate students may accept it if its utility is motivated and it is used in a reasonable way.

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